Communications in Partial Differential Equations, 31: 959–985, 2006

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300500358129

Quasi-Periodic Solutions of Completely Resonant Forced Wave Equations

MASSIMILIANO BERTI[1] AND MICHELA PROCESI[2]

[1]Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Napoli, Italy
[2]Dipartimento di Matematica, University of Rome, Rome, Italy

We prove existence of quasi-periodic solutions with two frequencies of completely resonant, periodically forced nonlinear wave equations with periodic spatial boundary conditions. We consider both the cases when the forcing frequency is: (Case A) a rational number and (Case B) an irrational number.

Keywords Infinite dimensional Hamiltonian systems; Lyapunov–Schmidt reduction; Nonlinear wave equation; Quasi-periodic solutions; Variational methods.

2000 AMS Subject Classification: 35L05; 37K50; 58E05.

1. Introduction

We prove existence of small amplitude quasi-periodic solutions for completely resonant forced nonlinear wave equations like

$$\begin{cases} v_{tt} - v_{xx} + f(\omega_1 t, v) = 0 \\ v(t, x) = v(t, x + 2\pi) \end{cases} \tag{1.1}$$

where the nonlinear forcing term

$$f(\omega_1 t, v) = a(\omega_1 t) v^{2d-1} + O(v^{2d}), \quad d > 1, d \in \mathbb{N}^+$$

is $2\pi/\omega_1$-periodic in time. We shall consider the following two cases:

A) the forcing frequency $\omega_1 \in \mathbb{Q}$;
B) the forcing frequency $\omega_1 \in \mathbb{R} \backslash \mathbb{Q}$.

The existence of periodic solutions for completely resonant forced wave equations was first proven in the pioneering articles Rabinowitz (1967, 1971) (with Dirichlet boundary conditions) if the forcing frequency is a rational number ($\omega_1 = 1$

Received February 1, 2005; Revised and Accepted August 1, 2005
Address correspondence to Massimiliano Berti, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, Monte S. Angelo, I-80126 Napoli, Italy; E-mail: berti@sissa.it

in Rabinowitz, 1967, 1971). This requires to solve an infinite dimensional bifurcation equation which lacks compactness property; see Brezis and Nirenberg (1978), Coron (1983), and Berti and Biasco (2005, to appear) and references therein for other results. If the forcing frequency is an irrational number, the existence of periodic solutions has been proven in Plotinikov and Yungermann (1988) and McKenna (1985): here the bifurcation equation is trivial but a "small divisors problem" appears.

To prove the existence of small amplitude quasi-periodic solutions for completely resonant PDEs like (1.1) one generally has to deal with a small divisor problem as well; however, the main difficulty is to understand from which solutions of the linearized equation at $v = 0$,

$$v_{tt} - v_{xx} = 0,$$

quasi-periodic solutions branch-off: such linearized equation possesses only 2π-periodic solutions $q_+(t+x) + q_-(t-x)$ where $q_+(\cdot)$, $q_-(\cdot)$ are 2π-periodic (completely resonant PDE).

Here is the main difference with regard to (w.r.t) nonresonant PDEs for which a developed existence theory of periodic and quasi-periodic solutions has been established, see e.g., Kuksin (2000), Wayne (1990), Craig and Wayne (1993), Pöschel (1996), and Bourgain (1998) and references therein.

For completely resonant autonomous PDEs, existence of periodic solutions has been proven in Lidskiĭ and Shul′man (1988), Bourgain (1999), Bambusi and Paleari (2001), Berti and Bolle (2003, 2004, to appear), Gentile et al. (2005), and Gentile and Procesi (2006), and quasi-periodic solutions with 2-frequencies have been recently obtained in Procesi (2005, Preprint) for the specific nonlinearities $f = u^3 + O(u^5)$. Here the bifurcation equation is solved by ODE methods.

In this article we prove the existence of quasi-periodic solutions with two frequencies ω_1, ω_2 for the completely resonant forced equation (1.1) in both cases: Case A) : $\omega_1 \in \mathbb{Q}$; Case B): $\omega_1 \in \mathbb{R}\backslash\mathbb{Q}$.

The more interesting case is $\omega_1 \in \mathbb{Q}$ (Case A) when the forcing frequency ω_1 enters in resonance with the linear frequency 1. To find out from which solutions of the linearized equation quasi-periodic solutions of (1.1) branch-off, it is required to solve an infinite dimensional bifurcation equation which cannot be solved in general by ODE techniques (it is a system of integro-differential equations). However, exploiting the variational nature of equation (1.1) like in Berti and Bolle (2003, 2004), the bifurcation problem can be reduced to finding critical points of a suitable action functional which, in this case, possesses the infinite dimensional linking geometry (Benci and Rabinowitz, 1979).

1.1. *Main Results*

We look for quasi-periodic solutions $v(t, x)$ of equation (1.1) of the form

$$\begin{cases} v(t,x) = u(\omega_1 t, \omega_2 t + x) \\ u(\varphi_1 + 2k_1\pi, \varphi_2 + 2k_2\pi) = u(\varphi_1, \varphi_2), \quad \forall k_1, k_2 \in \mathbb{Z} \end{cases} \tag{1.2}$$

with frequencies

$$\omega = (\omega_1, \omega_2) = (\omega_1, 1 + \varepsilon),$$

imposing the frequency $\omega_2 = 1 + \varepsilon$ to be close to the linear frequency 1.

Writing $\partial_{tt} - \partial_{xx} = (\partial_t - \partial_x) \circ (\partial_t + \partial_x)$ we get

$$\big[\omega_1 \partial_{\varphi_1} + (\omega_2 - 1)\partial_{\varphi_2}\big] \circ \big[\omega_1 \partial_{\varphi_1} + (\omega_2 + 1)\partial_{\varphi_2}\big] u + f(\varphi_1, u) = 0 \tag{1.3}$$

and therefore

$$\big[\omega_1^2 \partial_{\varphi_1}^2 + (\omega_2^2 - 1)\partial_{\varphi_2}^2 + 2\omega_1\omega_2 \partial_{\varphi_1}\partial_{\varphi_2}\big] u(\varphi) + f(\varphi_1, u) = 0. \tag{1.4}$$

We assume that the forcing term $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

$$f(\varphi_1, u) = a_{2d-1}(\varphi_1) u^{2d-1} + O(u^{2d}), \quad d \in \mathbb{N}^+, \ d > 1$$

is analytic in u but has only finite regularity in φ_1. More precisely,

> **(H)** $f(\varphi_1, u) := \sum_{k=2d-1}^{\infty} a_k(\varphi_1) u^k$, $d \in \mathbb{N}^+$, $d > 1$ and the coefficients $a_k(\varphi_1) \in H^1(\mathbb{T})$ verify, for some $r > 0$, $\sum_{k=2d-1}^{\infty} |a_k|_{H^1} r^k < \infty$. The function $f(\varphi_1, u)$ is not identically constant in φ_1.

We look for solutions u of (1.4) in the Banach space[1]

$$\mathscr{H}_{\sigma,s} := \left\{ u(\varphi) = \sum_{l \in \mathbb{Z}^2} \hat{u}_l e^{\mathrm{i} l \cdot \varphi} : \hat{u}_l^* = \hat{u}_{-l} \text{ and } |u|_{\sigma,s} := \sum_{l \in \mathbb{Z}^2} |\hat{u}_l| e^{|l_2|\sigma} [l_1]^s < +\infty \right\},$$

where $[l_1] := \max\{|l_1|, 1\}$ and $\sigma > 0$, $s \geq 0$.

The space $\mathscr{H}_{\sigma,s}$ is a Banach algebra with respect to multiplications of functions (see Lemma 4.1 in the Appendix), namely

$$u_1, u_2 \in \mathscr{H}_{\sigma,s} \Longrightarrow u_1 u_2 \in \mathscr{H}_{\sigma,s} \quad \text{and} \quad |u_1 u_2|_{\sigma,s} \leq C |u_1|_{\sigma,s} |u_2|_{\sigma,s}.$$

We shall prove the following theorems.

Theorem A. *Let* $\omega_1 = n/m \in \mathbb{Q}$. *Assume that* f *satisfies assumption* **(H)** *and* $a_{2d-1}(\varphi_1) \neq 0$, $\forall \varphi_1 \in \mathbb{T}$. *Let* $\mathscr{B}_\gamma$ *be the uncountable*[2] *zero-measure Cantor set*

$$\mathscr{B}_\gamma := \left\{ \varepsilon \in (-\varepsilon_0, \varepsilon_0) : |l_1 + \varepsilon l_2| > \frac{\gamma}{|l_2|}, \ \forall l_1, l_2 \in \mathbb{Z} \backslash \{0\} \right\},$$

where $0 < \gamma < 1/6$.

There exist constants $\bar{\sigma} > 0$, $\bar{s} > 2$, $\bar{\varepsilon} > 0$, $\overline{C} > 0$, *such that* $\forall \varepsilon \in \mathscr{B}_\gamma$, $|\varepsilon|\gamma^{-1} \leq \bar{\varepsilon}/m^2$, *there exists a classical solution* $u(\varepsilon, \varphi) \in \mathscr{H}_{\bar{\sigma},\bar{s}}$ *of* (1.4) *with* $(\omega_1, \omega_2) = (n/m, 1 + \varepsilon)$ *satisfying*

$$\left| u(\varepsilon, \varphi) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar{q}_\varepsilon(\varphi) \right|_{\bar{\sigma},\bar{s}} \leq \overline{C} \frac{m^2 |\varepsilon|}{\gamma \omega_1^3} |\varepsilon|^{\frac{1}{2(d-1)}} \tag{1.5}$$

[1]Given $z \in \mathbb{C}$, z^* denotes its complex conjugate.

[2]The proof that $\mathscr{B}_\gamma \cap (0, \varepsilon_0)$ and $\mathscr{B}_\gamma \cap (-\varepsilon_0, 0)$ are both uncountable $\forall \varepsilon_0 > 0$ is like in Bambusi and Paleari (2001).

for an appropriate function $\bar{q}_\varepsilon \in \mathscr{H}_{\sigma,s}\backslash\{0\}$ *of the form* $\bar{q}_\varepsilon(\varphi) = \bar{q}_+(\varphi_2) + \bar{q}_-(2m\varphi_1 - n\varphi_2)$.

As a consequence, equation (1.1) *admits the quasi-periodic solution* $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ *with two frequencies* $(\omega_1, \omega_2) = (n/m, 1+\varepsilon)$ *and the map* $t \to v(\varepsilon, t, \cdot) \in H^{\bar\sigma}(\mathbb{T})$ *has the form*[3]

$$\left| v(\varepsilon, t, x) - |\varepsilon|^{\frac{1}{2(d-1)}} \left[\bar{q}_+(x + (1+\varepsilon)t) + \bar{q}_-((1-\varepsilon)nt - nx) \right] \right|_{H^{\bar\sigma}(\mathbb{T})} = O\left(\frac{m^2}{\gamma\omega_1^3} |\varepsilon|^{\frac{2d-1}{2(d-1)}} \right).$$

At the first order the quasi-periodic solution $v(\varepsilon, t, x)$ of equation (1.1) is the superposition of two waves traveling in opposite directions (in general, both components q_+, q_- are nontrivial).

The bifurcation of quasi-periodic solutions looks quite different if ω_1 is irrational.

Theorem B. *Let* $\omega_1 \in \mathbb{R}\backslash\mathbb{Q}$. *Assume that* f *satisfies assumption* (**H**), $\int_0^{2\pi} a_{2d-1}(\varphi_1) d\varphi_1 \neq 0$ *and* $f(\varphi_1, u) \in H^s(\mathbb{T})$, $s \geq 1$, *for all* u.

Let $\mathscr{C}_\gamma \subset D \equiv (-\varepsilon_0, \varepsilon_0) \times (1, 2)$ *be the uncountable zero-measure Cantor set*[4]

$$\mathscr{C}_\gamma := \left\{ \begin{array}{r} (\varepsilon, \omega_1) \in D : \omega_1 \notin \mathbb{Q}, \dfrac{\omega_1}{\omega_2} \notin \mathbb{Q}, \ |\omega_1 l_1 + \varepsilon l_2| > \dfrac{\gamma}{|l_1| + |l_2|}, \\ |\omega_1 l_1 + (2+\varepsilon) l_2| > \dfrac{\gamma}{|l_1| + |l_2|}, \ \forall l_1, l_2 \in \mathbb{Z}\backslash\{0\} \end{array} \right\}. \tag{1.6}$$

Fix any $0 < \bar{s} < s - 1/2$. *There exist positive constants* $\bar\varepsilon$, $\overline{C}$, $\bar\sigma > 0$, *such that,* $\forall (\varepsilon, \omega_1) \in \mathscr{C}_\gamma$ *with* $|\varepsilon|\gamma^{-1} < \bar\varepsilon$ *and* $\varepsilon \int_0^{2\pi} a_{2d-1}(\varphi_1) d\varphi_1 > 0$, *there exists a nontrivial solution* $u(\varepsilon, \varphi) \in \mathscr{H}_{\bar\sigma,\bar{s}}$ *of equation* (1.4) *with* $(\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ *satisfying*

$$\left| u(\varepsilon, \varphi) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar{q}_\varepsilon(\varphi_2) \right|_{\bar\sigma,\bar{s}} \leq \overline{C} \frac{|\varepsilon|}{\gamma} |\varepsilon|^{\frac{1}{2(d-1)}} \tag{1.7}$$

for some function $\bar{q}_\varepsilon(\varphi_2) \in H^{\bar\sigma}(\mathbb{T})\backslash\{0\}$.

