EQUIVARIANT WAVE MAPS

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1. Introduction

1.1. **Wave maps.** Let (N, h) be a complete, smooth Riemannian manifold of dimension k with $\partial N = \emptyset$. Usually we will assume N to be compact. The Nash embedding theorem always permits to regard N as a submanifold of some Euclidean space $\mathbb{R}^n$. The projection π_N taking a point $p \in \mathbb{R}^n$ to its nearest neighbor $\pi_N(p) \in N$ then is uniquely defined and smooth in a tubular neighborhood of N.

The concept of a wave map to the "target" manifold N generalizes the standard wave equation. Denote space-time coordinates on $(m+1)$-dimensional Minkowski space $\mathbb{R}^{m+1}$ as $z = (t, x) = (x^\alpha), 0 \le \alpha \le m$. For a ($C^1$-)map $u\colon \mathbb{R}^{m+1} \to N \hookrightarrow \mathbb{R}^n$ and a space-time domain Q let

$$\mathcal{L}(u, Q) = \frac{1}{2} \int_Q \partial^\alpha u \cdot \partial_\alpha u \, dz, \tag{1}$$

denote the Lagrangean action of u on Q. Here, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$; moreover, we raise and lower indices with the Minkowski metric $(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, \dots, 1)$ and we tacitly sum over repeated indices. Finally, we use the standard Euclidean inner product to compute the expression $\partial^\alpha u \cdot \partial_\alpha u$.

Given a vector field $\varphi \in C_0^\infty(\mathbb{R}^{m+1}; \mathbb{R}^n)$, we may use π_N to define a 1-parameter variation of the map u through maps $u_\varepsilon = \pi_N(u + \varepsilon\varphi)\colon \mathbb{R}^{m+1} \to N$ for $|\varepsilon| < \varepsilon_0$ with

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} u_\varepsilon = d\pi_N(u)\varphi \tag{2}$$

and such that $u_\varepsilon \equiv u$ outside the support of φ.

Definition 1.1. *The map u is a* **wave map** *if u is stationary for $\mathcal{L}$ in the sense that*

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathcal{L}(u_\varepsilon, Q) = 0 \tag{3}$$

for any variation u_ε of u defined via a vector field $\varphi \in C_0^\infty(\mathbb{R}^{m+1}; \mathbb{R}^n)$ as above and any open bounded Q such that $supp(\varphi) \subset Q$.

For a wave map u of class C^2 we may integrate by parts to compute

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathcal{L}(u_\varepsilon, Q) = \int_Q \partial^\alpha u \cdot \partial_\alpha (d\pi_N(u)\varphi) \, dz = \int_Q d\pi_N(u)(\partial^\alpha \partial_\alpha u) \cdot \varphi \, dz \,. \tag{4}$$

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Since $\varphi \in C_0^\infty(\mathbb{R}^{m+1}); \mathbb{R}^n)$ is arbitrary, we conclude that

$$D^\alpha \partial_\alpha u = d\pi_N(u)(\partial^\alpha \partial_\alpha u) = 0, \tag{5}$$

with D denoting the pull-back of the covariant derivative on N. Equation (5) is independent of the embedding of N and may also be interpreted intrinsically.

Observe that in the case when $N = \mathbb{R}$ we have $\pi_N = id$ and equation (5) simply is the standard wave equation

$$\Box u = -\partial^\alpha \partial_\alpha u = u_{tt} - \Delta u = 0,$$

where $u_t = \partial_t u$ and with $\Delta = \sum_{i=1}^m \partial_i^2$ the spatial Laplacian.

Geometrically, equation (5) can easily be interpreted as saying that

$$\Box u \perp T_u N. \tag{6}$$

Thus, in the case when $N = S^k \hookrightarrow \mathbb{R}^{k+1}$ equation (5) takes the form $\Box u = \lambda u$ for some scalar function λ. Taking account of the fact that $|u|^2 \equiv 1$, we compute

$$\lambda = \Box u \cdot u = -\partial^\alpha(\partial_\alpha u \cdot u) + \partial_\alpha u \cdot \partial^\alpha u = \partial_\alpha u \cdot \partial^\alpha u = |\nabla u|^2 - |u_t|^2$$

and thus find the equation

$$\Box u = u_{tt} - \Delta u = (|\nabla u|^2 - |u_t|^2)u$$

for a wave map $u\colon \mathbb{R}^{m+1} \to S^k \hookrightarrow \mathbb{R}^{k+1}$.

For an arbitrary closed hypersurface $N \subset \mathbb{R}^n$ with unit normal vector field ν, letting $w = \nu \circ u$ and observing that $w \cdot \partial^\alpha u = 0$, likewise from (6) we have the equation $\Box u = \lambda w$ for some scalar function

$$\lambda = \Box u \cdot w = -\partial^\alpha(\partial_\alpha u \cdot w) + \partial_\alpha u \cdot \partial^\alpha w = \partial_\alpha u \cdot \partial^\alpha w.$$

With $B(p)\colon T_pN \times T_pN \to (T_pN)^\perp$ denoting the second fundamental form of $N \subset \mathbb{R}^n$ at $p \in N$, thus we obtain the extrinsic form of the wave map equation

$$\Box u = \lambda w = w\partial_\alpha u \cdot \partial^\alpha w = \nu(u)(\partial_\alpha u \cdot d\nu(u)\partial^\alpha u) = B(u)(\partial^\alpha u, \partial_\alpha u). \tag{7}$$

In components $u = (u^1, \ldots, u^n)$ and again using the fact that $w^j \partial_\alpha u^j = w \cdot \partial^\alpha u = 0$, following Hélein [7],and Rivière [17] we can also write this as

$$\Box u^i = w^i \partial_\alpha w^j \partial_\alpha u^j = (w^i \partial_\alpha w^j - w^j \partial_\alpha w^i)\partial_\alpha u^j = \Omega_\alpha^{ij} \partial_\alpha u^j, 1 \le i \le n, \tag{8}$$

with an anti-symmetric 1-form $\Omega = (\Omega^{ij}) = \Omega_\alpha dx^\alpha$, and similarly for arbitrary codimension of N.

Note that from any solution u to equation (5) or (7) we can obtain further solutions by scaling $u^R(t,x) = u(Rt, Rx)$ with a constant $R > 0$.

1.2. **The energy identity.** The geometric equation (6) immediately implies the conservation law

$$0 = u_t \cdot \Box u = \frac{d}{dt} e(u) - div(\nabla u \cdot u_t) \tag{9}$$

for the energy density and density of momentum

$$e(u) = \frac{1}{2}|du|^2 = \frac{|u_t|^2 + |\nabla u|^2}{2},\ m(u) = \nabla u \cdot u_t.$$

Since clearly $|m(u)| \le e(u)$, from (9) it follows that

$$\int_{\{t\}\times B_R(x_0)} e(u(t))dx \le \int_{\{0\}\times B_{R+t}(x_0)} e(u(t))dx \tag{10}$$

for all $x_0 \in \mathbb{R}^m$, $R > 0$, and $t > 0$. In particular, energy will spread with speed at most 1, and $u(t)$ will have compact support for any t whenever $u(0)$ and $u_t(0)$ have this property. In this case then, upon integrating equation (9) over the region $[0,t]\times\mathbb{R}^m$ we find the identity

$$E(u(t)) = \int_{\{t\}\times\mathbb{R}^m} e(u(t))dx = E(u(0)). \tag{11}$$

This H^1-energy is the only quantity known to be conserved in general.

1.3. **The Cauchy problem for wave maps.** We study the Cauchy problem for wave maps with initial data $(u,u_t)_{|t=0} \in \dot H^s \times \dot H^{s-1}(\mathbb{R}^m;TN)$, where $\dot H^s$ for any s denotes the homogenous Sobolev space. In view of the invariance

$$||(u,u_t)_{|t=0}||_{\dot H^{\frac{m}{2}}\times\dot H^{\frac{m}{2}-1}(\mathbb{R}^m;TN)} = ||(u^R,u_t^R)_{|t=0}||_{\dot H^{\frac{m}{2}}\times\dot H^{\frac{m}{2}-1}(\mathbb{R}^m;TN)}$$

under scaling $u^R(t,x) = u(Rt,Rx)$ the $\dot H^{\frac{m}{2}}\times\dot H^{\frac{m}{2}-1}$-regularity is critical.

In a long quest towards the full resolution of this problem, finally the initial value problem for (5) was shown to be globally well-posed for initial data

$$(u,u_t)_{|t=0} = (u_0,u_1) \in \dot H^{\frac{m}{2}}\times\dot H^{\frac{m}{2}-1}(\mathbb{R}^m;TN) \tag{12}$$

that are small in the critical norm.