As a consequence, equation (1.1) *admits the nontrivial quasi-periodic solution* $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ *with two frequencies* $(\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ *and the map* $t \to v(\varepsilon, t, \cdot) \in H^{\bar\sigma}(\mathbb{T})$ *has the form*

$$\left| v(\varepsilon, t, x) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar{q}_\varepsilon(x + (1+\varepsilon)t) \right|_{H^{\bar\sigma}(\mathbb{T})} = O\left(\gamma^{-1} |\varepsilon|^{\frac{2d-1}{2(d-1)}} \right).$$

Remark 1. Imposing in the definition of $\mathscr{C}_\gamma$ the condition $\omega_1/\omega_2 = \omega_1/(1+\varepsilon) \in \mathbb{Q}$ we obtain, by Theorem B the existence of periodic solutions of equation (1.1). They are reminiscent, in this completely resonant context, of the Birkhoff–Lewis periodic orbits with large minimal period accumulating at the origin, see Bambusi and Berti (2005) and Berti et al. (2004).

Remark 2 (Non Existence)**.** In Theorem B, existence of quasi-periodic solutions could follow by other hypotheses on f, see Remark 3. However, the hypothesis

[3]We denote $H^\sigma(\mathbb{T}) := \{u(\varphi) = \sum_{l\in\mathbb{Z}} \hat{u}_l e^{il\varphi} : \hat{u}_l^* = \hat{u}_{-l}, \ |u|_{H^\sigma(\mathbb{T})} := \sum_{l\in\mathbb{Z}} |\hat{u}_l| e^{\sigma|l|} < +\infty\}$.
[4]See Lemma 3.1.

that the leading term in the nonlinearity f is an odd power of u is not of purely technical nature. If $f(\varphi_1, u) = a(\varphi_1)u^D$ with D even and $\int_0^{2\pi} a(\varphi_1)d\varphi_1 \neq 0$, then, $\forall R > 0$ there exists $\varepsilon_0 > 0$ such that $\forall \sigma \geq 0$, $\bar{s} > s - 1/2$, $\forall(\varepsilon, \omega_1) \in \mathscr{C}_\gamma$ with $|\varepsilon| < \varepsilon_0$, equation (1.4) does not possess solutions $u \in \mathscr{H}_{\sigma,\bar{s}}$ in the ball $|u|_{\sigma,\bar{s}} \leq R|\varepsilon|^{1/(D-1)}$, see Proposition 3.

To prove Theorems A and B, instead of looking for solutions of equation (1.4) in a shrinking neighborhood of zero, it is a convenient devise to perform the rescaling

$$u \to \delta u \quad \text{with } \delta := |\varepsilon|^{1/2(d-1)}$$

enhancing the relation between the amplitude δ and the frequency $\omega_2 = 1 + \varepsilon$. We obtain the equation

$$\mathscr{L}_\varepsilon u + \varepsilon f(\varphi_1, u, \delta) = 0 \tag{1.8}$$

where, see (1.3),

$$\begin{aligned}\mathscr{L}_\varepsilon &:= \left[\omega_1\partial_{\varphi_1} + \varepsilon\partial_{\varphi_2}\right] \circ \left[\omega_1\partial_{\varphi_1} + (2+\varepsilon)\partial_{\varphi_2}\right] \\ &= \left[\omega_1^2\partial_{\varphi_1}^2 + 2\omega_1\partial_{\varphi_1}\partial_{\varphi_2}\right] + \varepsilon\left[(2+\varepsilon)\partial_{\varphi_2}^2 + 2\omega_1\partial_{\varphi_1}\partial_{\varphi_2}\right]\end{aligned}$$

and

$$f(\varphi_1, u, \delta) := \operatorname{sign}(\varepsilon)\frac{f(\varphi_1, \delta u)}{\delta^{2(d-1)}} = \operatorname{sign}(\varepsilon)\left(a_{2d-1}(\varphi_1)u^{2d-1} + \delta a_{2d}(\varphi_1)u^{2d} + \cdots\right) \tag{1.9}$$

and $\operatorname{sign}(\varepsilon) := 1$ if $\omega_2 > 1$ and $\operatorname{sign}(\varepsilon) := -1$ if $\omega_2 < 1$.

To find solutions of equation (1.8) we shall apply the Lyapunov–Schmidt decomposition method which leads to solve separately a "range equation" and a "bifurcation equation".

In order to solve the range equation (avoiding small divisor porblems) we restrict ε to the uncountable zero-measure set $\mathscr{B}_\gamma$ for Theorem A, respectively $(\varepsilon, \omega_1) \in \mathscr{C}_\gamma$ for Theorem B, and we apply the Contraction Mapping Theorem; similar nonresonance conditions have been employed, e.g., in Lidskiĭ and Shul′man (1988), Bambusi and Paleari (2001), Berti and Bolle (2003, 2004), McKenna (1985), and Procesi (2005).

To solve the *infinite* dimensional bifurcation equation we proceed in different ways in Case A) and Case B).

As already said, in Case A) we follow the variational approach of Berti and Bolle (2003, 2004) noting that the bifurcation equation is the Euler–Lagrange equation of a "reduced action functional" which turns out to have the geometry of the infinite dimensional linking theorem of Benci–Rabinowitz (Benci and Rabinowitz, 1979). However, we cannot directly apply the linking theorem because the reduced action functional is defined only in a ball centered at the origin (where the range equation is solved). Moreover, the infinite dimensional linking theorem of Benci and Rabinowitz (1979) requires the compactness of the gradient of the functional, property which is not preserved by extending the functional in the whole infinite dimensional space.

In order to overcome these difficulties, we perform a further finite dimensional reduction of a Galerkin type inspired to Berti and Bolle (to appear) on a subspace of dimension N, with N large but independent of ε, see the equations (2.3)–(2.5).

We shall have to solve the (2.4)–(2.5) equations in a sufficiently large domain of q_1 (Lemma 2.3), consistent with the $|\cdot|_{H^1}$ bounds on the solution q_1 of the bifurcation equation that can be obtained by the variational arguments, see Lemma 2.6.

Another advantage of this method is that allows to prove the analyticity of the solution u in the variable φ_2.

In Case B) the bifurcation equation could be solved through variational methods as in Case A). However, there is a simpler technique available. The bifurcation equation reduces, in the limit $\varepsilon \to 0$, to a super-quadratic Hamiltonian system with one degree of freedom. We prove the existence of a nondegenerate solution by phase-space analysis. Therefore, it can be continued by the Implicit Function Theorem to a solution of the complete bifurcation equation for ε small.

The article is organized as follows. For simplicity of exposition, we first prove Theorem A in the case $\omega_1 = 1$. We deal with the general case $\omega_1 = \frac{n}{m} \in \mathbb{Q}$ at the end of Section 2. In Section 3 we prove Theorem B.

2. Case A: $\omega_1 \in \mathbb{Q}$

Equation (1.8) becomes, for $\omega_1 = 1$,

$$\mathcal{L}_\varepsilon u + \varepsilon f(\varphi_1, u, \delta) = 0, \tag{2.1}$$

where

$$\begin{aligned}\mathcal{L}_\varepsilon &:= \left[\partial_{\varphi_1} + \varepsilon \partial_{\varphi_2}\right] \circ \left[\partial_{\varphi_1} + (2+\varepsilon)\partial_{\varphi_2}\right] \\ &= \left[\partial^2_{\varphi_1} + 2\partial_{\varphi_1}\partial_{\varphi_2}\right] + \varepsilon\left[(2+\varepsilon)\partial^2_{\varphi_2} + 2\partial_{\varphi_1}\partial_{\varphi_2}\right] \equiv L_0 + \varepsilon L_1.\end{aligned}$$

To fix notations we shall prove Theorem A in the case $a_{2d-1}(\varphi_1) > 0$ and $\varepsilon > 0$, i.e., $\mathrm{sign}(\varepsilon) = 1$.

By the assumption **(H)** on the nonlinearity f and by the Banach algebra property of $\mathcal{H}_{\sigma,s}$ the Nemitskii operator

$$u \to f(\varphi_1, u, \delta) \in C^\infty(B_\rho, \mathcal{H}_{\sigma,s}), \quad 0 < s < \frac{1}{2},$$

where B_ρ is the ball of radius $\rho\delta^{-1}$ in $\mathcal{H}_{\sigma,s}$ and ρ is connected to the analyticity radius r of f (note that since $a_k(\varphi_1) \in H^1(\mathbb{T})$, then $a_k(\cdot) \in \mathcal{H}_{\sigma,s}$, $\forall \sigma > 0$, $0 < s < 1/2$).

Equation (2.1) is the Euler–Lagrange equation of the *Lagrangian action functional* $\Psi_\varepsilon \in C^1(\mathcal{H}_{\sigma,s}, \mathbb{R})$ defined by

$$\begin{aligned}\Psi_\varepsilon(u) := &\int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1} u)^2 + (\partial_{\varphi_1} u)(\partial_{\varphi_2} u) + \frac{\varepsilon(2+\varepsilon)}{2}(\partial_{\varphi_2} u)^2 \\ &+ \varepsilon(\partial_{\varphi_1} u)(\partial_{\varphi_2} u) - \varepsilon F(\varphi_1, u, \delta) \\ \equiv\ &\Psi_0(u) + \varepsilon\Gamma(u, \delta),\end{aligned}$$

where $F(\varphi_1, u, \delta) := \int_0^u f(\varphi_1, \xi, \delta)d\xi$ and

$$\Psi_0(u) := \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1} u)^2 + (\partial_{\varphi_1} u)(\partial_{\varphi_2} u),$$
$$\Gamma(u, \delta) := \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} u)^2 + (\partial_{\varphi_1} u)(\partial_{\varphi_2} u) - F(\varphi_1, u, \delta).$$

To find critical points of Ψ_ε we perform a variational Lyapunov–Schmidt reduction inspired by Berti and Bolle (2003, 2004), see also Ambrosetti and Badiale (1998).

2.1. *The Variational Lyapunov–Schmidt Reduction*

The unperturbed functional $\Psi_0 : \mathscr{H}_{\sigma,s} \to \mathbb{R}$ possesses an infinite dimensional linear space Q of critical points which are the solutions q of the equation

$$L_0 q = \partial_{\varphi_1}\big(\partial_{\varphi_1} + 2\partial_{\varphi_2}\big)q = 0.$$

The space Q can be written as

$$Q = \left\{ q = \sum_{l \in \mathbb{Z}^2} \hat{q}_l e^{\mathrm{i} l \cdot \varphi} \in \mathscr{H}_{\sigma,s} \,|\, \hat{q}_l = 0 \text{ for } l_1(l_1 + 2l_2) \neq 0 \right\}.$$

In view of the variational argument that we shall use to solve the bifurcation equation, we split Q as

$$Q = Q_+ \oplus Q_0 \oplus Q_-$$

where[5]

$$Q_+ := \{q \in Q : \hat{q}_l = 0 \text{ for } l \notin \Lambda_+\} = \{q_+ := q_+(\varphi_2) \in H_0^\sigma(\mathbb{T})\}$$
$$Q_0 := \{q_0 \in \mathbb{R}\}$$
$$Q_- := \{q \in Q : \hat{q}_l = 0 \text{ for } l \notin \Lambda_-\} = \{q_- := q_-(2\varphi_1 - \varphi_2),\, q_-(\cdot) \in H_0^{\sigma,s}(\mathbb{T})\}$$

and

$$\Lambda_+ := \{l \in \mathbb{Z}^2 : l_1 = 0, l \neq 0\}, \quad \Lambda_- := \{l \in \mathbb{Z}^2 : l_1 + 2l_2 = 0, l \neq 0\}. \qquad (2.2)$$

We shall also use in Q the norm

$$|q|_{H^1}^2 = |q_+|_{H^1(\mathbb{T})}^2 + q_0^2 + |q_-|_{H^1(\mathbb{T})}^2 \sim \sum_{l \in \Lambda_- \cup \{0\} \cup \Lambda_+} \hat{q}_l^2(|l|^2 + 1).$$

[5] $H_0^\sigma(\mathbb{T})$ denotes the functions of $H^\sigma(\mathbb{T})$ with zero average. $H^{\sigma,s}(\mathbb{T}) := \{u(\varphi) = \sum_{l \in \mathbb{Z}} \hat{u}_l e^{\mathrm{i} l \varphi} : \hat{u}_l^* = \hat{u}_{-l},\, |u|_{H^{\sigma,s}(\mathbb{T})} := \sum_{l \in \mathbb{Z}} |\hat{u}_l| e^{\sigma|l|}[l]^s < +\infty\}$ and $H_0^{\sigma,s}(\mathbb{T})$ its functions with zero average.

We decompose the space $\mathscr{H}_{\sigma,s} = Q \oplus P$ where

$$P := \left\{ p = \sum_{l \in \mathbb{Z}^2} \hat{p}_l e^{\mathrm{i} l \cdot \varphi} \in \mathscr{H}_{\sigma,s} \mid \hat{p}_l = 0 \text{ for } l_1(2l_2 + l_1) = 0 \right\}.$$

Projecting equation (2.1) onto the closed subspaces Q and P, setting $u = q + p \in \mathscr{H}_{\sigma,s}$ with $q \in Q$ and $p \in P$, we obtain

$$\begin{cases} L_1[q] + \Pi_Q f(\varphi_1, q + p, \delta) = 0 & (Q) \\ \mathscr{L}_\varepsilon[p] + \varepsilon \Pi_P f(\varphi_1, q + p, \delta) = 0 & (P) \end{cases}$$

where $\Pi_Q : \mathscr{H}_{\sigma,s} \to Q$, $\Pi_P : \mathscr{H}_{\sigma,s} \to P$ are the projectors respectively onto Q and P.

In order to prove analyticity of the solutions and to highlight the compactness of the problem we perform a *finite* dimensional Lyapunov–Schmidt reduction, introducing the decomposition

$$Q = Q_1 \oplus Q_2,$$

where

$$Q_1 := Q_1(N) := \left\{ q = \sum_{|l| \le N} \hat{q}_l e^{\mathrm{i} l \cdot \varphi} \in Q \right\}, \quad Q_2 := Q_2(N) := \left\{ q = \sum_{|l| > N} \hat{q}_l e^{\mathrm{i} l \cdot \varphi} \in Q \right\}.$$

Setting $q = q_1 + q_2$ with $q_1 \in Q_1$ and $q_2 \in Q_2$, we finally get

$$L_1[q_1] + \Pi_{Q_1}[f(\varphi_1, q_1 + q_2 + p, \delta)] = 0 \Longleftrightarrow d\Psi_\varepsilon(u)[h] = 0 \quad \forall h \in Q_1 \quad (Q_1) \tag{2.3}$$

$$L_1[q_2] + \Pi_{Q_2}[f(\varphi_1, q_1 + q_2 + p, \delta)] = 0 \Longleftrightarrow d\Psi_\varepsilon(u)[h] = 0 \quad \forall h \in Q_2 \quad (Q_2) \tag{2.4}$$

$$\mathscr{L}_\varepsilon[p] + \varepsilon \Pi_P[f(\varphi_1, q_1 + q_2 + p, \delta)] = 0 \Longleftrightarrow d\Psi_\varepsilon(u)[h] = 0 \quad \forall h \in P \quad (P) \tag{2.5}$$

where $\Pi_{Q_i} : \mathscr{H}_{\sigma,s} \to Q_i$ are the projectors onto Q_i $(i = 1, 2)$.

We shall solve first the (Q_2)–(P)-equations for all $|q_1|_{H^1} \le 2R$, provided ε belongs to a suitable Cantor-like set, $|\varepsilon| \le \varepsilon_0(R)$ is sufficiently small and $N \ge N_0(R)$ is large enough (see Lemma 2.3).

Next we shall solve the (Q_1)-equation by means of a variational linking argument, see Subsection 2.4.

2.2. *The* (Q_2)–(P)*-Equations*

We first prove that $\mathscr{L}_\varepsilon$ restricted to P has a bounded inverse when ε belongs to the uncountable zero measure set

$$\mathscr{B}_\gamma := \left\{ \varepsilon \in (-\varepsilon_0, \varepsilon_0) : |l_1 + \varepsilon l_2| > \frac{\gamma}{|l_2|}, \forall l_1, l_2 \in \mathbb{Z} \setminus \{0\} \right\},$$

where $0 < \gamma < 1/6$. $\mathscr{B}_\gamma$ accumulates at zero both from the right and from the left, see Bambusi and Paleari (2001).

The operator $\mathscr{L}_\varepsilon$ is diagonal in the Fourier basis $\{e^{\mathrm{i} l \cdot \varphi}, l \in \mathbb{Z}^2\}$ with eigenvalues $D_l := (l_1 + \varepsilon l_2)(l_1 + (2 + \varepsilon) l_2)$.

Lemma 2.1. *For $\varepsilon \in \mathscr{B}_\gamma$ the eigenvalues D_l of $\mathscr{L}_\varepsilon$ restricted to P, satisfy*

$$|D_l| = |l_1 + \varepsilon l_2|\,|(l_1 + 2l_2) + \varepsilon l_2| > \gamma \quad \forall l_1 \neq 0,\ \ l_1 + 2l_2 \neq 0.$$

As a consequence, the operator $\mathscr{L}_\varepsilon : P \to P$ has a bounded inverse $\mathscr{L}_\varepsilon^{-1}$ satisfying

$$\left|\mathscr{L}_\varepsilon^{-1}[h]\right|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{\gamma}, \quad \forall h \in P. \tag{2.6}$$

Proof. Denoting by $[x]$ the nearest integer close to x and $\{x\} = x - [x]$, we have that $D_l > 1$ if both $l_1 \neq -[\varepsilon l_2]$ and $l_1 + 2l_2 \neq -[\varepsilon l_2]$. If $l_1 = -[\varepsilon l_2]$ then

$$|D_l| \geq \frac{\gamma}{|l_2|}(|2l_2| - \{\varepsilon l_2\}) \geq \gamma.$$

In the same way, if $l_1 + 2l_2 = -[\varepsilon l_2]$ we have $|D_l| \geq \frac{\gamma}{|l_2|}(|2l_2| - \{\varepsilon l_2\}) \geq \gamma$. $\square$

Lemma 2.2. *The operator $L_1 : Q_2 \to Q_2$ has bounded inverse L_1^{-1} which satisfies*

$$\left|L_1^{-1}[h]\right|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{N^2}. \tag{2.7}$$

Proof. L_1 is diagonal in the Fourier basis of $Q : e^{\mathrm{i}l\cdot\varphi}$ with $l \in \Lambda_+ \cup \{0\} \cup \Lambda_-$ (recall (2.2)) with eigenvalues

$$d_l = (2+\varepsilon)l_2^2 \ \text{ if } l_1 = 0 \qquad \text{and} \qquad d_l = (-2+\varepsilon)l_2^2 \ \text{ if } l_1 + 2l_2 = 0. \tag{2.8}$$

The eigenvalues of L_1 restricted to $Q_2(N)$ verify $|d_l| \geq (2-\varepsilon)N^2$ and (2.7) holds. $\square$

Fixed points of the nonlinear operator $\mathscr{G} : Q_2 \oplus P \to Q_2 \oplus P$ defined by

$$\mathscr{G}(q_2, p; q_1) := \left(-L_1^{-1}\Pi_{Q_2} f(\varphi_1, q_1 + q_2 + p, \delta), -\varepsilon\mathscr{L}_\varepsilon^{-1}\Pi_P f(\varphi_1, q_1 + q_2 + p, \delta)\right)$$

are solutions of the (Q_2)–(P)-equations.

Using the Contraction Mapping Theorem we can prove the following lemma.

Lemma 2.3 (Solution of the (Q_2)–(P) Equations). $\forall R > 0$. *There exist an integer $N_0(R) \in \mathbb{N}^+$ and positive constants $\varepsilon_0(R) > 0$, $C_0(R) > 0$ such that*:

$$\forall |q_1|_{H^1} \leq 2R, \quad \forall \varepsilon \in B_\gamma, \quad |\varepsilon|\gamma^{-1} \leq \varepsilon_0(R), \quad \forall N \geq N_0(R) : 0 \leq \sigma N \leq 1; \tag{2.9}$$

there exists a unique solution $(q_2(q_1), p(q_1)) := (q_2(\varepsilon, N, q_1), p(\varepsilon, N, q_1)) \in Q_2 \oplus P$ of the (Q_2)–(P) equations satisfying

$$|q_2(\varepsilon, N, q_1)|_{\sigma,s} \leq \frac{C_0(R)}{N^2}, \quad |p(\varepsilon, N, q_1)|_{\sigma,s} \leq C_0(R)|\varepsilon|\gamma^{-1}. \tag{2.10}$$

Moreover, the map $q_1 \to (q_2(q_1), p(q_1))$ is in $C^1(B_{2R}, Q_2 \oplus P)$ and

$$|p'(q_1)[h]|_{\sigma,s} \leq C_0(R)|\varepsilon|\gamma^{-1}|h|_{H^1}, \qquad |q_2'(q_1)[h]|_{\sigma,s} \leq \frac{C_0(R)}{N^2}|h|_{H^1} \quad \forall h \in Q_1. \tag{2.11}$$

Proof. See the Appendix. □

2.3. *The (Q_1)-Equation*

Once the (Q_2)–(P)-equations have been solved by $(q_2(q_1),\ p(q_1)) \in Q_2 \oplus P$, there remains the finite dimensional (Q_1)-equation

$$L_1[q_1] + \Pi_{Q_1} f(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta) = 0. \tag{2.12}$$

The geometric interpretation of the construction of $(q_2(q_1), p(q_1))$ is that on the finite dimensional submanifold $Z \equiv \{q_1 + q_2(q_1) + p(q_1) : |q_1| < 2R\}$, diffeomorphic to the ball

$$B_{2R} := \{q_1 \in Q_1 : |q_1|_{H^1} < 2R\},$$

the partial derivatives of the action functional Ψ_ε with respect to the variables (q_2, p) vanish. We claim that at a critical point of Ψ_ε restricted to Z, also the partial derivative of Ψ_ε w.r.t. the variable q_1 vanishes and therefore that such point is critical also for the nonrestricted functional $\Psi_\varepsilon : \mathscr{H}_{\sigma,s} \to \mathbb{R}$.

Actually, the bifurcation equation (2.12) is the Euler–Lagrange equation of the reduced Lagrangian action functional

$$\Phi_{\varepsilon,N} : B_{2R} \subset Q_1 \to \mathbb{R}, \quad \Phi_{\varepsilon,N}(q_1) := \Psi_\varepsilon(q_1 + q_2(q_1) + p(q_1)).$$

Lemma 2.4. $\Phi_{\varepsilon,N} \in C^1(B_{2R}, \mathbb{R})$ *and a critical point* $q_1 \in B_{2R}$ *of* $\Phi_{\varepsilon,N}$ *is a solution of the bifurcation equation* (2.12). *Moreover,* $\Phi_{\varepsilon,N}$ *can be written as*

$$\Phi_{\varepsilon,N}(q_1) = \mathrm{const} + \varepsilon(\Gamma(q_1) + \mathscr{R}_{\varepsilon,N}(q_1)), \tag{2.13}$$

where

$$\begin{aligned}\Gamma(q_1) &:= \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} q_1)^2 + (\partial_{\varphi_1} q_1)(\partial_{\varphi_2} q_1) - a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d}\\ \mathscr{R}_{\varepsilon,N}(q_1) &:= \int_{\mathbb{T}^2} F(\varphi_1, q_1, \delta = 0) - F(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)\\ &\quad + \frac{1}{2} f(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)(q_2(q_1) + p(q_1))\end{aligned}$$

and, for some positive constant $C_2(R) \geq C_1(R)$,

$$|\mathscr{R}_{\varepsilon,N}(q_1)| \leq C_2(R)\left(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\right) \tag{2.14}$$

$$\left|\mathscr{R}'_{\varepsilon,N}(q_1)[h]\right| \leq C_2(R)\left(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\right)|h|_{H^1}, \quad \forall h \in Q_1. \tag{2.15}$$

Proof. See the Appendix. □

The problem of finding nontrivial solutions of the (Q_1)-equation is reduced to finding nontrivial critical points of the reduced action functional $\Phi_{\varepsilon,N}$ in B_{2R}.

By (2.13), this is equivalent to finding critical points of the rescaled functional (still denoted $\Phi_{\varepsilon,N}$ and called the *reduced action functional*)

$$\Phi_{\varepsilon,N}(q_1) = \Gamma(q_1) + \mathcal{R}_{\varepsilon,N}(q_1) \equiv \left(\mathcal{A}(q_1) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d}\right) + \mathcal{R}_{\varepsilon,N}(q_1), \qquad (2.16)$$

where the quadratic form

$$\mathcal{A}(q) := \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} q)^2 + (\partial_{\varphi_1} q)(\partial_{\varphi_2} q)$$

is positive definite on Q_+, negative definite on Q_-, and zero-definite on Q_0. For $q_1 = q_+ + q_0 + q_- \in Q_1$,

$$\mathcal{A}(q_1) = \mathcal{A}(q_+ + q_0 + q_-) = \mathcal{A}(q_+) + \mathcal{A}(q_-) = \frac{\alpha_+}{2}|q_+|^2_{H^1} - \frac{\alpha_-}{2}|q_-|^2_{H^1} \qquad (2.17)$$

for suitable positive constants α_+, α_-, bounded away from zero by constants independent of ε.

We shall prove the existence of critical points of $\Phi_{\varepsilon,N}$ in B_{2R} of the "linking type".

2.4. *Linking Critical Points of the Reduced Action Functional* $\Phi_{\varepsilon,N}$

We cannot directly apply the linking Theorem because $\Phi_{\varepsilon,N}$ is defined only in the ball B_{2R}. Therefore, our first step is to extend $\Phi_{\varepsilon,N}$ to the whole space Q_1.