The break-through was achieved by Tao [27], [28] in the case when $N = S^k$, first only for $m \ge 5$ and finally for all $m \ge 2$. For $m \ge 5$, by a variant of Tao's method, Klainerman-Rodnianski [10] were able to extend his results to general targets. These results rely on sophisticated microlocalisation techniques and seem highly technical. Independently and almost simultaneously with [10], jointly with Shatah [21] we established well-posedness in any dimension $m \ge 4$ and for any complete Riemannian target manifold N with bounded curvature. Moreover, our proof proceeds directly in configuration space and does not require any tools from harmonic analysis other than the Strichartz estimates and its recent improvement by Keel-Tao [9]. Similar results are due to Nahmod - Stefanov - Uhlenbeck [16]. In the low-dimensional cases $2 \le m \le 3$ global well-posedness of the Cauchy problem for (5), (12) for initial data of small critical energy was obtained by Krieger [12], [13] for wave maps $u\colon \mathbb{R}^{m+1} \to H^2$ to hyperbolic space H^2, and, finally, by Tataru [33] for general targets. Previous work of Tataru [31], [32] already had shown the problem to be well-posed for initial data of small energy in a critical Besov space.

For completeness, we recall our results with Shatah [21] in Appendix B.

1.4. **The two-dimensional case.** Since the H^1-energy is the only known conserved quantity for the wave map system, the case when $m = 2$ is particularly interesting. In this dimension the H^1-energy is critical and one may hope to obtain also global results and a characterization of singularities. Indeed, this is possible in the case of co-rotational wave maps from $(1+2)$-dimensional Minkowski space into a target surface of revolution and for rotationally symmetric wave maps on $\mathbb{R}^{1+2}$.

In the following we review the known results in these cases and describe some recent improvements.

2. Co-rotational wave maps

2.1. The co-rotational setting. Let N be a surface of revolution with metric

$$ds^2 = d\rho^2 + g^2(\rho)d\theta^2,$$

where $\theta \in S^1$ and with $g \in C^\infty(\mathbb{R})$ satisfying $g(0) = 0$, $g'(0) = 1$. Moreover, we assume that g is odd and either

$$g(\rho) > 0 \text{ for all } \rho > 0 \tag{13}$$

with

$$\int_0^\infty |g(\rho)|\, d\rho = \infty, \tag{14}$$

or, if N is compact, that g has a first zero $\rho_1 > 0$ where $g'(\rho_1) = -1$, and that g is periodic with period $2\rho_1$. Note that in this second case assumption (14) is trivially satisfied. The case (13) corresponds to non-compact surfaces; condition (14) is a technical assumption needed to rule out that N contains a "sphere at infinity".

We regard (ρ, θ) as polar coordinates on N. Letting (r, ϕ) be the usual polar coordinates on $\mathbb{R}^2$, we then consider equivariant wave maps $u\colon \mathbb{R} \times \mathbb{R}^2 \to N$ given by

$$\rho = h(t, r), \theta = \phi.$$

The equation (5) or (7) for a wave map $u\colon \mathbb{R}^{2+1} \to N$ in this co-rotational case simplifies to the nonlinear scalar equation

$$\Box h + \frac{f(h)}{r^2} = 0, \tag{15}$$

where

$$\Box h = h_{tt} - \Delta h = h_{tt} - \frac{1}{r}\big(rh_r\big)_r = h_{tt} - h_{rr} - \frac{h_r}{r}$$

and with $f(h) = g(h)g'(h)$. If $N = S^2$, for example, we have $g(h) = sin(h)$ and $f(h) = \frac{1}{2}sin(2h)$

2.2. Results. In [22], Shatah and Tahvildar-Zadeh showed that the initial value problem for (7) with smooth equivariant data

$$(u, u_t)_{|t=0} = (u_0, u_1) \tag{16}$$

of finite energy admits a unique smooth solution for small time, which may be extended for all time if the target surface N is geodesically convex.

The latter condition is equivalent to the assumption $g'(\rho) \geq 0$ for all $\rho > 0$. This condition was later weakened by Grillakis [4] who showed that it suffices to assume

$$(g(\rho)\rho)' = g(\rho) + g'(\rho)\rho > 0 \text{ for } \rho > 0\,.$$

Note that this hypothesis, in particular, implies conditions (13) and (14).

In [24] we improve these results and show that conditions (13) and (14) already suffice for proving global well-posedness of the Cauchy problem for (15). In fact, we show that for general target surfaces N satisfying (14) the appearance of a singularity in (15) is related to the existence of a non-constant harmonic map

$\overline{u}\colon S^2 \to N$, thereby confirming a long-standing conjecture about wave maps in this special, co-rotational case. But if N also satisfies (13), any co-rotational harmonic map $\overline{u}\colon S^2 \to N$ is constant, and global well-posedness follows.

On the other hand, when $N = S^2$ on the basis of numerical work of Bizon et al. [1] and Isenberg-Liebling [8] it had been conjectured that for suitable initial data equivariant wave maps $u\colon \mathbb{R} \times \mathbb{R}^2 \to S^2$ indeed may develop singularities in finite time. In a penetrating analysis, Krieger-Schlag-Tataru [14] and Rodnianski-Sterbenz [18] recently were able to confirm this conjecture also theoretically and give a rigorous proof of blow-up.

2.3. **Blow-up criterion.** By the results of Shatah-Tahvildar-Zadeh [22] singularities of co-rotational maps may be detected by measuring their energy

$$E(u(t), R) = \frac{1}{2} \int_{B_R(0)} |Du(t)|^2 \, dx,$$

with $|Du|^2 = |u_t|^2 + |\nabla u|^2$. In terms of $h = h(t)$ we have

$$E(u(t), R) = \pi \int_0^R \big(|Dh|^2 + \frac{g^2(h)}{r^2} \big) \, r dr.$$

We also let

$$E(u(t)) = \lim_{R \to \infty} E(u(t), R).$$

By [22] there exists a number $\varepsilon_0 = \varepsilon_0(N) > 0$ such that the Cauchy problem for co-rotational wave maps for smooth data with energy $E(u(0)) < \varepsilon_0$ admits a global smooth solution. By finite speed of propagation, similarly we obtain well-posedness of the Cauchy problem for time $t \le R$, provided $E(u(0), R) < \varepsilon_0$.

Conversely, let $u\colon [0, t_0[\times \mathbb{R}^2 \to N$ be a smooth co-rotational wave map. Then $z_0 = (t_0, 0)$ is a (first) singularity and t_0 is the blow-up time of u if and only if there holds

$$\inf_{0 \le t < t_0} E(u(t), t_0 - t) \ge \varepsilon_0 > 0. \tag{17}$$

In fact, for any map u satisfying (17) the space-time gradient Du cannot be bounded near the origin $(0, 0)$. On the other hand, negating condition (17) we can find a time $t < t_0$ such that

$$E(u(t), R) < \varepsilon_0$$

for some $R > t_0 - t$ and the results quoted above will allow us to extend u smoothly as a solution to (5) on a neighborhood of $z_0 = (t_0, 0)$. Observe that, by symmetry, u can only blow up at the origin.

2.4. **Characterization of blow-up and well-posedness.** We can now state the main result from [24].

Theorem 2.1. *Let u be a smooth co-rotational solution to* (7) *blowing up at time t_0. Then there exist sequences $R_i \downarrow 0, t_i \uparrow t_0 (i \to \infty)$ such that*

$$u_i(t, x) = u(t_i + R_i t, R_i x) \to u_\infty(t, x)$$

strongly in $H^1_{loc}(]-1, 1[\times \mathbb{R}^2)$, where u_∞ is a non-constant, time-independent solution of (7) *giving rise to a non-constant, smooth co-rotational harmonic map $\overline{u}\colon S^2 \to N$.*

As a consequence, for target manifolds that do not admit non-constant co-rotational harmonic spheres in [24] we obtain global existence of smooth solutions to the Cauchy problem (7), (12) for smooth co-rotational data.

Theorem 2.2. *Suppose N is a surface of revolution with metric $ds^2 = d\rho^2 + g^2(\rho)d\theta^2$ satisfying* (13) *and* (14). *Then for any smooth co-rotational data the Cauchy problem* (7), (12) *admits a unique global smooth solution.*

As we shall see below, similar results also hold true in the case of radially symmetric wave maps $u = u(t,r)$ from $\mathbb{R}^{1+2}$ to an arbitrary closed target manifold; confer [25], [26].

We now recall the proofs of Theorems 2.1 and 2.2.

2.5. **Notation.** Let $u\colon [0,t_0[\times\mathbb{R}^2 \to N$ be a smooth co-rotational wave map blowing up at time t_0 and let $h = h(t,r)$ be the associated solution of (15).

For convenience we shift and reverse time and then scale our space-time coordinate $z = (t,x)$ so that in our new coordinates u is an equivariant solution to (7) on $]0,1] \times \mathbb{R}^2$ blowing up at the origin.

Letting

$$K^T = \{z = (t,x); 0 \le |x| \le t \le T\}$$

be the forward light cone with vertex at the origin, truncated at height T, with lateral boundary

$$M^T = \{(t,x) \in K^T; |x| = t\},$$

we also introduce the flux

$$\mathrm{Flux}(u,T) = \frac{1}{2}\int_{M^T} |D^{||}u|^2\, do = \pi \int_0^T \left(|h_t + h_r|^2 + \frac{g^2(h)}{r^2} \right)\Big|_{t=r} r\, dr.$$

Here, $|D^{||}u|^2$ denotes the energy of all derivatives in directions tangent to M^T.

2.6. **Basic estimates.** We recall the energy bounds and decay estimates for (7) from [22]; these can also be found in [20], Chapter 8.1. Upon integrating the conservation law (9) over a truncated cone $K^{T_0} \setminus K^T$ for $0 < T \le T_0 \le 1$ we then find the identity

$$\int_{\{T\}\times B_T(0)} e\, dx + \frac{1}{2}\int_{M^{T_0}\setminus M^T} |D^{||}u|^2\, do = \int_{\{T_0\}\times B_{T_0}(0)} e\, dx\,.$$

From this we deduce the *energy inequality*

$$E(u(t),R) \le E(u(t+\tau), R + |\tau|). \tag{18}$$

for any $t, \tau, R > 0$. (Of course, in the present case we only consider values such that $0 < t, t+\tau \le 1$.)

Moreover, we conclude that

$$\lim_{T\downarrow 0} \int_{\{T\}\times B_T(0)} e\, dx$$

exists and we have *decay of the flux*

$$\mathrm{Flux}(u,T) \to 0 \text{ as } T \downarrow 0. \tag{19}$$

Condition (14) together with the energy inequality implies the *uniform bounds*

$$\sup_{r<R} |h(t,r)| \le C(E(u(t),R)) \text{ for any } R>0 \tag{20}$$

for the function h associated with u, where $C(s) \to 0$ as $s \to 0$. Indeed, let

$$G(s)) = \int_0^s |g(\rho)|\, d\rho.$$

Since (14) implies that $G(s) \to \infty$ as $s \to \infty$ it then suffices to estimate

$$\begin{aligned} G(|h(t,R)|) &= \int_0^R (G(|h(t,r)|)_r \, dr \le \int_0^R |g(h(t,r))||h_r(t,r)|\, dr \\ &\le \frac{1}{2}\int_0^R \big(|h_r|^2 + \frac{g^2(h(t,r))}{r^2}\big) r dr \le CE(u(t),R)\,. \end{aligned}$$

Moreover we have *exterior energy decay:* For any $0<\lambda\le 1$ as $t\to 0$ there holds

$$E(u(t),t) - E(u(t),\lambda t) \to 0. \tag{21}$$

An immediate consequence of (21) is the *decay of time derivatives:* Suppose that N satisfies (14). Then

$$\frac{1}{T}\int_{K^T} |u_t|^2 \, dz \to 0 \text{ as } T \to 0. \tag{22}$$

These estimates seem particular to the equivariant setting. The (lengthy) proof of (21) and the derivation of (22) are given in the appendix.

Finally, as is also well-known, in view of the uniform energy bounds (18) above, we have uniform Hölder continuity away from $x=0$.

Lemma 2.3. *For any $r_0>0$, any (t,r) and (s,q) with $2r_0 \le q \le s < t \le 1, 2r_0 \le r \le t$ there holds*

$$|h(t,r) - h(s,q)|^2 \le C(|r-q| + |t-s|) \tag{23}$$

with a constant C depending only on the energy $E(u(1),1)$ and r_0.

Proof. Given $r_0>0$, for any t and $r_0 \le r' < r \le t \le 1$ by Hölder's inequality and (18) we have

$$|h(t,r)-h(t,r')|^2 \le \left(\int_{r'}^r |h_r|\, dr''\right)^2 \le \frac{r-r'}{r'}\cdot \int_{r'}^r |h_r|^2\, r''\, dr'' \le C\frac{r-r'}{r_0},$$

while for any $s<t$ and $r_0 \le r' \le s$ we find

$$|h(s,r')-h(t,r')|^2 \le \left(\int_s^t |h_t(t',r')|\, dt'\right)^2 \le \frac{t-s}{r_0}\int_s^t |h_t(t',r')|^2\, r'\, dt'.$$

Combining these inequalities, for any (t,r) and (s,q) with $2r_0 \le q \le s < t \le 1, 2r_0 \le r \le t$ and any r' with $r_0 \le r' \le r_1 := \inf\{q,r\}$ we find

$$|h(t,r)-h(s,q)|^2 \le C\frac{r-r'+q-r'}{r_0} + 2\frac{t-s}{r_0}\int_s^t |h_t(t',r')|^2\, r'\, dt'.$$

Taking the average with respect to $r' \in [r_1 - \min\{r_0, |r-q|+|t-s|\}, r_1]$, we obtain the claim. $\square$

2.7. **Proofs of Theorems 2.1 and 2.2.** Fix a number $\varepsilon_1 = \varepsilon_1(N) > 0$ to be determined below. For $0 < t \leq 1$ then choose $R = R(t) > 0$ so that

$$\varepsilon_1 \leq E(u(t), 6R(t)) \leq 2\varepsilon_1. \tag{24}$$

Applying the energy inequality (18), for any $|\tau| \leq 5R$ we have

$$E(u(t+\tau), R) \leq E(u(t), 6R) \leq 2\varepsilon_1 \tag{25}$$

and similarly

$$\varepsilon_1 \leq E(u(t+\tau), 6R + |\tau|) \leq E(u(t+\tau), 11R). \tag{26}$$

We will choose ε_1 so that $2\varepsilon_1 < \varepsilon_0$. Then, in particular, from (17) and (24) we deduce the inequality

$$6R(t) < t \tag{27}$$

for all t. In fact, we obtain a much stronger result.

Lemma 2.4. *$R(t)/t \to 0$ as $t \to 0$.*

Proof. Suppose by contradiction that for some sequence $t_i \downarrow 0$ $(i \to \infty)$ with associated radii $R_i = R(t_i)$ there holds $6R_i \geq \lambda t_i$ for some constant $\lambda > 0$. Then from (17) and (24) we deduce that

$$0 < \varepsilon_0 - 2\varepsilon_1 \leq E(u(t_i), t_i) - E(u(t_i), 6R_i) \leq E(u(t_i), t_i) - E(u(t_i), \lambda t_i),$$

contradicting (21) for large $i \in \mathbb{N}$. □

The following lemma is the main new technical ingredient in our work [24].

Consider the intervals $\Lambda_{R(t)}(t) =]t - R(t), t + R(t)[$, $0 < t \leq 1$. By Vitali's theorem we can find a countable subfamily of disjoint intervals $\Lambda_i = \Lambda_{R(t_i)}(t_i)$, $i \in \mathbb{N}$, such that $]0,1] \subset \cup_{i=1}^\infty \Lambda_i^*$, where $\Lambda_i^* = \Lambda_{5R(t_i)}(t_i)$. Observe that (27) implies

$$\inf \Lambda_i^* = t_i - 5R(t_i) > R(t_i) =: R_i \tag{28}$$

for each i. For any $\tau > 0$ the interval $[\tau, 1]$ is covered by finitely many intervals Λ_i^* which, however, fail to cover $]0,1]$ completely in view of (28). Therefore, we may assume that $t_i \to 0$ as $i \to \infty$.

Lemma 2.5. *With the above notations there holds*

$$\liminf_{i\to\infty} \frac{1}{R_i} \int_{\Lambda_i} \int_{B_t(0)} |u_t|^2 \, dx \, dt = 0.$$

Proof. Negating the assertion, we can find a number $\delta > 0$ and an index $i_0 \in \mathbb{N}$ such that

$$\int_{\Lambda_i} \int_{B_t(0)} |u_t|^2 \, dx \, dt \geq \delta R_i \quad \text{for } i \geq i_0. \tag{29}$$

Given $0 < T < \inf \cup_{i<i_0} \Lambda_i^*$, let $I_0 = \{i \,;\, \inf \Lambda_i^* < T\} \subset \{i_0, i_0+1, \dots\}$. Observe that

$$]0, T[\subset \cup_{i\in I_0} \Lambda_i^* \,.$$

By (28) we have

$$R_i < \inf \Lambda_i^* = t_i - 5R_i < T$$

and therefore

$$t_i + R_i < T + 6R_i < 7T$$

for all $i \in I_0$. It follows that

$$\cup_{i \in I_0} \Lambda_i \subset]0, 7T]. \tag{30}$$

By choice of I_0, our assumption (29), and in view of (30) we now obtain that

$$\begin{aligned} \delta T \le \delta \sum_{i \in I_0} \operatorname{diam} \Lambda_i^* &= 10\,\delta \sum_{i \in I_0} R_i \le 10 \sum_{i \in I_0} \int_{\Lambda_i} \int_{B_t(0)} |u_t|^2\,dx\,dt \\ &= 10 \int_{\cup_{i \in I_0} \Lambda_i} \int_{B_t(0)} |u_t|^2\,dx\,dt \le 10 \int_{K^{7T}} |u_t|^2\,dz, \end{aligned} \tag{31}$$

where we also used the fact that the intervals Λ_i are disjoint. But for small $T > 0$ this contradicts (22), thus proving the lemma. ☐

Proof of Theorem 2.1. i) Letting

$$u_i(t, x) = u(t_i + R_i t, R_i x), i \in \mathbb{N},$$

from Lemma 2.5 for a suitable subsequence we obtain

$$\int_{-1}^{1} \int_{B_{r_i}(0)} |\partial_t u_i|^2\,dx\,dt \to 0 \text{ as } i \to \infty, \tag{32}$$

where $r_i = t_i/R_i - 1 \to \infty$ as $i \to \infty$ on account of Lemma 2.4. Relabelling, we may assume that (32) holds true for the original sequence (u_i).