Step 1. *Extension of* $\Phi_{\varepsilon,N}$. We define the extended action functional $\widetilde{\Phi}_{\varepsilon,N} \in C^1(Q_1, \mathbb{R})$ as

$$\widetilde{\Phi}_{\varepsilon,N}(q_1) := \Gamma(q_1) + \widetilde{\mathcal{R}}_{\varepsilon,N}(q_1),$$

where $\widetilde{\mathcal{R}}_{\varepsilon,N} : Q_1 \to \mathbb{R}$ is

$$\widetilde{\mathcal{R}}_{\varepsilon,N}(q_1) := \lambda\left(\frac{|q_1|^2_{H_1}}{R^2}\right)\mathcal{R}_{\varepsilon,N}(q_1),$$

and $\lambda : [0, +\infty) \to [0, 1]$ is a smooth, nonincreasing, cut-off function such that

$$\begin{cases} \lambda(x) = 1 & |x| \le 1 \\ \lambda(x) = 0 & |x| \ge 4 \end{cases} \quad |\lambda'(x)| < 1.$$

By definition $\widetilde{\Phi}_{\varepsilon,N} \equiv \Phi_{\varepsilon,N}$ on $B_R := \{q_1 \in Q_1 : |q_1|_{H^1} \le R\}$ and $\widetilde{\Phi}_{\varepsilon,N} \equiv \Gamma$ outside B_{2R}.

Moreover, by (2.14) and (2.15), there is a constant $C_3(R) \ge C_2(R) > 0$ such that $\forall |q_1|_{H^1} \le 2R$

$$|\widetilde{\mathcal{R}}_{\varepsilon,N}(q_1)| \le C_3(R)\left(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\right) \qquad (2.18)$$

$$\left|\widetilde{\mathcal{R}}'_{\varepsilon,N}(q_1)[h]\right| \le C_3(R)\left(\delta + |\varepsilon|\gamma^{-1} + \tfrac{1}{N^2}\right)|h|_{H^1}, \quad \forall h \in Q_1. \qquad (2.19)$$

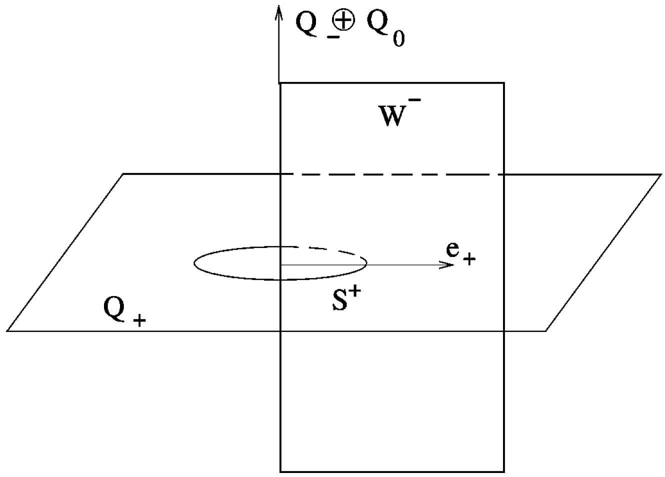

Figure 1. The cylinder W^- and the sphere S^+ link.

In the sequel we shall always assume

$$C_3(R)\left(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\right) \leq 1.$$

Step 2. $\widetilde{\Phi}_{\varepsilon,N}$ *verifies the geometrical hypotheses of the linking Theorem* (see Figure 1).

Lemma 2.5. *There exist ε-N-γ-independent positive constants ρ, ω, r_1, $r_2 > \rho$, and $0 < \varepsilon_1(R) \leq \varepsilon_0(R)$, $N_1(R) \geq N_0(R)$ such that, $\forall |\varepsilon|\gamma^{-1} \leq \varepsilon_1(R)$, $\forall N \geq N_1(R)$*

(i) $\widetilde{\Phi}_{\varepsilon,N}(q_1) \geq \omega > 0$, $\forall q_1 \in S^+ := \{q_1 \in Q_1 \cap Q_+ : |q_1|_{H_1} = \rho\}$,
(ii) $\widetilde{\Phi}_{\varepsilon,N}(q_1) \leq \omega/2$, $\forall q_1 \in \partial W^-$ *where W^- is the cylinder*

$$W^- := \left\{q_1 = q_0 + q_- + re^+, |q_0 + q_-|_{H_1} \leq r_1, q_- \in Q_1 \cap Q_-, q_0 \in \mathbb{R}, r \in [0, r_2]\right\}$$

and $e_+ := \cos(\varphi_2) \in Q_1 \cap Q_+$. Note that ρ, ω are independent of R.

In what follows, κ_i, $\kappa_\pm$ will denote positive constants *independent* on R, N, ε, and γ.

Proof. (i) $\forall q_+ \in Q_1 \cap Q_+$ with $|q_+|_{H^1} = \rho < R$ we have

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon,N}(q_+) = \Phi_{\varepsilon,N}(q_+) &= \mathscr{A}(q_+) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_+^{2d}}{2d} + \mathscr{R}_{\varepsilon,N}(q_+) \\ &\geq \frac{\alpha_+}{2}\rho^2 - \kappa_1\rho^{2d} - \left(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\right)C_3(R). \quad (2.20)\end{aligned}$$

Now we fix $\rho > 0$ small such that $(\alpha_+\rho^2/2) - \kappa_1\rho^{2d} \geq \alpha_+\rho^2/4$. Next, for $(\delta + |\varepsilon|\gamma^{-1} + N^{-2})C_3(R) \leq \alpha_+\rho^2/8$ we get by (2.20)

$$\widetilde{\Phi}_{\varepsilon,N}(q_+) \geq \frac{\alpha_+}{8}\rho^2 =: \omega > 0, \quad \forall q_+ \in Q_1 \cap Q^+ \text{ with } |q_+| = \rho.$$

(ii) Let

$$B_1 := \{q_1 = q_0 + q_- + r_2 e_+ \text{ with } |q_0 + q_-|_{H^1} \le r_1,\, q_- \in Q_1 \cap Q_-\} \subset \partial W^-$$
$$B_2 := \{q_1 = q_0 + q_- + re_+ \text{ with } |q_0 + q_-|_{H^1} = r_1,\, q_- \in Q_1 \cap Q_-,\, r \in [0, r_2]\} \subset \partial W^-$$

and choose $r_1, r_2 > 2R$. For $q_1 = q_0 + q_- + re_+ \in B_1 \cup B_2$

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon,N}(q_1) = \Gamma(q_1) &= \mathscr{A}(q_1) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)(q_0 + q_- + re_+)^{2d} \\ &= -\frac{\alpha_-}{2}|q_-|_{H^1}^2 + r^2\mathscr{A}(e_+) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{(q_0 + q_- + re_+)^{2d}}{2d} \\ &\le -\frac{\alpha_-}{2}|q_-|_{H^1}^2 + r^2\mathscr{A}(e_+) - \alpha\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^{2d} \qquad (2.21)\end{aligned}$$

because $a_{2d-1}(\varphi_1)/2d \ge \alpha > 0$. Now, by Hölder inequality and orthogonality

$$\begin{aligned}\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^{2d} &\ge \kappa_2\Big(\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^2\Big)^d \\ &= \kappa_2\Big(\int_{\mathbb{T}^2} q_0^2 + q_-^2 + r^2 e_+^2\Big)^d \\ &\ge \kappa_3\big(q_0^2 + r^2\big)^d \ge \kappa_3\big(q_0^{2d} + r^{2d}\big)\end{aligned}$$

and by (2.21) we deduce

$$\widetilde{\Phi}_{\varepsilon,N}(q_0 + q_- + re_+) \le \big(\kappa_+ r^2 - \kappa_3 r^{2d}\big) - \left(\frac{\alpha_-}{2}|q_-|_{H^1}^2 + \kappa_3 q_0^{2d}\right).$$

Now we fix r_2 large such that $\kappa_+ r_2^2 - \kappa_3 r_2^{2d} \le 0$ and therefore

$$\widetilde{\Phi}_{\varepsilon,N}(q_1) \le \kappa_+ r_2^2 - \kappa_3 r_2^{2d} \le 0 \quad \forall q_1 \in B_1.$$

Next, setting $M := \max_{r\in[0,r_2]}\big(\kappa_+ r^2 - \kappa_3 r^{2d}\big)$, we fix r_1 large such that

$$\frac{\alpha_-}{2}|q_-|_{H^1}^2 + \kappa_3 q_0^{2d} \ge M \quad \forall |q_- + q_0| = r_1$$

and therefore

$$\widetilde{\Phi}_{\varepsilon,N}(q_1) \le M - \left(\frac{\alpha_-}{2}|q_-|_{H^1}^2 + \kappa_3 q_0^{2d}\right) \le 0 \quad \forall q_1 \in B_2.$$

Finally, if $q_1 = q_- + q_0$,

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon,N}(q_1) &= \mathscr{A}(q_-) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d} + \widetilde{\mathscr{R}}_{\varepsilon,N}(q_1) \\ &\le |\widetilde{\mathscr{R}}_{\varepsilon,N}(q_1)| \le C_3(R)(\delta + |\varepsilon|\gamma^{-1} + N^{-2}) \qquad (2.22)\end{aligned}$$

and so $\widetilde{\Phi}_{\varepsilon,N}(q_1) \le \omega/2$ if $C_3(R)(\delta + |\varepsilon|\gamma^{-1} + N^{-2}) \le \omega/2$. $\square$

We introduce the minimax class

$$\mathcal{S} := \left\{\psi \in C(\overline{W^-}, Q) \,|\, \psi = \mathrm{Id} \text{ on } \partial W^-\right\}.$$

The maps of $\mathcal{S}$ have an important intersection property, see, e.g., Proposition 5.9 of Rabinowitz (1986).

Proposition 1 (S^+ and W^- Link with Respect to $\mathcal{S}$)**.**

$$\psi \in \mathcal{S} \Longrightarrow \psi(W^-) \cap S^+ \neq \emptyset.$$

Define the minimax linking level

$$\mathcal{K}_{\varepsilon,N} := \inf_{\psi\in\mathcal{S}} \max_{q\in W^-} \widetilde{\Phi}_{\varepsilon,N}(\psi(q_1)).$$

By the intersection property of Proposition 1 and Lemma 2.5(i),

$$\max_{q_1\in W^-} \widetilde{\Phi}_{\varepsilon,N}(\psi(q_1)) \geq \min_{q_1\in S^+} \widetilde{\Phi}_{\varepsilon,N}(q_1) \geq \omega > 0 \quad \forall \psi \in \mathcal{S}$$

and therefore,

$$\mathcal{K}_{\varepsilon,N} > \omega > 0.$$

Moreover, since $\mathrm{Id} \in \mathcal{S}$ and (2.18)

$$\begin{aligned} \mathcal{K}_{\varepsilon,N} &\leq \max_{q_1\in W^-} \widetilde{\Phi}_{\varepsilon,N}(q_1) \leq \max_{q_1\in W^-} \left(\Gamma(q_1) + \widetilde{\mathcal{R}}_{\varepsilon,N}(q_1)\right) \\ &\leq \max_{q_1\in W^-} \left(\frac{\alpha_+}{2}|q_+|^2_{H^1} + \frac{\alpha_-}{2}|q_-|^2_{H^1} + \int_{\mathbb{T}^2} \kappa q_1^{2d}\right) + 1 \leq \mathcal{K}_\infty < +\infty, \end{aligned} \tag{2.23}$$

where $\mathcal{K}_\infty$ is independent of N, ε, γ since the constants r_1, r_2 in the definition of W^- are independent of N, ε, γ.

We deduce, by the linking theorem the existence of a (Palais–Smale) sequence $(q_j) \in Q_1$ at the level $\mathcal{K}_{\varepsilon,N}$, namely

$$\widetilde{\Phi}_{\varepsilon,N}(q_j) \to \mathcal{K}_{\varepsilon,N}, \quad \widetilde{\Phi}'_{\varepsilon,N}(q_j) \to 0. \tag{2.24}$$

Step 3. *Existence of a nontrivial critical point.* Our final aim is to prove that the Palais–Smale sequence q_j converges, up to subsequence, to some nontrivial critical point $\bar{q}_1 \neq 0$ in some open ball of Q_1 where $\widetilde{\Phi}_{\varepsilon,N}$ and $\Phi_{\varepsilon,N}$ coincide.