Moreover, the energy inequality (18) implies the uniform bound

$$E(u_i(t), r_i) \le E(u(1), 1) =: E_0 \tag{33}$$

for all $i \in \mathbb{N}$ and $|t| \le 1$.

Hence we may extract a further subsequence such that $u_i \rightharpoonup u_\infty$ weakly in H^1_{loc} and locally uniformly away from $x = 0$ on $[-1, 1] \times \mathbb{R}^2$ as $i \to \infty$, and similarly for the associated functions h_i. Their limit h_∞ then is associated with u_∞ and is a time-independent solution of (15) away from $x = 0$. It follows that $u_\infty(t, x) = \overline{u}(x)$ is a time-independent solution of (7) on $]-1, 1[\times(\mathbb{R}^2 \setminus \{0\})$; that is, $\overline{u}\colon \mathbb{R}^2 \setminus \{0\} \to N$ is a smooth, co-rotational harmonic map with finite energy

$$E(\overline{u}) = \int_{\mathbb{R}^2} |\nabla u|^2\,dx \le \liminf_{i \to \infty} \sup_{|t| \le 1} E(u_i(t), r_i) \le E_0.$$

By [19] then $\overline{u}$ extends to a smooth harmonic map $\overline{u}\colon \mathbb{R}^2 \to N$. Since $\mathbb{R}^2$ is conformal to $S^2 \setminus \{p_0\}$ by stereographic projection from any point $p_0 \in S^2$ and since the composition of a harmonic map with a conformal transformation again yields a harmonic map with the same energy, we may thus regard $\overline{u}$ as a harmonic map from $S^2 \setminus \{p_0\}$ to N. Finally, recalling that $E(\overline{u}) < \infty$ and again using [19], we see that the map $\overline{u}$ extends to a smooth equivariant harmonic map $\overline{u}\colon S^2 \to N$.

ii) To show that $\overline{u}$ is non-constant we now establish strong convergence

$$u_i \to u_\infty \text{ in } H^1_{loc}(]-1, 1[\times\mathbb{R}^2)$$

as $i \to \infty$. Recalling (25), we have

$$E(u_i(t), 1) \le 2\varepsilon_1, \quad E(u_\infty(t), 1) \le 2\varepsilon_1$$

uniformly in i and $|t| \le 1$. Hence, from (20) for sufficiently small $\varepsilon_1 > 0$ the images of $B_1(0)$ under $u_i(t)$ or u_∞ are all contained in a fixed coordinate system around the center of symmetry $O \in N$. In addition, we can achieve that

$$\sup_{|t|,|x|\le 1} |B(u_i)||u_i - u_\infty| \le \frac{1}{4} \tag{34}$$

uniformly in $i \in \mathbb{N}$, provided $\varepsilon_1 > 0$ is chosen sufficiently small.

For any $\varphi \in C_0^\infty(]-1,1[\times\mathbb{R}^2)$ with $0 \le \varphi \le 1$ then, upon multiplying the equation (7) for u_i by $(u_i - u_\infty)\varphi$ and integrating by parts we obtain

$$\int_{\mathbb{R}^{1+2}} |D(u_i - u_\infty)|^2 \varphi\, dz \le \int_{\mathbb{R}^{1+2}} |B(u_i)||Du_i|^2 |u_i - u_\infty|\varphi\, dz + I, \tag{35}$$

with error

$$\begin{aligned} |I| &\le C \int_{\mathbb{R}^{1+2}} (|\partial_t u_i|^2 \varphi + |Du_i||u_i - u_\infty||D\varphi|)\, dz \\ &\quad + \sum_\alpha \Big| \int_{\mathbb{R}^{1+2}} \partial_\alpha u_\infty \partial_\alpha (u_i - u_\infty)\varphi\, dz \Big| \to 0 \text{ as } i \to \infty \end{aligned}$$

in view of (32) and since $u_i \to u_\infty$ strongly in L^2_{loc} by Rellich's theorem.

Now we estimate

$$|Du_i|^2 \le 2|D(u_i - u_\infty)|^2 + 2|Du_\infty|^2$$

and observe that

$$\int_{\mathbb{R}^{1+2}} |Du_\infty|^2 |u_i - u_\infty|\varphi\, dz \to 0$$

as $i \to \infty$ by bounded almost everywhere convergence $u_i \to u_\infty$ and Lebesgue's theorem on dominated convergence. Also recalling (34), we thus may absorb the first term on the right of (35) on the left to obtain that

$$\int_{\mathbb{R}^{1+2}} |D(u_i - u_\infty)|^2 \varphi\, dz \to 0$$

as $i \to \infty$. Since φ as above is arbitrary, this yields the desired convergence $u_i \to u_\infty$ in $H^1_{loc}(]-1,1[\times\mathbb{R}^2)$.

But, recalling (26), we also have the uniform lower bound

$$\varepsilon_1 \le E(u_i(t), 11)$$

for all $i \in \mathbb{N}$ and $|t| \le 1$ and we conclude that $u_\infty \not\equiv \mathit{const}$, as claimed. Therefore, also $\overline{u}\colon S^2 \to N$ is non-constant, and the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. In view of Theorem 2.1 it suffices to show that any co-rotational harmonic map $\overline{u}\colon S^2 \to N$ with finite energy is constant. Let $\overline{u}$ be such a map, viewed as a map $\overline{u}\colon \mathbb{R}^2 \to N$. Also consider the associated distance function $\rho = \overline{h}(r)$, a time-independent solution of (15). The image $\overline{u}(S^2)$ being compact there exists $r_0 > 0$ such that

$$|\overline{h}(r_0)| = \max_{r>0} |\overline{h}(r)|.$$

Hence $\overline{h}_r(r_0) = 0$ and therefore $\overline{u}_r(x) = 0$ for any $x \in \partial B_{r_0}(0)$.

Since any harmonic map $\overline{u}\colon \mathbb{R}^2 \to N$ with finite energy is conformal, the vanishing of $\overline{u}_r$ implies that also $\overline{u}_\phi$ vanishes along $\partial B_{r_0}(0)$, and we conclude that

$\overline{u} \equiv const$ on $\partial B_{r_0}(0)$. Equivariance of $\overline{u}$ then implies that $g(\overline{h}(r_0)) = 0$ and hence $\overline{h}(r_0) = 0$ on account of (13). But then $\overline{h} \equiv 0$ by choice of r_0, and $\overline{u} \equiv const \equiv O$, as desired. □

3. Radially symmetric wave maps

Next, we show that the Cauchy problem for radially symmetric wave maps $u(t,x) = u(t,|x|)$ from the $(1+2)$-dimensional Minkowski space to an arbitrary smooth, compact Riemannian manifold without boundary is globally well-posed for arbitrary smooth, radially symmetric data.

3.1. **The result.** Again let N be a smooth, compact Riemmanian k-manifold without boundary, isometrically embedded in $\mathbb{R}^n$. Given smooth, radially symmetric data $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \colon \mathbb{R}^2 \to TN$, by a result of Christodoulou-Tahvildar-Zadeh [2] there is a unique smooth solution $u = (u^1, \ldots, u^n) = u(t,|x|)$ for small time to the Cauchy problem for the equation

$$\Box u = u_{tt} - \Delta u = B(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N, \tag{36}$$

with initial data

$$(u, u_t)_{|t=0} = (u_0, u_1). \tag{37}$$

Here B again denotes the second fundamental form of N.

As shown by Christodoulou-Tahvildar-Zadeh [2], the solution may be extended globally, if the energy of u is small or if the range of u is contained in a convex part of the target N. Either condition, however, turns out to be unnecessary. In fact, by using the blow-up analysis from [24] that we presented in the second chapter, in [25], [26] we showed that the local solution may be extended globally for any target manifold.

Theorem 3.1. *Let $N \subset \mathbb{R}^n$ be a smooth, compact Riemannian manifold without boundary. Then for any radially symmetric data $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \in C^\infty(\mathbb{R}^2; TN)$ there exists a unique, smooth solution $u = u(t,|x|)$ to the Cauchy problem* (36)*,* (37)*, defined for all time.*

The regularity requirements on the data may be relaxed; we consider smooth data mainly for ease of exposition.