Lemma 2.6. *There exists a constant $R_* > 0$, independent on R-ε-N-γ, and functions $0 < \varepsilon_2(R) \leq \varepsilon_1(R)$, $N_2(R) \geq N_1(R)$ such that for all $|\varepsilon|\gamma^{-1} \leq \varepsilon_2(R)$, $N \geq N_2(R)$ the functional $\widetilde{\Phi}_{\varepsilon,N}$ possesses a nontrivial critical point $\bar{q}_1 \in Q_1$ with critical value $\widetilde{\Phi}_{\varepsilon,N}(\bar{q}_1) = \mathcal{K}_{\varepsilon,N}$, satisfying $|\bar{q}_1|_{H_1} \leq R_*$.*

Proof. Writing $\widetilde{\Phi}_{\varepsilon,N}(q) = \Gamma(q) + \widetilde{\mathcal{R}}_{\varepsilon,N}(q)$ we derive, by (2.18) and (2.19)

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon,N}(q_j) - \frac{1}{2}\widetilde{\Phi}'_{\varepsilon,N}(q_j)[q_j] &= \Gamma(q_j) - \frac{1}{2}\Gamma'(q_j)[q_j] + \left(\widetilde{\mathcal{R}}_{\varepsilon,N}(q_j) - \frac{1}{2}\widetilde{\mathcal{R}}'_{\varepsilon,N}(q_j)[q_j]\right)\\ &= \left(\frac{1}{2} - \frac{1}{2d}\right)\int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)q_j^{2d} + \left(\widetilde{\mathcal{R}}_{\varepsilon,N}(q_j) - \frac{1}{2}\widetilde{\mathcal{R}}'_{\varepsilon,N}(q_j)[q_j]\right)\\ &\geq \alpha\left(\frac{1}{2} - \frac{1}{2d}\right)\int_{\mathbb{T}^2} q_j^{2d} - (\delta + |\varepsilon|\gamma^{-1} + N^{-2})C_3(R).\end{aligned}$$

Therefore, by (2.23) and (2.24),

$$\mathcal{K}_\infty + 1 + |q_j|_{H_1} \geq \kappa_1 \int_{\mathbb{T}^2} q_j^{2d} := \kappa_1 |q_j|_{L^{2d}}^{2d}. \tag{2.25}$$

We also deduce, by (2.25), the Hölder inequality, and orthogonality,

$$\begin{aligned}\mathcal{K}_\infty + 1 + |q_j|_{H_1} &\geq \kappa_2\Big(\int_{\mathbb{T}^2}\big(q_{+,j} + q_{0,j} + q_{-,j}\big)^2\Big)^d\\ &= \kappa_2\Big(\int_{\mathbb{T}^2} q_{+,j}^2 + q_{0,j}^2 + q_{-,j}^2\Big)^d \geq \kappa_3 (q_{0,j})^{2d}\end{aligned}$$

and therefore,

$$|q_{0,j}| \leq \kappa_4 (1 + |q_j|_{H_1})^{1/2d}. \tag{2.26}$$

By (2.18) and (2.19), and the Hölder inequality,

$$\begin{aligned}\widetilde{\Phi}'_{\varepsilon,N}(q_j)[q_{+,j}] &= \alpha_+ |q_{+,j}|_{H_1}^2 - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1) q_j^{2d-1} q_{+,j} + \widetilde{\mathcal{R}}'_{\varepsilon,N}(q_j)[q_{+,j}]\\ &\geq \alpha_+ |q_{+,j}|_{H_1}^2 - \kappa_5 |q_{+,j}|_{H_1}\int_{\mathbb{T}^2} |q_j|^{2d-1} - (\delta + \gamma^{-1}|\varepsilon| + N^{-2})C_3(R)|q_{+,j}|_{H_1}\\ &\geq \kappa_6 |q_{+,j}|_{H_1}\big(|q_{+,j}|_{H_1} - |q_j|_{L^{2d}}^{2d-1} - 1\big).\end{aligned} \tag{2.27}$$

By (2.27) and (2.25), using that $\widetilde{\Phi}'_{\varepsilon,N}(q_j) \to 0$ and simple inequalities, we conclude

$$|q_{+,j}|_{H_1} \leq \kappa_7\big(1 + |q_j|_{H_1}^{(2d-1)/2d}\big).$$

Estimating analogously $\widetilde{\Phi}'_{\varepsilon,N}(q_j)[q_{-,j}]$ we derive

$$|q_{-,j}|_{H_1} \leq \kappa_8\big(1 + |q_j|_{H_1}^{(2d-1)/2d}\big)$$

and by (2.26) we finally deduce

$$|q_j|_{H_1} = |q_{0,j}| + |q_{+,j}|_{H_1} + |q_{-,j}|_{H_1} \leq \kappa_9\big(1 + |q_j|_{H_1}^{1/2d} + |q_j|_{H_1}^{(2d-1)/2d}\big).$$

We conclude that $|q_j|_{H_1} \leq R_*$ for a suitable positive constant R_* independent of ε, N, R, and γ.

Since Q_1 is finite dimensional q_j converges, up to subsequence, to some critical point $\bar{q}_1$ of $\widetilde{\Phi}_{\varepsilon,N}$ with $|\bar{q}_1|_{H_1} \leq R_*$. Finally, since $\widetilde{\Phi}_{\varepsilon,N}(\bar{q}_1) = \mathscr{K}_{\varepsilon,N} \geq \omega > 0$, we conclude that $\bar{q}_1 \neq 0$. □

We are now ready to prove Theorem A in the case $\omega_1 = 1$.

Proof of Theorem A for $\omega_1 = 1$. Let us fix

$$\overline{R} := R_* + 1 \quad \text{and take } |\varepsilon|\gamma^{-1} \leq \varepsilon_2(\overline{R}) := \bar{\varepsilon}.$$

Set $\overline{N} := N_2(\overline{R}) \geq N_0(\overline{R})$.

Applying Lemma 2.3 we obtain, for

$$0 < \sigma \leq \frac{1}{N_2(\overline{R})}$$

a solution $(q_2(q_1), p(q_1)) \in (Q_2(\overline{N}) \oplus P) \cap \mathscr{H}_{\sigma,s}$ of the (Q_2)–(P) equations $\forall |q_1|_{H_1} \leq 2\overline{R}$. By Lemma 2.6, the extended functional $\widetilde{\Phi}_{\varepsilon,N}(q_1)$ possesses a critical point $\bar{q}_1 \neq 0$ with $|\bar{q}_1|_{H_1} \leq R_* < \overline{R}$. Since $\widetilde{\Phi}_{\varepsilon,N}(q_1)$ coincides with $\Phi_{\varepsilon,N}(q_1)$ on the ball $B_{\overline{R}}$ we get, by Lemma 2.4, the existence of a nontrivial weak solution $\bar{q}_1 + q_2(\bar{q}_1) + p(\bar{q}_1) \in \mathscr{H}_{\sigma,s}$ of equation (2.1). Finally,

$$u = |\varepsilon|^{1/2(d-1)}\big[\bar{q}_1 + q_2(\bar{q}_1) + p(\bar{q}_1)\big] \equiv |\varepsilon|^{1/2(d-1)}\big[\bar{q}_\varepsilon + p(\bar{q}_1)\big]$$

solves equation (1.4).

The solution $\bar{q}_\varepsilon := \bar{q}_1 + q_2(\bar{q}_1)$ of the (Q)-equation belongs to $Q \cap \mathscr{H}_{\sigma,s+2}$ by the regularizing properties of L_1^{-1}, see in Lemma 2.2 formula (2.8).

Since $\bar{p} := p(\bar{q}_1)$ solves

$$\big(\partial^2_{\varphi_1} + 2(1+\varepsilon)\partial_{\varphi_1}\partial_{\varphi_2}\big)\bar{p} = -\varepsilon\big[(2+\varepsilon)\partial^2_{\varphi_2}\bar{p} + \Pi_P f(\varphi_1, u, \delta)\big] \in \mathscr{H}_{\sigma',s} \quad \forall\, 0 < \sigma' < \sigma \tag{2.28}$$

and the eigenvalues of $\partial^2_{\varphi_1} + 2(1+\varepsilon)\partial_{\varphi_1}\partial_{\varphi_2}$ restricted to P satisfy, for $\varepsilon \in \mathscr{B}_\gamma$,

$$\big|l_1[(l_1 + 2l_2) + \varepsilon 2 l_2]\big| \geq \gamma \frac{|l_1|}{2|l_2|} \quad \forall l_1 + 2l_2 \neq 0, \;\; l_2 \neq 0$$

and we deduce that $\bar{p} \in \mathscr{H}_{\sigma'',s+1}$ for all $0 < \sigma'' < \sigma'$ and $|\partial_{\varphi_1}\bar{p}|_{\sigma'',s} = O(|\varepsilon|\gamma^{-1})$. Now, again by (2.28),

$$\partial^2_{\varphi_1}\bar{p} = -2(1+\varepsilon)\partial_{\varphi_2}\partial_{\varphi_1}\bar{p} - \varepsilon\big[(2+\varepsilon)\partial^2_{\varphi_2}\bar{p} + \Pi_P f(\varphi_1, u, \delta)\big] \in \mathscr{H}_{\bar{\sigma},s} \quad \forall\, 0 < \bar{\sigma} < \sigma''.$$

Therefore $\bar{p} \in \mathscr{H}_{\bar{\sigma},s+2}$ and $|\bar{p}|_{\bar{\sigma},s+2} = O(|\varepsilon|\gamma^{-1})$. (1.5) follows with $\bar{s} := s + 2 > 2$.

By (1.2), the function $v(\varepsilon, t, x) = u(\varepsilon, t, x + (1+\varepsilon)t)$ is a solution of equation (1.1) with $\omega_1 = 1$. The frequency $\omega_2 = 1 + \varepsilon \notin \mathbb{Q}$ since $\varepsilon \in \mathscr{B}_\gamma$. To show that $v(\varepsilon, t, x)$ is quasi-periodic it remains to prove that u depends on both the variables (φ_1, φ_2) independently.

We claim that $\bar{q}_1 \notin Q_0 \oplus Q_-$, i.e. $\bar{q}_+(\varphi_2) \in Q_+\backslash\{0\}$, and therefore u depends on φ_2. Indeed by Lemma 2.6 we know that $\widetilde{\Phi}_{\varepsilon,N}(\bar{q}_1) > \omega > 0$ and $|\bar{q}_1|_{H^1(\mathbb{T})} < \overline{R}$. On the other hand, by (2.22) in Lemma 2.5 $\widetilde{\Phi}_{\varepsilon,N}(q_- + q_0) < \omega/2$, for all $|q_- + q_0|_{H^1} \le \overline{R}$, so that necessarily $\bar{q}_1 \notin Q_0 \oplus Q_-$.

We claim that any solution u of (2.1) depending only on φ_2, namely solving

$$(2+\varepsilon)u''(\varphi_2) + f(\varphi_1, u(\varphi_2), \delta) = 0, \tag{2.29}$$

is $u(\varphi_2) \equiv 0$. Indeed, by definition,

$$\delta^{2(d-1)} f(\varphi_1, u, \delta) = f(\varphi_1, \delta u) = \sum_{k=2d-1}^{\infty} a_k(\varphi_1)(\delta u)^k$$

(recall $\operatorname{sign}(\varepsilon) = 1$). Consider now a smooth zero mean function $g(\varphi_1)$ such that $\int_0^{2\pi} a_k(\varphi_1) g(\varphi_1) \neq 0$ for some k (recall that by assumption **(H)** some of the $a_k(\varphi_2)$ are not constant). By (2.29) we have

$$(2+\varepsilon)u''(\varphi_2)\int_0^{2\pi} g(\varphi_1)d\varphi_1 + \int_0^{2\pi} f(\varphi_1, u(\varphi_2), \delta) g(\varphi_1) d\varphi_1 = 0,$$

which implies, by the assumption **(H)** on f,

$$\sum_{k=2d-1}^{\infty} [\delta u(\varphi_2)]^k \int_0^{2\pi} a_k(\varphi_1) g(\varphi_1) d\varphi_1 = 0. \tag{2.30}$$

The function $G(z) := \sum_{k=2d-1}^{\infty} b_k z^k$ with $b_k := \int_0^{2\pi} a_k(\varphi_1) g(\varphi_1) d\varphi_1$ is a nontrivial analytic function. Therefore equation (2.30), i.e., $G(\delta u(\varphi_2)) = 0$, cannot have a sequence of zeros accumulating to zero. So, for δ small enough, $u(\varphi_2) \equiv 0$. $\square$

Proof of Theorem A for any Rational Frequency $\omega_1 = \frac{n}{m} \in \mathbb{Q}$. Consider now equation (1.8) with $\omega_1 = n/m$ where n, m are coprime integers.

The space Q, formed by the solutions of $\partial_{\varphi_1}\left(\frac{n}{m}\partial_{\varphi_1} + 2\partial_{\varphi_2}\right)q = 0$ can be written as

$$Q = \left\{ q = \sum_{l \in \mathbb{Z}^2} \hat{q}_l e^{il\cdot\varphi} \in \mathcal{H}_{\sigma,s} \mid \hat{q}_l = 0 \text{ for } l_1(nl_1 + 2ml_2) \neq 0 \right\}$$

and is composed by functions of the form

$$q(\varphi) = q_+(\varphi_2) + q_-(2m\varphi_1 - n\varphi_2) + q_0.$$

Let P the supplementary space to Q and perform the Lyapunov–Schmidt decomposition like in (2.3)–(2.5).

For ε in the Cantor set $\mathcal{B}_\gamma$, the eigenvalues

$$D_l = \left(\frac{n}{m} l_1 + \varepsilon l_2\right)\left(\frac{n}{m} l_1 + 2l_2 + \varepsilon l_2\right)$$

of the linear operator $\mathcal{L}_\varepsilon$ can be bounded, arguing as in Lemma 2.1, by

$$|D_l| = \frac{|(nl_1 + \varepsilon m l_2)(nl_1 + 2ml_2 + \varepsilon m l_2)|}{m^2} > \frac{\gamma}{m^2} \quad \forall l_1 \neq 0, \;\; nl_1 + 2ml_2 \neq 0.$$

As a consequence,

$$\left|\mathscr{L}_\varepsilon^{-1}[h]\right|_{\sigma,s} \le \frac{m^2|h|_{\sigma,s}}{\gamma}, \quad \forall h \in P,$$

and, in solving the (Q_2)–(P) equations as in Lemma 2.3, we obtain the new restriction

$$\gamma^{-1}|\varepsilon| \le \frac{\varepsilon_0(R)}{m^2}, \quad N \ge N_0(R)$$

and the bound (compare with (2.10)) $|p(q_1)|_{\sigma,s} \le C_0(R)|\varepsilon|\gamma^{-1}m^2$.