Summarizing the ideas of the proof, as in the co-rotational symmetric setting of [24] that we described in the second chapter, again we argue indirectly. Thus, we suppose that the local solution u to (36), (37) becomes singular in finite time. As before we then obtain a sequence of rescaled solutions u_l on the region $]-1,1[\times\mathbb{R}^2$ with energy bounds and such that $\partial_t u_l \to 0$ in $L^2_{loc}(]-1,1[\times\mathbb{R}^2)$. Finally, rephrasing the wave map equation intrinsically and imposing the exponential gauge, we establish energy decay. But this contradicts the blow-up criterion of Christodoulou and Tahvildar-Zadeh [2] and completes the proof. (The use of the exponential gauge was suggested to me by Jalal Shatah.)

3.2. **Basic estimates.** Let $u = u(t, |x|)\colon [0, t_0[\times\mathbb{R}^2 \to N \subset \mathbb{R}^n$ be a smooth radially symmetric wave map blowing up at time t_0. Necessarily, blow-up occurs at $x = 0$. As before, upon shifting and reversing time and then scaling our space-time coordinates suitably, we may assume that u is a smooth radial solution to (36) on $]0, 1] \times \mathbb{R}^2$ blowing up at the origin. Again let

$$K^T = \{z = (t, x); 0 \le |x| \le t \le T\}$$

be the truncated forward light cone from the origin with lateral boundary

$$M^T = \{(t, x) \in K^T; |x| = t\}.$$

Denoting as

$$e = \frac{1}{2}|Du|^2 = \frac{1}{2}(|u_t|^2 + |u_r|^2), \quad f = \frac{1}{2}|D^{||}u|^2 = \frac{1}{2}|u_t + u_r|^2$$

the energy and flux density of u, and letting

$$E(u, R) = \int_{B_R(0)} e\, dx, \quad \mathrm{Flux}(u, T) = \int_{M^T} f\, do$$

be the local energy and the flux through M^T, then from [2], [22] we have the following results just as in the co-rotational setting. The identity (9) again leads to the *energy inequality:* For any $t, \tau, R > 0$ there holds

$$E(u(t), R) \le E(u(t + \tau), R + |\tau|). \tag{38}$$

(Again, we only consider values such that $0 < t, t + \tau \le 1$. Together with [2] this yields the *blow-up criterion:* There exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that

$$E(u(t), t) \ge \varepsilon_0 \text{ for all } 0 < t \le 1\,. \tag{39}$$

Moreover, we have *flux decay:*

$$Flux(u, T) \to 0 \text{ as } T \to 0. \tag{40}$$

As shown in the Appendix, similar to (21) and (22) we also have *exterior energy decay* and *decay of time derivatives:* For any $0 < \lambda \le 1$ as $t \to 0$ there holds

$$E(u(t), t) - E(u(t), \lambda t) \to 0, \tag{41}$$

and

$$\frac{1}{T}\int_{K^T} |u_t|^2\, dz \to 0 \text{ as } T \to 0. \tag{42}$$

Moreover, as shown in Lemma 2.3, the function u is locally uniformly Hölder continuous on $]0, 1] \times B_1(0)$ away from $x = 0$.

Fix a number $0 < \varepsilon_1 = \varepsilon_1(N) < \varepsilon_0/2$ as determined below. For $0 < t \le 1$ we again choose $R = R(t)$ so that

$$\varepsilon_1 \le E(u(t), 6R) \le 2\varepsilon_1. \tag{43}$$

Then from (38) for any $|\tau| \le 5R$ we have

$$E(u(t + \tau), R) \le E(u(t), 6R) \le 2\varepsilon_1 < \varepsilon_0 \tag{44}$$

and similarly

$$\varepsilon_1 \le E(u(t + \tau), 6R + |\tau|) \le E(u(t + \tau), 11R). \tag{45}$$

In particular, combining (39) and (43) we deduce the inequality

$$6R(t) \le t \tag{46}$$

for all t. In fact, from (39), (41), and (43) as in Lemma 2.4 we even obtain that

$$R(t)/t \to 0 \text{ as } t \downarrow 0. \tag{47}$$

As in Lemma 2.5 we consider the intervals $\Lambda_{R(t)}(t) =]t - R(t), t + R(t)[, 0 < t \le 1$. An application of Vitali's covering theorem and (42) then yields a sequence $t_l \to 0$ with corresponding radii $R_l = R(t_l)$ such that

$$\frac{1}{R_l} \int_{\Lambda_l} \left(\int_{B_t(0)} |u_t|^2 \, dx \right) dt \to 0$$

as $l \to \infty$, where $\Lambda_l = \Lambda_{R_l}(t_l), l \in \mathbb{N}$. Rescale, letting

$$u_l(t,x) = u(t_l + R_l t, R_l x), l \in \mathbb{N}.$$

Observe that u_l solves (36) on $[-1,1] \times \mathbb{R}^2$ with

$$\int_{-1}^{1} \left(\int_{D_l(t)} |\partial_t u_l|^2 \, dx \right) dt \to 0 \text{ as } l \to \infty, \tag{48}$$

where

$$D_l(t) = \{x; R_l|x| \le t_l + R_l t\}$$

exhausts $\mathbb{R}^2$ as $l \to \infty$ uniformly in $|t| \le 1$ on account of (47).

Moreover, from (38), (39), (44), and (45) we have the uniform energy estimates

$$\frac{1}{2} E(u_l(t), 1) \le \varepsilon_1 \le E(u_l(t), 11) \tag{49}$$

and

$$\varepsilon_0 \le \frac{1}{2} \int_{D_l(t)} |Du_l|^2 \, dx = E(u(t_l + R_l t), t_l + R_l t) \le E(u(1), 1) =: E_0, \tag{50}$$

uniformly for $|t| \le 1$ and sufficiently large $l \in \mathbb{N}$. Hence, we may assume that $u_l \rightharpoonup u_\infty$ weakly in $H^1_{loc}(]-1,1[\times\mathbb{R}^2)$ and locally uniformly away from $x = 0$, where $u_\infty(t,x) = u_\infty(|x|)$ is a time-independent radial map $u_\infty : \mathbb{R}^2 \to N$ with finite energy $E(u_\infty) \le E_0$.

Lemma 3.2. *We have $u_\infty \equiv const$, and $Du_l \to 0$ in $L^2_{loc}(]-1,1[\times(\mathbb{R}^2 \setminus \{0\})$ as $l \to \infty$.*

Proof. We claim that u_∞ is smooth and harmonic. Indeed, fix any function $\varphi \in C_0^\infty(]-1,1[\times\mathbb{R}^2)$ vanishing near $x = 0$. Upon multiplying (36) by $(u_l - u_\infty)\varphi$ and integrating by parts, we then have

$$\int_{\mathbb{R}^{1+2}} |D(u_l - u_\infty)|^2 \varphi \, dz = \int_{\mathbb{R}^{1+2}} \langle B(u_l)(\partial_\alpha u_l, \partial^\alpha u_l), u_l - u_\infty \rangle \varphi \, dz + I,$$

where

$$\begin{aligned} |I| \le 2 \int_{\mathbb{R}^{1+2}} |\partial_t u_l|^2 \varphi \, dz + \int_{\mathbb{R}^{1+2}} |Du_l||u_l - u_\infty||D\varphi| \, dz \\ + \left| \int_{\mathbb{R}^{1+2}} Du_\infty \cdot D(u_l - u_\infty) \varphi \, dz \right| \to 0 \end{aligned}$$

as $l \to \infty$. Observing that $(u_l - u_\infty)\varphi \to 0$ uniformly, moreover, we have

$$\int_{\mathbb{R}^{1+2}} \langle B(u_l)(\partial_\alpha u_l, \partial^\alpha u_l), u_l - u_\infty \rangle \varphi \, dz \to 0$$

as $l \to \infty$, and $u_l \to u_\infty$ strongly in $H^1_{loc}(]-1,1[\times \mathbb{R}^2 \setminus \{0\})$. Thus, we may pass to the distribution limit in equation (36) for u_l and find that u_∞ is weakly harmonic on $\mathbb{R}^2 \setminus \{0\}$. Since u_∞ has finite energy, by results of [19] then u_∞ is smooth and extends to a smooth, radially symmetric harmonic map $u_\infty \colon \mathbb{R}^2 \to N$.

Next recall that a harmonic map $u_\infty \colon \mathbb{R}^2 \to N$ with finite energy is conformal; in particular, there holds $|\partial_r u_\infty| = \frac{1}{r}|\partial_\phi u_\infty| \equiv 0$, and u_∞ must be constant. □

Finally we note the following estimate similar to [2], Lemma 4.

Lemma 3.3. *For any $\psi = \psi(t) \in C_0^\infty(]-1,1[)$ there holds*

$$\int_{-1}^{1}\int_{B_1(0)} |\partial_t u_l|^2 \psi |\log|x||\, dx\, dt = \int_{-1}^{1}\int_{B_1(0)} e(u_l)\psi\, dx\, dt + o(1),$$

where $o(1) \to 0$ as $l \to \infty$.

Proof. In radial coordinates $r = |x|$, equation (36) for $u = u_l$ may be written in the form

$$u_{tt} - \frac{1}{r}\,\partial_r(r u_r) \perp T_u N. \tag{51}$$

Multiplying by $u_r \psi r^2 \log r$, we obtain

$$\begin{aligned} 0 = &\frac{d}{dt}\left(\langle u_t, u_r\rangle \psi r^2 \log r\right) - \frac{d}{dr}\left(\frac{|u_t|^2 + |u_r|^2}{2}\psi r^2 \log r\right) \\ &+ |u_t|^2 \psi r \log r - \langle u_t, u_r\rangle \psi_t r^2 \log r + e(u) r \psi. \end{aligned}$$

Upon integrating this identity over the domain $0 < r < 1, |t| < 1$ and observing that the boundary terms vanish, we find

$$\int_{-1}^{1}\int_{0}^{1} |u_t|^2 \psi r \log r\, dr\, dt + \int_{-1}^{1}\int_{0}^{1} e(u) r \psi\, dr\, dt = \int_{-1}^{1}\int_{0}^{1} \langle u_t, u_r\rangle \psi_t r^2 \log r\, dr\, dt.$$

In view of (48), (50), and Hölder's inequality the last term may be estimated

$$\begin{aligned} \left|\int_{-1}^{1}\int_{0}^{1} \langle u_t, u_r\rangle \psi_t r^2 \log r\, dr\, dt\right|^2 &= \left|\frac{1}{2\pi}\int_{-1}^{1}\int_{B_1(0)} \langle u_t, u_r\rangle \psi_t r \log r\, dx\, dt\right|^2 \\ &\le C \int_{-1}^{1}\int_{B_1(0)} |u_t|^2\, dx\, dt \cdot \int_{-1}^{1}\int_{B_1(0)} |u_r|^2\, dx\, dt \to 0 \text{ as } l \to \infty, \end{aligned}$$

proving the claim. □

3.3. **Intrinsic setting.**

Recalling (5), in terms of the pull-back covariant derivative D in u^*TN we may write equation (51) as

$$D_t u_t - \frac{1}{r} D_r(r u_r) = 0. \tag{52}$$

As was observed by Christodoulou-Tahvildar-Zadeh [2] and Hélein [7], with no loss of generality, we may assume that TN is parallelizable; that is, there exist smooth vector fields $\overline{e}_1, \ldots, \overline{e}_k$ such that at each $p \in N$ the collection $\overline{e}_1(p), \ldots, \overline{e}_k(p)$ is an orthonormal basis for T_pN; see [2], [7]. Given a (smooth) solution map $u = u(t,r)$ of (52) we then obtain a frame $e_i = R_i^j(\overline{e}_j \circ u), 1 \le i \le k$, for the pull-back bundle, where $R = R(t,r) = (R_i^j)$ is a smooth map from $\mathbb{R}^{1+2}$ into $SO(k)$.

Denoting

$$De_i = A_i^j e_j \tag{53}$$

with a matrix-valued connection 1-form $A = A_0\,dt + A_1\,dr$, we compute the curvature F of D via the commutation relation

$$\begin{aligned} D_\alpha D_\beta e_a - D_\beta D_\alpha e_a &= D_\alpha(A^b_{a,\beta} e_b) - D_\beta(A^b_{a,\alpha} e_b) \\ &= (\partial_\alpha A^c_{a,\beta} - \partial_\beta A^c_{a,\alpha} + A^c_{b,\alpha} A^b_{a,\beta} - A^c_{b,\beta} A^b_{a,\alpha}) e_c = F^c_{a,\alpha\beta} e_c, \end{aligned}$$

or, more concisely,

$$dA + \frac{1}{2}[A, A] = F.$$

Moreover, we now impose the "exponential gauge" condition $A_1 = 0$. This yields the relation

$$*dA = -\partial_r A_0 = F_{01}.$$

If we normalize $A_0(t, 1) = 0$ for all t, from this relation we obtain

$$A_0 = \int_r^1 F_{01}\,ds.$$

Observing that

$$|F_{01}| \le C|du|^2, \tag{54}$$

from (49) we then deduce the estimate

$$|A_0| \le a_0 := \int_r^1 |F_{01}|\,ds \le C \int_r^1 |du|^2\,ds \le C\varepsilon_1 r^{-1}.$$

Note that in the exponential gauge for any fixed time t the frame field $e = e(t, r)$ is obtained by parallel transport along the curve $\gamma(r) = u(t, r)$ from the frame $e(t, 1)$ at $r = 1$.

Expressing du as

$$du = u_t\,dt + u_r\,dr = q^i e_i,$$

where $q = q_0\,dt + q_1\,dr$ is a vector-valued 1-form with coefficients $q = (q^i)_{1\le i\le k}$, from (53) we have

$$D_\alpha \partial_\beta u = D_\alpha(q^a_\beta e_a) = (\partial_\alpha q^c_\beta + A^c_{a,\alpha} q^a_\beta) e_c.$$

With the shorthand notation

$$D_\alpha q_\beta = (\partial_\alpha + A_\alpha) q_\beta$$

we then may write equation (52) in the form

$$D_t q_0 - \frac{1}{r} D_r(r q_1) = \partial_t q_0 + A_0 q_0 - \frac{1}{r}\partial_r(r q_1) = 0. \tag{55}$$

Moreover, the commutation relation $D_\alpha \partial_\beta u - D_\beta \partial_\alpha u = 0$ translates into the equation $D_r q_0 = D_t q_1$; that is,

$$\partial_r q_0 = \partial_t q_1 + A_0 q_1. \tag{56}$$

Finally there holds

$$|q_0| = |u_t|, |q_1| = |u_r|. \tag{57}$$

3.4. **Proof of Theorem 3.1.** By using Lemma 3.3 we show that (48) for sufficiently small $\varepsilon_1 > 0$ leads to a contradiction with (49).

Fix a cut-off function $0 \le \varphi = \varphi(r) \le 1$ in $C_0^\infty([0,1[)$ such that $\varphi(r) = 1$ for $r \le 1/2$. Also fix $0 \le \psi = \psi(t) \le 1$ in $C_0^\infty(]-1,1[)$ such that $\psi(t) = 1$ for $|t| \le 1/2$. For $u = u_l$ with associated 1-forms q, let

$$Q = Q_l = \int_r^1 q_1 \varphi \, ds.$$

Note that by Hölder's inequality and (49) we can estimate

$$|Q|^2 \le \Big(\int_r^1 |q|\, ds\Big)^2 \le \int_r^1 s|q|^2\, ds \cdot \int_r^1 \frac{ds}{s} \le 4\varepsilon_1 \log(\frac{1}{r}). \tag{58}$$

We will also use the bound

$$\Big(\int_0^r s|q|\varphi\, ds\Big)^2 \le \int_0^r s|q|^2\, ds \cdot \int_0^r s\, ds \le 2\varepsilon_1 r^2 \tag{59}$$

resulting from (49). Similarly, we have

$$\Big(\int_0^r s|q_0||\log s|^{1/2}\varphi\, ds\Big)^2 \le \frac{r^2}{2}\int_0^1 s|q_0|^2|\log s|\, ds,$$

which in view of (49), (54), and Lemma 3.3 allows to estimate

$$\begin{aligned}
&\int_{-1}^1\int_0^1 \Big(\int_0^r s|q_0||\log s|^{1/2}\varphi\, ds\Big)|F_{01}|\psi\, dr\, dt \\
&\quad\le \int_{-1}^1 \Big(\int_0^1 s|q_0|^2|\log s|\, ds\Big)^{1/2}\Big(\int_0^1 r|F_{01}|\, dr\Big)\psi\, dt \\
&\quad\le C\varepsilon_1\Big(\int_{-1}^1\int_0^1 s|q_0|^2|\log s|\psi\, ds\ dt\Big)^{1/2} \le C\varepsilon_1^{3/2}.
\end{aligned} \tag{60}$$

Also note that Lemma 3.2 implies

$$\int_{-1}^1\int_0^1 r|\log r|^{1/2}|q|\psi\, dr\, dt \le C\Big(\int_{-1}^1\int_{B_1(0)} r|\log r||Du|^2\psi\, dx\, dt\Big)^{1/2} \to 0 \tag{61}$$

as $l \to \infty$.