The corresponding reduced action functional has again the form (2.13)–(2.16) with the different quadratic part

$$\mathscr{A}(q_1) = \mathscr{A}(q_+ + q_0 + q_-) = \mathscr{A}(q_+) + \mathscr{A}(q_-) = \frac{\alpha_+}{2}|q_+|_{H^1}^2 - n^2\frac{\alpha_-}{2}|q_-|_{H^1}^2$$

and therefore it will still possess a linking critical point $\bar{q}_1 \in Q_1$.

To prove the bound (1.5) note that the eigenvalues of $\omega_1^2\partial_{\varphi_1}^2 + 2\omega_1(1+\varepsilon)\partial_{\varphi_1}\partial_{\varphi_2}$ $(\omega_1 = n/m)$ restricted to P satisfy, for $\varepsilon \in \mathscr{B}_\gamma$,

$$\omega_1\left|l_1\frac{(nl_1 + 2l_2m) + \varepsilon 2l_2m}{m}\right| \ge \frac{\omega_1|l_1|\gamma}{2|l_2|m^2} \quad \forall\, nl_1 + 2ml_2 \ne 0, \;\; l_2 \ne 0$$

and therefore $\bar{p} \in \mathscr{H}_{\bar{\sigma},s+2}$ and $|\bar{p}|_{\bar{\sigma},s+2} = O(|\varepsilon|m^2/\omega_1^3\gamma)$. □

3. Case B: $\omega_1 \notin \mathbb{Q}$

We now look for solutions of equation (1.8) when the forcing frequency ω_1 is an irrational number.

To fix notations, we shall prove Theorem B when $\int_0^{2\pi} a_{2d-1}(\varphi_1)d\varphi_1 > 0$ and therefore $\varepsilon > 0$, i.e. $\mathrm{sign}(\varepsilon) = 1$.

Fixed $0 < \bar{s} < s - 1/2$, the Nemitskii operator $u \to f(\varphi_1, u, \delta) \in C^\infty(B_\rho, \mathscr{H}_{\sigma,\bar{s}})$ since, if $a_k(\varphi_1) \in H^s(\mathbb{T})$, then $a_k(\cdot) \in \mathscr{H}_{\sigma,\bar{s}}$, $\forall \sigma > 0$, $0 < \bar{s} < s - 1/2$.

For $\varepsilon = 0$, equation (1.8) reduces to

$$\omega_1\partial_{\varphi_1}(\omega_1\partial_{\varphi_1} + 2\partial_{\varphi_2})q = 0 \tag{3.1}$$

and its solutions q form, by the irrationality of ω_1, the infinite dimensional subspace

$$Q := \{q \in \mathscr{H}_{\sigma,\bar{s}} : \partial_{\varphi_1}q \equiv 0\} = \{q = q(\varphi_2) \in H^\sigma(\mathbb{T})\}. \tag{3.2}$$

To find solutions of (1.8) for $\varepsilon \ne 0$, we perform a Lyapunov–Schmidt reduction and we decompose the space

$$\mathscr{H}_{\sigma,\bar{s}} = Q \oplus P,$$

where $Q \equiv H^\sigma(\mathbb{T})$ and

$$P := \left\{ p = \sum_{l \in \mathbb{Z}^2} \hat{p}_l e^{il\cdot\varphi} \in \mathscr{H}_{\sigma,\bar{s}} \mid \hat{p}_l = 0 \text{ for } l_1 = 0 \right\}.$$

Projecting equation (1.8) onto the closed subspaces Q and P, setting $u = q + p \in \mathscr{H}_{\sigma,\bar{s}}$ with $q \in Q$, $p \in P$ we obtain

$$(2+\varepsilon)\ddot{q} + \Pi_Q[f(\varphi_1, q+p, \delta)] = 0 \quad (Q) \tag{3.3}$$

$$\mathscr{L}_\varepsilon[p] + \varepsilon \Pi_P[f(\varphi_1, q+p, \delta)] = 0 \quad (P) \tag{3.4}$$

where $\ddot{q} = \partial^2_{\varphi_2} q$, $\Pi_Q : \mathscr{H}_{\sigma,\bar{s}} \to Q$ is the projector onto Q,

$$(\Pi_Q u)(\varphi_2) := \frac{1}{2\pi} \int_0^{2\pi} u(\varphi_1, \varphi_2) d\varphi_1,$$

and $\Pi_P = \mathrm{Id} - Q$ is the projector onto P.

We could proceed now as in the previous section performing a finite dimensional reduction and applying variational methods. However, in this case, the infinite dimensional (Q)-equation can be directly solved by the Implicit Function Theorem in a space of analytic functions.

For this, it is useful to consider the parameter δ (and $\varepsilon = \delta^{2(d-1)}$) in the right-hand side of (3.4), as an independent parameter $\delta = \eta$, $\varepsilon = \eta^{2(d-1)}$, and to solve the equation

$$\mathscr{L}_\varepsilon[p] + \eta^{2(d-1)} \Pi_P[f(\varphi_1, q+p, \eta)] = 0 \quad (P_\eta) \tag{3.5}$$

for (ε, ω_1) in the Cantor set $\mathscr{C}_\gamma$ and *for all* η small. In this way we highlight the smoothness of the solution $p(\eta, \varepsilon, \cdot)$ of the (P_η)-equation (3.5) in the variable η.

3.1. *Solution of the* (P_η)*-Equation*

We first prove that the operator $\mathscr{L}_\varepsilon : P \to P$ has a bounded inverse when (ε, ω_1) belongs to the Cantor set $\mathscr{C}_\gamma$ defined in (1.6).

Lemma 3.1. *For any $\varepsilon_0 > 0$ the Cantor set $\mathscr{C}_\gamma$ is uncountable.*

Proof. Consider the set $\overline{\mathscr{C}}$ of couples $x_1, x_2 \in \mathscr{B}_\gamma$ such that

$$x_1 \in (-\varepsilon_1, \varepsilon_1), \quad x_2 \in (1+\varepsilon_1, 2-\varepsilon_1), \quad x_1 + x_2 \notin \mathbb{Q}, \;\; x_1 - x_2 \notin \mathbb{Q},$$

where $\varepsilon_1 = \varepsilon_0/2$. $\overline{\mathscr{C}}$ is an uncountable subset of $\mathbb{R}^2$ since for all $x_1 \in \mathscr{B}_\gamma$ the conditions $x_1 \pm x_2 \notin \mathbb{Q}$ exclude only a countable set of values x_2. The lemma follows since $\mathscr{C}_\gamma$ contains $\psi^{-1}\overline{\mathscr{C}}$ where $\psi : (\varepsilon, \omega_1) \to (\varepsilon/\omega_1, (2+\varepsilon)/\omega_1)$ is an invertible map for $(\varepsilon, \omega_1) \in (-\varepsilon_0, \varepsilon_0) \times (1, 2)$. $\square$

The operator $\mathscr{L}_\varepsilon$ has eigenvalues $D_l = (\omega_1 l_1 + \varepsilon l_2)(\omega_1 l_1 + 2l_2 + \varepsilon l_2)$.

Lemma 3.2. *For $(\varepsilon, \omega_1) \in \mathscr{C}_\gamma$ the eigenvalues D_l of $\mathscr{L}_\varepsilon$ restricted to P satisfy*

$$|D_l| = \left|(\omega_1 l_1 + \varepsilon l_2)(\omega_1 l_1 + 2l_2 + \varepsilon l_2)\right| > \gamma, \quad \forall l_1 \neq 0. \tag{3.6}$$

As a consequence, the operator $\mathscr{L}_\varepsilon : P \to P$ has a bounded inverse $\mathscr{L}_\varepsilon^{-1}$ satisfying

$$\left|\mathscr{L}_\varepsilon^{-1}[p]\right|_{\sigma,\bar{s}} \leq \frac{|p|_{\sigma,\bar{s}}}{\gamma}, \quad \forall p \in P. \tag{3.7}$$

Proof. Estimate (3.6) is trivially satisfied if $-l_1 \neq \frac{\varepsilon}{\omega_1} l_2$ and $-l_1 \neq \frac{2+\varepsilon}{\omega_1} l_2$. Now, if $-l_1 = \left[\frac{\varepsilon}{\omega_1} l_2\right]$, then $|(2+\varepsilon)l_2 + \omega_1 l_1| > |(2+\varepsilon)l_2 - \varepsilon l_2| - \frac{1}{2} > |l_2|$. Therefore, using $|\omega_1 l_1 + \varepsilon l_2| > \gamma/|l_2|$, we get (3.6). The same estimate (3.6) holds if $-l_1 = \left[\frac{2+\varepsilon}{\omega_1} l_2\right]$ since, in this case, $|\omega_1 l_1 + \varepsilon l_2| > |(2+\varepsilon)l_2 - \varepsilon l_2| - \frac{1}{2} > |l_2|$. □

Fixed points of the nonlinear operator $\mathscr{G} : P \to P$ defined by

$$\mathscr{G}(\eta, p) := -\eta^{2(d-1)} \mathscr{L}_\varepsilon^{-1} \Pi_P f(\varphi_1, q + p, \eta)$$

are solutions of the (P_η)-equation.

Lemma 3.3. *Assume $(\varepsilon, \omega_1) \in \mathscr{C}_\gamma$. $\forall R > 0$ there exists $\eta_0(R)$, $C_0(R) > 0$ such that $\forall |q|_{H^\sigma(\mathbb{T})} \leq R$, $0 < \eta\gamma^{-c} \leq \eta_0(R)$, with $c = 1/2(d-1)$, there exists a unique $p(\eta, q) \in P \cap \mathscr{H}_{\sigma,\bar{s}}$ solving the (P_η)-equation* (3.5) *and satisfying*

$$|p(\eta, q)|_{\sigma,\bar{s}} \leq C_0(R)\eta^{2(d-1)}\gamma^{-1} \tag{3.8}$$

and the equivariance property

$$p(\eta, q_\theta)(\varphi_1, \varphi_2) = p(\eta, q)(\varphi_1, \varphi_2 - \theta), \quad \forall \theta \in \mathbb{T} \tag{3.9}$$

where $q_\theta(\varphi_1, \varphi_2) := q(\varphi_1, \varphi_2 - \theta)$. Moreover $p(\cdot, \cdot) \in C^1((0, \eta_0(R)) \times Q; P)$.

Proof. See the Appendix. □

3.2. *The (Q)-Equation*

Once the (P_η)-equation has been solved by $p(\eta, q) \in P$ there remains the infinite dimensional bifurcation equation

$$(2+\varepsilon)\ddot{q} + \Pi_Q[f(\varphi_1, q + p(\eta, q), \eta)] = 0. \tag{3.10}$$

Recalling (1.9), the (Q)-equation (3.10) evaluated at $\eta = 0$ reduces to the ordinary differential equation

$$(2+\varepsilon)\ddot{q} + \langle a_{2d-1}\rangle q^{2d-1} = 0, \tag{3.11}$$

where $\langle a_{2d-1}\rangle := (1/2\pi)\int_0^{2\pi} a_{2d-1}(\varphi_1)d\varphi_1$.

Equation (3.11) is a superlinear autonomous Hamiltonian system with one degree of freedom and can be studied by a direct phase-space analysis.

Lemma 3.4. *There exists $\bar{\sigma} > 0$ such that, equation* (3.11) *possesses a 2π-periodic, analytic solution $\bar{q}(\varphi_2) \in H^{\bar{\sigma}}(\mathbb{T})$. Moreover, $\bar{q}(\varphi_2)$ is nondegenerate up to time translations, i.e., the linearized equation on $\bar{q}$*

$$(2+\varepsilon)\ddot{h} + (2d-1)\langle a_{2d-1}\rangle \bar{q}^{2(d-1)} h = 0 \tag{3.12}$$

possesses a one-dimensional space of 2π-periodic solutions, spanned by $\dot{\bar{q}}$.

Proof. Up to a rescaling, equation (3.11) can be written as $\ddot{x} = -V'(x)$ with potential energy $V(x) := x^{2d}$. All solutions of such system are analytic and periodic with period

$$T(E) = 4\int_0^{E^{\frac{1}{2d}}} \frac{dx}{\sqrt{2(E - x^{2d})}} = 4E^{\frac{1}{2d}-\frac{1}{2}} \int_0^1 \frac{dx}{\sqrt{2(1 - x^{2d})}}.$$

The equation $T(E) = 2\pi$ has a solution $\bar{q}(\varphi_2)$ which is in $H^{\bar{\sigma}}(\mathbb{T})$ for some appropriate $\bar{\sigma} > 0$. The nondegeneracy of the corresponding 2π-periodic solution follows by

$$\frac{dT}{dE} = 2\left(\frac{1}{d} - 1\right) E^{\frac{1}{2d}-\frac{3}{2}} \int_0^1 \frac{dx}{\sqrt{2(1 - x^{2d})}} \neq 0$$

and the next proposition proven in the Appendix. $\square$

Proposition 2. *Suppose the autonomous second order equation $-\ddot{x} = V'(x)$, $x \in \mathbb{R}$, possesses a continuous family of nonconstant periodic solutions $x(E, t)$ with energy E and period $T(E)$ satisfying the anysocronicity condition $\frac{dT(E)}{dE} \neq 0$. Then $x(E, t)$ is nondegenerate up to time translations, i.e., the $T(E)$-periodic solutions of the linearized equation*

$$-\ddot{h} = D^2 V(x(E, t))h \tag{3.13}$$

form a one dimensional subspace spanned by $(\partial_t x)(E, t)$.