Using the function $Q\varphi\psi r$ as a multiplier, from (56) then we obtain

$$\begin{aligned}
\int_{-1}^1\int_0^1 \partial_t q_0 Q\varphi\psi r\, dr\, dt &= -\int_{-1}^1\int_0^1 q_0\Big(\int_r^1 \partial_t q_1\varphi\, ds\Big)\varphi\psi r\, dr\, dt + I \\
&= \int_{-1}^1\int_0^1 |q_0|^2\varphi^2\psi r\, dr\, dt + \int_{-1}^1\int_0^1 q_0\Big(\int_r^1 A_0 q_1\varphi\, ds\Big)\varphi\psi r\, dr\, dt + II,
\end{aligned}$$

where, in view of (58), and (61),

$$|I| = |\int_{-1}^1\int_0^1 q_0 Q\varphi\psi_t r\, dr\, dt| \le C\int_{-1}^1\int_0^1 r|q_0||\log r|^{1/2}|\psi_t|\, dr\, dt \to 0$$

as $l \to \infty$. Similarly,

$$|II| \le |I| + |\int_{-1}^{1}\int_0^1 q_0\Big(\int_r^1 q_0\partial_r\varphi\,ds\Big)\varphi\psi r\,dr\,dt|$$
$$\le C\int_{-1}^{1}\int_0^1 r|q_0||\log r|^{1/2}\psi\,dr\,dt \to 0.$$

On the other hand, noting that

$$\frac{1}{r}\partial_r(rq_1)rQ = \partial_r(rq_1Q) + r|q_1|^2\varphi,$$

we obtain

$$\int_{-1}^{1}\int_0^1 \frac{1}{r}\partial_r(rq_1)rQ\varphi\psi\,dr\,dt = \int_{-1}^{1}\int_0^1 r|q_1|^2\varphi^2\psi\,dr\,dt + III,$$

where, by (58) and (61),

$$|III| \le \int_{-1}^{1}\int_0^1 r|q_1||Q||\varphi_r|\psi\,dr\,dt \le C\int_{-1}^{1}\int_0^1 r|\log r|^{1/2}|q_1|\psi\,dr\,dt \to 0$$

as $l \to \infty$. Thus, from (55) we deduce the identity

$$\int_{-1}^{1}\int_0^1 r(|q_1|^2 - |q_0|^2)\varphi^2\psi\,dr\,dt + o(1)$$
$$= \int_{-1}^{1}\int_0^1 q_0\Big(\int_r^1 A_0q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt + \int_{-1}^{1}\int_0^1 A_0q_0\Big(\int_r^1 q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt,$$

where $o(1) \to 0$ as $l \to \infty$. Using (59), (54) and repeated integration by parts, we find

$$\int_{-1}^{1}\int_0^1 q_0\Big(\int_r^1 A_0q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt = \int_{-1}^{1}\int_0^1 \Big(\int_0^r q_0\varphi s\,ds\Big)A_0q_1\varphi\psi\,dr\,dt$$
$$\le C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 ra_0|q_1|\varphi\psi\,dr\,dt = C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 r|q_1|\varphi\Big(\int_r^1 |F_{01}|\,ds\Big)\psi\,dr\,dt$$
$$= C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 \Big(\int_0^r s|q_1|\varphi\,ds\Big)|F_{01}|\psi\,dr\,dt \le C\varepsilon_1\int_{-1}^{1}\int_0^1 r|F_{01}|\psi\,dr\,dt$$
$$\le C\varepsilon_1\int_{-1}^{1}\int_0^1 r|du|^2\psi\,dr\,dt \le C\varepsilon_1^2.$$

Similarly, we estimate, now using (58) and (60),

$$\int_{-1}^{1}\int_0^1 A_0q_0\Big(\int_r^1 q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt \le C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 a_0|q_0||\log r|^{1/2}\varphi\psi r\,dr\,dt$$
$$= C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 r|q_0||\log r|^{1/2}\varphi\Big(\int_r^1 |F_{01}|\,ds\Big)\psi\,dr\,dt$$
$$= C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 \Big(\int_0^r s|q_0||\log s|^{1/2}\varphi\,ds\Big)|F_{01}|\psi\,dr\,dt \le C\varepsilon_1^2.$$

But then from (49), Lemma 3.2, and (48), with error $o(1) \to 0$ as $l \to \infty$ we obtain

$$\begin{aligned}\varepsilon_1 &\le \frac{1}{2}\int_{-1}^{1}\int_{B_{11}(0)} |Du|^2\psi\,dx\,dt \le \pi\int_{-1}^{1}\int_0^1 r|q|^2\varphi^2\psi\,dr\,dt + o(1)\\ &\le \pi\int_{-1}^{1}\int_0^1 r(|q_1|^2-|q_0|^2)\varphi^2\psi\,dr\,dt + o(1) \le C\varepsilon_1^2 + o(1),\end{aligned}$$

which is impossible for sufficiently small $\varepsilon_1 > 0$ and large l. The proof of Theorem 3.1 is complete.

4. Open problem

The above results naturally give rise to the question whether it is possible to characterize blow-up of smooth wave maps $u\colon [0,T[\times\mathbb{R}^2 \to N$ at a first blow-up point (T,x_0) in a fashion similar to Theorem 2.1, leading to global well-posedness of the Cauchy problem for arbitrarily large smooth data in cases where the target manifold N does not support a non-constant harmonic sphere. Recent results of Tao [29], [30] also point in this direction.

Appendix A: Exterior energy decay

In this Appendix we recall the proof of the following lemma which is fundamental for the treatment of the equivariant and rotationally symmetric case.

Lemma 5.1. *Let u be a radially symmetric solution of* (36) *or a co-rotational wave map on $K = K^1$ which is smooth away from the origin. Then for any $0 < \lambda \le 1$ as $t \to 0$ there holds*

$$E(u(t),t) - E(u(t),\lambda t) \to 0.$$

Proof. We follow the presentation in [20]. Therefore in the following we change time t to $-t$.

With the notation

$$e = \frac{1}{2}(|u_r|^2 + |u_t|^2),\ m = u_r\cdot u_t,\ l = \frac{1}{2}(|u_r|^2 - |u_t|^2) \tag{62}$$

for a radially symmetric solution u of (36) we compute

$$\frac{\partial}{\partial t}(rm) - \frac{\partial}{\partial r}(re) = ru_r\cdot(u_{tt} - \frac{1}{r}(ru_r)_r) + l = l, \tag{63}$$

thereby observing the geometric interpretation (51) of (36) and the fact that $u_r \in T_uN$. Moreover, recalling the equation (9) we have

$$\frac{\partial}{\partial t}(re) - \frac{\partial}{\partial r}(rm) = 0. \tag{64}$$

Similarly, for a co-rotational wave map u with associated function h solving (15) we let

$$\begin{aligned} e &= \frac{1}{2}(|u_r|^2+|u_t|^2) = \frac{1}{2}(|h_r|^2+|h_t|^2+\frac{g^2(h)}{r^2}),\ m = h_r\cdot h_t,\\ L &= \frac{1}{2}(|h_r|^2+\frac{g^2(h)}{r^2}-|h_t|^2) - \frac{2}{r}f(h)h_r\end{aligned} \tag{65}$$

and we compute

$$\partial_t(re) - \partial_r(rm) = 0,\ \partial_t(rm) - \partial_r(re) = L. \tag{66}$$

Changing coordinates to

$$\eta = t + r\,, \qquad \xi = t - r\,, \tag{67}$$

and introducing

$$\mathcal{A}^2 = r(e+m),\ \mathcal{B}^2 = r(e-m)\,,$$

identities (63), (64) turn into

$$\begin{aligned}\partial_\xi \mathcal{A}^2 &= l\,,\\ \partial_\eta \mathcal{B}^2 &= -l\,,\end{aligned}$$

where

$$r^2 l^2 \le \mathcal{A}^2\mathcal{B}^2\,.$$

Likewise, (66) can be written as

$$\begin{aligned}\partial_\xi \mathcal{A}^2 &= L\,,\\ \partial_\eta \mathcal{B}^2 &= -L\,,\end{aligned}$$

where now, with $F = g^2/2$, and using the fact that $|h| \le C(E_0)$ by (20) to bound $f^2(h) \le CF(h)$,

$$\begin{aligned}L^2 &\le \frac{3}{4}\left(h_t^2 - h_r^2\right)^2 + \frac{12}{r^2}h_r^2 f^2(h) + \frac{3}{r^4}F^2(h)\\ &\le C\left[\frac{1}{4}(h_t^2 - h_r^2)^2 + \frac{1}{r^2}(h_t^2 + h_r^2)F(h) + \frac{1}{r^4}F^2(h)\right]\\ &= \frac{C}{r^2}\mathcal{A}^2\mathcal{B}^2\,.\end{aligned}$$

Thus in both cases we get the inequalities

$$|\partial_\xi \mathcal{A}| \le \frac{C}{r}\mathcal{B}\,, \quad |\partial_\eta \mathcal{B}| \le \frac{C}{r}\mathcal{A}\,. \tag{68}$$

Upon integrating (68) on a rectangle $\Gamma = [\eta, 0] \times [\xi_0, \xi]$, as shown in Figure 1, we obtain

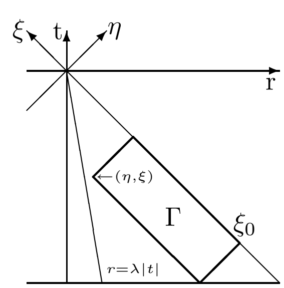

FIGURE 1. Domain of integration Γ.