From now on, we fix $\overline{R} := |\bar{q}|_{H^{\bar{\sigma}}(\mathbb{T})} + 1$ in Lemma 3.3 and take $0 < \eta\gamma^{-c} \leq \eta_0(\overline{R})$.
By Lemma 3.4 and (3.9), we can construct solutions of the infinite dimensional bifurcation equation (3.10) by means of the Implicit Function Theorem.

Lemma 3.5. *There exist $0 < \eta_1 \leq \eta_0(\overline{R})$, $C_1 > 0$ such that for all $0 < \eta\gamma^{-c} \leq \eta_1$, equation* (3.10) *has a unique* (*up to translations*) *solution $\bar{q}_\eta(\varphi_2) \in H^{\bar{\sigma}}(\mathbb{T})$ satisfying*

$$|\bar{q}_\eta - \bar{q}|_{H^{\bar{\sigma}}(\mathbb{T})} \leq C_1 |\eta|.$$

Proof of Theorem B. Setting again $\delta \equiv \eta$, and calling (by abuse of notation) $\bar{q}_\varepsilon(\varphi_2) + p(\varepsilon, \bar{q}_\varepsilon)$ solves (1.8) and

$$u(\varepsilon, \varphi) = |\varepsilon|^{1/2(d-1)}[\bar{q}_\varepsilon(\varphi_2) + p(\varepsilon, \bar{q}_\varepsilon)]$$

is a nontrivial solution of (1.4). The bound (1.7) follows by (3.8). As in Theorem A, the solution u depends on both the variables (φ_1, φ_2). Finally, the solution

$v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ of (1.1) is quasi-periodic since, by the definition of $\mathscr{C}_\gamma$, $\omega_1/\omega_2 = \omega_1/(1+\varepsilon) \notin \mathbb{Q}$. $\square$

Remark 3. To prove existence of solutions of (1.8), i.e., (1.1), it is sufficient that the second order equation (3.11) possessess a continuous, nonisocronous family of nonconstant periodic orbits one of them having period $2\pi/j$, see Proposition 2.

The hypothesis that the leading term in the nonlinearity f is an odd power of u is not of technical nature: the following nonexistence result holds.

Proposition 3 (Nonexistence). *Let* $f(\varphi_1, u) = a(\varphi_1)u^D$ *with* D *even and* $\int_0^{2\pi} a(\varphi_1)d\varphi_1 \neq 0$. $\forall R > 0$, *there exists* $\varepsilon_0 > 0$ *such that* $\forall \sigma \geq 0$, $\bar{s} > s > -\frac{1}{2}$, $\forall(\varepsilon, \omega_1) \in \mathscr{C}_\gamma$ *with* $|\varepsilon| < \varepsilon_0$ *equation* (1.4) *does not possess solutions* $u \in \mathscr{H}_{\sigma,\bar{s}}$ *in the ball* $|u|_{\sigma,\bar{s}} \leq R|\varepsilon|^{1/(D-1)}$.

Proof. We first rescale equation (1.4) with $u \to |\varepsilon|^{1/(D-1)}u$ obtaining

$$\mathscr{L}_\varepsilon u + |\varepsilon| a(\varphi_1)u^D = 0. \tag{3.14}$$

Write any solution $u_\varepsilon \in B_{\sigma,\bar{s}}(R) := \{u \in \mathscr{H}_{\sigma,\bar{s}} : |u|_{\sigma,\bar{s}} \leq R\}$ of (3.14) as $u_\varepsilon = q_\varepsilon + p_\varepsilon$ with $q_\varepsilon \in Q$, $p_\varepsilon \in P$. p_ε satisfies the (P)-equation $\mathscr{L}_\varepsilon p + |\varepsilon|\Pi_P a(\varphi_1)u^D = 0$ and therefore $|p_\varepsilon|_{\sigma,\bar{s}} \leq C(R)|\varepsilon|$. Then, for ε small enough, $p_\varepsilon \equiv p(\varepsilon, q_\varepsilon)$ where $p(\varepsilon, q_\varepsilon)$ is constructed as in Lemma 3.3 and satisfies the estimate $|p(\varepsilon, q_\varepsilon)|_{\sigma,\bar{s}} \leq C|\varepsilon||q_\varepsilon|^D_{H^\sigma(\mathbb{T})}$.

The projection q_ε satisfies the (Q)-equation

$$(2+\varepsilon)\ddot{q}_\varepsilon + \mathrm{sign}(\varepsilon)\Pi_Q[a(\varphi_1)(q_\varepsilon + p(\varepsilon, q_\varepsilon))^D] = 0, \tag{3.15}$$

and therefore $|q_\varepsilon|_{H^{\sigma,2}(\mathbb{T})} \leq C(R)$.

We claim that $q_\varepsilon \to 0$ in $H^\sigma(\mathbb{T})$ (and so in $\mathscr{H}_{\sigma,\bar{s}}$) for $\varepsilon \to 0$. Indeed, from any subsequence q_ε, we can extract by the compact embedding $H^{\sigma,2}(\mathbb{T}) \hookrightarrow H^\sigma(\mathbb{T})$ another convergent subsequence q_{ε_n} such that $q_{\varepsilon_n} \to \bar{q} \in H^\sigma(\mathbb{T})$. By (3.15), we deduce that

$$2\ddot{\bar{q}} + \mathrm{sign}(\varepsilon)\langle a\rangle \bar{q}^D = 0,$$

where $\langle a\rangle := \int_0^{2\pi} a(\varphi_1)d\varphi_1 \neq 0$. Such equation does not possess nontrivial periodic solutions for both $\mathrm{sign}(\varepsilon) = \pm 1$, i.e., $\varepsilon > 0$ and $\varepsilon < 0$, and we conclude that $\bar{q} = 0$.

We finally prove that equation (3.15) does not possess nontrivial periodic solutions in a small neighborhood of the origin.

Linearizing equation (3.15) at $q = 0$ we get $(2+\varepsilon)\ddot{h} = 0$ whose solutions in $H^\sigma(\mathbb{T})$ are the constants. We can perform another Lyapunov–Schmidt reduction close to zero decomposing $H^\sigma(\mathbb{T}) = \{\text{constants}\} \oplus \{\text{zero average functions}\}$, namely, $q_\varepsilon = \rho + w$. By the Implicit Function Theorem we get that for any constant $|\rho| \leq \rho_0$ small enough (independently of ε) there exists a unique zero average function w_ρ with $|w_\rho|_{H^\sigma(\mathbb{T})} = O(\rho^D)$ solving

$$(2+\varepsilon)\ddot{w}_\rho + \big[a(\varphi_1)(\rho + w_\rho + p(\varepsilon, q_\varepsilon))^D - \big\langle a(\varphi_1)(\rho + w_\rho + p(\varepsilon, q_\varepsilon))^D\big\rangle\big] = 0.$$

Hence ρ is such that

$$0 = \left\langle a(\varphi_1)(\rho + w_\rho + p(\varepsilon, q_\varepsilon))^D \right\rangle = \langle a \rangle \rho^D + o(\rho^D).$$

This implies $\rho = 0$ since $\langle a \rangle \neq 0$ and so $q_\varepsilon = \rho + w_\rho = 0$. $\square$

4. Appendix

Lemma 4.1. *$\mathcal{H}_{\sigma,s}$ is a Banach algebra for $\sigma, s \geq 0$.*

Proof. By the product Cauchy formula,

$$uv = \sum_{j \in \mathbb{Z}^2} \left(\sum_{k \in \mathbb{Z}^2} u_{j-k} v_k \right) e^{\mathrm{i} j \cdot \varphi}$$

and therefore,

$$\begin{aligned} |uv|_{\sigma,s} &:= \sum_{j \in \mathbb{Z}^2} e^{\sigma |j_2|} [j_1]^s \left| \sum_{k \in Z^2} u_{j-k} v_k \right| \leq \sum_{j \in \mathbb{Z}^2} e^{\sigma |j_2|} [j_1]^s \sum_{k \in \mathbb{Z}^2} |u_{j-k}| \, |v_k| \\ &\leq \sum_{k \in \mathbb{Z}^2} |v_k| \sum_{j \in \mathbb{Z}^2} |u_{j-k}| e^{\sigma |j_2|} [j_1]^s \\ &\leq 2^s \sum_{k \in \mathbb{Z}^2} |v_k| e^{\sigma |k_2|} [k_1]^s \sum_{j \in \mathbb{Z}^2} |u_{j-k}| e^{\sigma |j_2 - k_2|} [j_1 - k_1]^s := 2^s |u|_{\sigma,s} |v|_{\sigma,s} \end{aligned}$$

since $e^{\sigma |j_2|} \leq e^{\sigma |j_2 - k_2|} e^{\sigma |k_2|}$ and $[j_1] \leq 2[j_1 - k_1][k_1]$ for all $k, j \in \mathbb{Z}^2$. $\square$

Proof of Lemma 2.3. Let us consider

$$B := \left\{ (q_2, p) \in Q_2 \oplus P : \; |q_2|_{\sigma,s} \leq \rho_1, |p|_{\sigma,s} \leq \rho_2 \right\}$$

with norm $|(q_2, p)|_{\sigma,s} := |q_2|_{\sigma,s} + |p|_{\sigma,s}$. We claim that, under the assumptions (2.9), there exists $0 < \rho_1, \; \rho_2 < 1$, see (4.6), such that the map $(q_2, p) \to \mathcal{G}(q_2, p; q_1)$ is a contraction in B, i.e.,

(i) $(q_2, p) \in B \Longrightarrow \mathcal{G}(q_2, p; q_1) \in B$, and
(ii) $|\mathcal{G}(q_2, p; q_1) - \mathcal{G}(\tilde{q}_2, \tilde{p}; q_1)|_{\sigma,s} \leq (1/2)|(q_2, p) - (\tilde{q}_2, \tilde{p})|_{\sigma,s}, \; \forall (q_2, p), (\tilde{q}_2, \tilde{p}) \in B$.

In the following κ_i will denote positive constants *independent* on R, N, and ε (i.e., on $\delta := |\varepsilon|^{1/2(d-1)}$).

By (2.7) and the Banach algebra property of $\mathcal{H}_{\sigma,s}$

$$\begin{aligned} |\mathcal{G}_1(q_2, p; q_1)|_{\sigma,s} &= \left| L_1^{-1} \Pi_{Q_2} f(\varphi_1, q_1 + q_2 + p, \delta) \right|_{\sigma,s} \\ &\leq \frac{\kappa_1}{N^2} \left(|q_1|_{\sigma,s}^{2d-1} + |q_2|_{\sigma,s}^{2d-1} + |p|_{\sigma,s}^{2d-1} \right), \end{aligned} \tag{4.1}$$

provided that $0 \leq \delta \leq \delta_0(R)$. Similarly, for $\varepsilon \in \mathcal{B}_\gamma$, by (2.6),

$$\begin{aligned} |\mathcal{G}_2(q_2, p; q_1)|_{\sigma,s} &= \left| \varepsilon \mathcal{L}_\varepsilon^{-1} \Pi_P f(\varphi_1, q_1 + q_2 + p, \delta) \right|_{\sigma,s} \\ &\leq \kappa_2 |\varepsilon| \gamma^{-1} \left(|q_1|_{\sigma,s}^{2d-1} + |q_2|_{\sigma,s}^{2d-1} + |p|_{\sigma,s}^{2d-1} \right). \end{aligned} \tag{4.2}$$

For all $q_1 \in Q_1(N)$ and since $0 \le s < 1/2$

$$
\begin{aligned}
|q_1|_{\sigma,s} &= \sum_{|l_2|\le N} |\hat{q}_{0,l_2}| e^{\sigma|l_2|} + |\hat{q}_{-2l_2,l_2}| e^{\sigma|l_2|}[-2l_2]^s \\
&\le e^{\sigma N} \sum_{|l_2|\le N} |\hat{q}_{0,l_2}| + |\hat{q}_{-2l_2,l_2}|[-2l_2]^s \\
&\le \kappa_3 \Bigg[\Big(\sum_{|l_2|\le N} |\hat{q}_{0,l_2}|^2 [l_2]^2 \Big)^{1/2} \Big(\sum_{l_2\in\mathbb{Z}} \frac{1}{[l_2]^2} \Big)^{1/2} \\
&\qquad + \Big(\sum_{|l_2|\le N} |\hat{q}_{-2l_2,l_2}|^2 [l_2]^2 \Big)^{1/2} \Big(\sum_{l_2\in\mathbb{Z}} \frac{1}{[l_2]^{2(1-s)}} \Big)^{1/2} \Bigg] \\
&\le \kappa_4 |q_1|_{H^1} \qquad (4.3)
\end{aligned}
$$

whenever $0 \le \sigma N \le 1$.

Substituting in (4.1)–(4.2), we get $\forall |q_1|_{H^1} \le 2R$, $\forall |q_2|_{\sigma,s} \le \rho_1$, $\forall |p|_{\sigma,s} \le \rho_2$

$$
|\mathscr{G}_1(q_2, p; q_1)|_{\sigma,s} \le \kappa_5 N^{-2}\big(R^{2d-1} + \rho_1^{2d-1} + \rho_2^{2d-1}\big) \qquad (4.4)
$$

$$
|\mathscr{G}_2(q_2, p; q_1)|_{\sigma,s} \le \kappa_5 |\varepsilon|\gamma^{-1}\big(R^{2d-1} + \rho_1^{2d-1} + \rho_2^{2d-1}\big)\ . \qquad (4.5)
$$

Now, setting $C_0(R) := \kappa_5 R^{2d-1}$, we define

$$
\rho_1 := \frac{2C_0(R)}{N^2} \quad \rho_2 := 2|\varepsilon|\gamma^{-1} C_0(R). \qquad (4.6)
$$

By (4.4) and (4.5), there exists $N_0(R) \in \mathbb{N}^+$ and $\varepsilon_0(R) > 0$ such that $\forall N \ge N_0(R)$ and $\forall |\varepsilon|\gamma^{-1} \le \varepsilon_0(R)$

$$
|\mathscr{G}_1(q_2, p; q_1)|_{\sigma,s} \le \rho_1, \quad |\mathscr{G}_2(q_2, p; q_1)|_{\sigma,s} \le \rho_2
$$

proving (i). Item (ii) is obtained with similar estimates.

By the Contraction Mapping Theorem there exists a unique fixed point $(q_2(q_1), p(q_1)) := (q_2(\varepsilon, N, q_1), p(\varepsilon, N, q_1))$ of $\mathscr{G}$ in B. The bounds (2.10) follow by (4.6).

Since $\mathscr{G} \in C^1(Q_2 \oplus P \times Q_1; Q_2 \oplus P \times Q_1)$ the Implicit Function Theorem implies that the maps $Q_1 \ni q_1 \to (q_2(\varepsilon, N, q_1), p(\varepsilon, N, q_1))$ are C^1.

Differentiating $(q_2(q_1), p(q_1)) = \mathscr{G}(q_2(q_1), p(q_1), q_1)$

$$
\begin{aligned}
q_2'(q_1)[h] &= -L_1^{-1}\Pi_{Q_2}(\partial_u f)(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)\big(h + q_2'(q_1)[h] + p'(q_1)[h]\big) \\
p'(q_1)[h] &= -\varepsilon \mathscr{L}_\varepsilon^{-1}\Pi_{Q_2}(\partial_u f)(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)\big(h + q_2'(q_1)[h] + p'(q_1)[h]\big)
\end{aligned}
$$

and using (2.7), (2.6), and the Banach algebra property of $\mathscr{H}_{\sigma,s}$

$$
\begin{aligned}
|q_2'(q_1)[h]|_{\sigma,s} &\le C(R)N^{-2}\big(|h|_{\sigma,s} + |q_2'(q_1)[h]|_{\sigma,s} + |p'(q_1)[h]|_{\sigma,s}\big) \\
|p'(q_1)[h]|_{\sigma,s} &\le C(R)|\varepsilon|\gamma^{-1}\big(|h|_{\sigma,s} + |q_2'(q_1)[h]|_{\sigma,s} + |p'(q_1)[h]|_{\sigma,s}\big),
\end{aligned}
$$

which implies the bounds (2.11) since

$$\det \begin{vmatrix} 1 - C(R)N^{-2} & -C(R)N^{-2} \\ -C(R)|\varepsilon|\gamma^{-1} & 1 - C(R)|\varepsilon|\gamma^{-1} \end{vmatrix} \geq \frac{1}{2}$$

for $C(R)(|\varepsilon|\gamma^{-1} + N^{-2})$ small enough and (4.3).

Proof of Lemma 2.4. By (2.4) and (2.5), we have that, at $u := q_1 + q_2(q_1) + p(q_1)$,

$$d\Psi_\varepsilon(u)[h] = 0 \quad \forall h \in Q_2 \qquad \text{and} \qquad d\Psi_\varepsilon(u)[h] = 0 \quad \forall h \in P. \tag{4.7}$$

Since $q_2'(q_1)[k] \in Q_2$ and $p'(q_1)[k] \in P$ $\forall k \in Q_1$, we deduce

$$d\Phi_{\varepsilon,N}(q_1)[k] = d\Psi_\varepsilon(u)\big[h + q_2'(q_1)[k] + p'(q_1)[k]\big] = d\Psi_\varepsilon(u)[k] \quad \forall k \in Q_1$$

and therefore $u := q_1 + p(q_1) + q_2(q_1)$ solves also the (Q_1)-equation (2.3).

Write $\Psi_\varepsilon(u) = \Psi_\varepsilon^{(2)}(u) - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta)$ where

$$\Psi_\varepsilon^{(2)}(u) := \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1} u)^2 + (1+\varepsilon)(\partial_{\varphi_1} u)(\partial_{\varphi_2} u) + \frac{\varepsilon(2+\varepsilon)}{2}(\partial_{\varphi_2} u)^2$$

is an homogeneous functional of degree two. By homogeneity,

$$\Psi_\varepsilon(u) = \frac{1}{2} d\Psi_\varepsilon^{(2)}(u)[u] - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta). \tag{4.8}$$

By (2.4) and (2.5) (i.e., (4.7)),

$$d\Psi_\varepsilon^{(2)}(q_1 + q_2(q_1) + p(q_1))[q_2(q_1) + p(q_1)] = \varepsilon \int_{\mathbb{T}^2} f(\varphi_1, u, \delta)(q_2(q_1) + p(q_1)). \tag{4.9}$$

Substituting in (4.8) we obtain, at $u = q_1 + q_2(q_1) + p(q_1)$

$$\begin{aligned} \Phi_{\varepsilon,N}(q_1) &= \Psi_\varepsilon(q_1 + p(q_1) + q_2(q_1)) \\ &= \frac{1}{2} d\Psi_\varepsilon^{(2)}(u)[q_1 + p(q_1) + q_2(q_1)] - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta) \\ &= \frac{1}{2} d\Psi_\varepsilon^{(2)}(q_1)[q_1] - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta) + \frac{1}{2} f(\varphi_1, u, \delta)(q_2(q_1) + p(q_1)) \\ &= \Psi_0(q_1) + \varepsilon \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} q_1)^2 + (\partial_{\varphi_1} q_1)(\partial_{\varphi_2} q_1) - F(\varphi_1, u, \delta) \\ &\quad + \frac{1}{2} f(\varphi_1, u, \delta)(q_2(q_1) + p(q_1)) = \text{const} + \varepsilon(\Gamma(q_1) + \mathcal{R}_{\varepsilon,N}(q_1)) \end{aligned}$$

because $\Psi_0(q_1) \equiv \text{const}$.

By (2.10) the bounds (2.14) and (2.15) follow.

Proof of Lemma 3.3. The existence of $p(\eta, q) \in \mathcal{H}_{\sigma,\bar{s}}$ can be proven as in Lemma 2.3 using the Contraction Mapping Theorem. The smoothness of $p(\eta, q)$ follows by the Implicit Function Theorem since $\mathcal{G}(\eta, p)$ is smooth in η and q.

By the invariance of equation (3.5) under translations in the φ_2 variable the function $p(\eta, q)(\varphi_1, \varphi_2 - \theta)$ solves

$$p(\eta, q)(\varphi_1, \varphi_2 - \theta) + \eta^{2(d-1)}\mathscr{L}_\varepsilon^{-1}\Pi_P f(\varphi_1, q_\theta + p(\eta, q)(\varphi_1, \varphi_2 - \theta), \eta) = 0$$

and, therefore, by uniqueness (3.9) holds.

Proof of Proposition 2. Write $x(E, t) = y(\omega(E)t, E)$ where $y(\varphi, E)$ is 2π-periodic in φ and $\omega(E) := 2\pi/T(E)$. The functions $(\partial_t x)(E, t)$ and

$$(\partial_E x)(E, t) = t\frac{d\omega(E)}{dE}(\partial_\varphi y)(\omega(E)t, E) + (\partial_E y)(\omega(E)t, E) \tag{4.10}$$

are two linearly independent solutions of the linearized equation (3.13). $(\partial_t x)(E, t)$ is 2π-periodic while, since

$$\frac{d\omega(T)}{dT} = 2\pi T(E)^{-2}\frac{dE(T)}{dT} \neq 0 \quad \text{and} \quad (\partial_\varphi y)(\varphi, E) \not\equiv 0$$

(if not $x(E, t)$ would be constant in t), $(\partial_E x)(E, t)$ is not 2π-periodic. We conclude that the space of $T(E)$-periodic solutions of (3.13) form a 1-dimensional linear space spanned by $(\partial_t x)(E, t)$.

Acknowledgments

This article is supported by M.I.U.R. "Variational Methods and Nonlinear Differential Equations". The authors thank Luca Biasco and Philippe Bolle for useful comments. Part of this article was written when the authors were at SISSA.

References

Ambrosetti, A., Badiale, M. (1998). Homoclinics: Poincaré–Melnikov type results via a variational approach. *Annales I. H. P. – Analyse Nonlin.* 15(2):233–252.

Bambusi, D., Paleari, S. (2001). Families of periodic solutions of resonant PDE's. *J. Nonlinear Sci.* 11(1):69–87.

Bambusi, D., Berti, M. (2005). A Birkhoof–Lewis type theorem for some Hamiltonian PDE's. *SIAM J. Math. Anal.* 37(1):83–102.

Benci, V., Rabinowitz, P. (1979). Critical point theorems for indefinite functionals. *Invent. Math.* 52(3):241–273.

Berti, M., Biasco, L. (2005). Periodic solutions of nonlinear wave equations with non-monotone forcing terms. *Rend. Mat. Acc. Naz. Lincei*. 16(2):117–124.

Berti, M., Biasco, L. Forced vibrations of wave equations with non-monotone nonlinearities. To appear in *Ann. Inst. Poincare*.

Berti, M., Bolle, P. (2003). Periodic solutions of nonlinear wave equations with general nonlinearities. *Comm. Math. Phys.* 243(2):315–328.

Berti, M., Bolle, P. (2004). Multiplicity of periodic solutions of nonlinear wave equations. *Nonlinear Anal.* 56:1011–1046.

Berti, M., Bolle, P. Cantor families of periodic solutions for completely resonant non linear wave equations. To appear in *Duke Math Journal*.

Berti, M., Biasco, L., Valdinoci, E. (2004). Periodic orbits close to elliptic tori and applications to the three-body problem. *Annali Scuola Normale Sup. di Pisa, Cl. Sci.* 3(V):87–138.

Bourgain, J. (1998). Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. *Ann. Math.* 148(2):363–439.

Bourgain, J. (1999). *Periodic Solutions of Nonlinear Wave Equations*. Chicago Lectures in Math. Chicago, IL: Univ. Chicago Press.

Brezis, H., Nirenberg, L. (1978). Forced vibrations for a nonlinear wave equation. *Comm. Pure Appl. Math.* 31(1):1–30.

Coron, J. M. (1983). Periodic solutions of a nonlinear wave equation without assumption of monotonicity. *Math. Ann.* 262(2):273–285.

Craig, W., Wayne, E. (1993). Newton's method and periodic solutions of nonlinear wave equations. *Comm. Pure Appl. Math.* 46:1409–1498.

Gentile, G., Procesi, M. (2006). Conservation of resonant periodic solutions for the one-dimensional nonlinear Schrödinger equation. *Comm. Math. Phys.* 262(3):533–553.

Gentile, G., Mastropietro, V., Procesi, M. (2005). Periodic solutions of completely resonant nonlinear wave equations. *Comm. Math. Phys.* 256(2):437–490.

McKenna, P. J. (1985). On solutions of a nonlinear wave question when the ratio of the period to the length of the interval is irrational. *Proc. Amer. Math. Soc.* 93(1):59–64.

Kuksin, S. B. (2000). *Analysis of Hamiltonian PDEs*. Oxford Lecture Series in Mathematics and its Applications, 19. Oxford University Press.

Lidskiĭ, B. V., Shul′man, E. I. (1988). Periodic solutions of the equation $u_{tt} - u_{xx} + u^3 = 0$. *Funct. Anal. Appl.* 22(4):332–333.

Plotinikov, P. I., Yungermann, L. N. (1988). Periodic solutions of a weakly nonlinear wave equation with an irrational relation of period to interval length. *Transl. in Diff. Eq.* 24(9):1059–1065.

Pöschel, J. (1996). Quasi-periodic solutions for a nonlinear wave equation. *Comment. Math. Helv.* 71(2):269–296.

Procesi, M. (2005). Quasi-periodic solutions for completely resonant wave equations in 1D and 2D. *Discr. Cont. Dyn. Syst.* 13(3):541–552.

Procesi, M. Families of quasi-periodic solutions for a completely resonant wave equation. Preprint.

Rabinowitz, P. (1967). Periodic solutions of nonlinear hyperbolic partial differential equations. *Comm. Pure Appl. Math.* 20:145–205.

Rabinowitz, P. (1971). Time periodic solutions of nonlinear wave equations. *Manuscripta Math.* 5:165–194.

Rabinowitz, P. H. (1986). *Minimax Methods in Critical Point Theory with Applications to Differential Equations*. CBMS Regional Conference Series in Mathematics, 65. American Mathematical Society.

Wayne, C. E. (1990). Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Comm. Math. Phys.* 127(3):479–528.