$$\mathcal{A}(\eta,\xi) \le \mathcal{A}(\eta,\xi_0) + C\int_{\xi_0}^{\xi} \frac{\mathcal{B}(0,\xi')}{\eta - \xi'}\,d\xi' + C^2\int_{\xi_0}^{\xi}\int_{\eta}^{0} \frac{\mathcal{A}(\eta',\xi')}{(\eta-\xi')(\eta'-\xi')}\,d\eta'\,d\xi'\,.$$

First we estimate the second term on the right.

$$\begin{aligned}\int_{\xi_0}^{\xi} \frac{\mathcal{B}(0,\xi')}{\eta-\xi'}\, d\xi' &\le \left(\int_{\xi_0}^{\xi} \mathcal{B}^2(0,\xi')\, d\xi'\right)^{1/2} \left(\int_{\xi_0}^{\xi} \frac{d\xi'}{(\eta-\xi')^2}\right)^{1/2} \\ &= (\mathrm{Flux}(\xi_0)-\mathrm{Flux}(\xi))^{1/2} \sqrt{\frac{1}{\eta-\xi}-\frac{1}{\eta-\xi_0}} \\ &\le C\sqrt{\frac{\mathrm{Flux}(\xi_0)}{|\eta-\xi|}}\,.\end{aligned}$$

Letting

$$a(\eta,\xi) = \sup_{\eta\le\eta'\le 0} \sqrt{\eta'-\xi}\,\mathcal{A}(\eta',\xi)\,, \tag{69}$$

the third term may be bounded

$$\begin{aligned}\int_{\xi_0}^{\xi}\int_{\eta}^{0} \frac{\mathcal{A}(\eta',\xi')}{(\eta-\xi')(\eta'-\xi')}\, d\eta'\, d\xi' &\le \int_{\xi_0}^{\xi}\int_{\eta}^{0} \frac{a(\eta,\xi')}{(\eta-\xi')(\eta'-\xi')^{3/2}}\, d\eta'\, d\xi' \\ \le \int_{\xi_0}^{\xi} \frac{a(\eta,\xi')}{\eta-\xi'}\left(\frac{1}{\sqrt{\eta-\xi'}}-\frac{1}{\sqrt{-\xi'}}\right) d\xi' &\le \int_{\xi_0}^{\xi} a(\eta,\xi')\frac{\eta}{\xi'(\eta-\xi')^{3/2}}\, d\xi'\,.\end{aligned} \tag{70}$$

Also observing that

$$\sup_{\eta\le\eta'\le 0} \sqrt{\eta'-\xi}\,\mathcal{A}(\eta',\xi_0) \le \sup_{\eta\le\eta'\le 0} \frac{\sqrt{\eta'-\xi}}{\sqrt{\eta'-\xi_0}} a(\eta,\xi_0) = \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) \tag{71}$$

with constants C_1, C_2 we then obtain

$$a(\eta,\xi) \le \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)} + C_2 \int_{\xi_0}^{\xi} a(\eta,\xi')\frac{\eta}{\xi'(\eta-\xi')}\, d\xi'\,.$$

Setting

$$\rho(\xi') = \frac{\eta}{\xi'(\eta-\xi')}, \tag{72}$$

and letting

$$F(\xi) = \int_{\xi_0}^{\xi} a(\eta,\xi')\rho(\xi')\, d\xi',\ G(\xi) = \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)}\,, \tag{73}$$

for any fixed η we then find the differential inequality

$$F' \le G\rho + C_2 F\rho \text{ in } [\xi_0,\lambda'\eta]\,, \tag{74}$$

where $\lambda' = (1+\lambda)/(1-\lambda) > 1$. Applying Gronwall's lemma we obtain

$$F(\xi) \le \int_{\xi_0}^{\xi} G(\xi')\rho(\xi')e^{C_2\int_{\xi'}^{\xi}\rho(\xi'')d\xi''}\, d\xi'\,. \tag{75}$$

But for $\xi_0 \le \xi' \le \xi = \lambda'\eta$ we have

$$\int_{\xi'}^{\xi} \rho(\xi'')d\xi'' = \int_{\xi'}^{\xi} \frac{\eta}{\xi''(\eta-\xi'')}\, d\xi'' = \log\frac{\xi(\eta-\xi')}{\xi'(\eta-\xi)} = \log\frac{\xi(\xi-\lambda'\xi')}{\xi'(\xi-\lambda'\xi)} \le \log\frac{\lambda'}{\lambda'-1}\,.$$

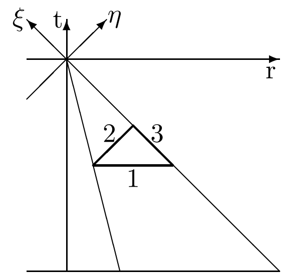

FIGURE 2. Triangular region Δ.

Hence we can estimate

$$
\begin{aligned}
a(\eta,\xi) &\leq G + C_2 F \\
&\leq \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)} \\
&\quad + C_3 \int_{\xi_0}^{\xi} \left(\frac{\sqrt{-\xi'}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)} \right) \frac{\eta}{\xi'(\eta-\xi')}\, d\xi' ,
\end{aligned} \tag{76}
$$

where $C_3 = e^{C_2 \log \frac{\lambda'}{\lambda'-1}}$. We also know that

$$
a(\eta,\xi_0) \leq \sup_{\eta\leq\eta'\leq 0} \sqrt{\eta'-\xi_0} \sup_{\eta\leq\eta'\leq 0} \mathcal{A}(\eta',\xi_0) \leq C(\xi_0)\sqrt{-\xi_0}\,,
$$

because u is assumed to be regular away from the origin, implying that $\mathcal{A}$ is bounded by a constant depending on ξ_0. Now, given $\epsilon > 0$, we can fix $\xi_0 < 0$ small enough such that $C\sqrt{\mathrm{Flux}(\xi_0)} < \epsilon$. Then,

$$
\begin{aligned}
a(\xi/\lambda',\xi) &\leq C(\xi_0)\sqrt{-\xi} + \epsilon + C(\xi_0) \int_{\xi_0}^{\xi} \frac{\xi/\lambda'}{\sqrt{-\xi'}(\xi/\lambda'-\xi')}\, d\xi' + C\epsilon \\
&\leq C(\xi_0)\sqrt{-\xi} + C\epsilon \leq C\epsilon
\end{aligned}
$$

for $\xi < 0$ small enough. Therefore,

$$
\mathcal{A}(\eta,\xi) \leq \frac{a(\xi/\lambda',\xi)}{\sqrt{\eta-\xi}} \leq \frac{C\epsilon}{\sqrt{\eta-\xi}}
$$

for (η,ξ) small enough inside $K^{\lambda}_{\mathrm{ext}}$. Hence,

$$
\int_{\eta}^{0} \mathcal{A}^2(\eta',\xi)d\eta' \leq C\epsilon^2 \int_{\xi/\lambda'}^{0} \frac{d\eta'}{\eta'-\xi} = C\epsilon^2 \log\frac{1}{(\lambda'-1)} = C\epsilon^2\,.
$$

Finally, if we integrate the energy identity (64)) on the triangle Δ (as shown in Figure 2 with vertices at (η,ξ), $(0,\xi)$, and $(0,\eta+\xi)$, with $\eta = \xi/\lambda'$ as before), we obtain

$$
0 = -\int_{\lambda|t|}^{|t|} e(r,t) r\, dr - \int_{\eta}^{0} r(e+m)d\eta' + \int_{\xi+\eta}^{\xi} r(e-m)d\xi' = \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
$$

As $t \to 0$ we proved that $\mathrm{II} \to 0$; moreover, $\mathrm{III} \to 0$ because it is the flux, and therefore $\mathrm{I} \to 0$. $\square$

As consequence we obtain the decay of time derivatives.

Corollary 5.2. *Let u be a radially symmetric solution of* (36) *or a co-rotational wave map on $K = K^1$ which is smooth away from the origin. In the latter case also suppose that N satisfies* (14). *Then*

$$\frac{1}{T}\int_{K^T} |u_t|^2\, dz \to 0 \text{ as } T \to 0.$$

Proof. Again we change time t to $-t$. Multiply the identity (63), (66), respectively, by r and integrate on the truncated cone

$$K_T^{-\epsilon} = \{(t,x); t \le -\epsilon, |x| \le -t \le -T\},$$

and let $\epsilon \to 0$ to obtain

$$\left| \iint_{K_T^0} u_t^2 r\, dr\, dt - \int_0^{|T|} (u_t u_r)\big|_{t=T} r^2\, dr \right| \le C|T|\operatorname{Flux}(T)\,.$$

Therefore, for any $\lambda \in]0,1[$ we have

$$\begin{aligned}
\frac{1}{|T|}\int_T^0\int_0^{-t} u_t^2 r\, dr\, dt &\le \frac{1}{|T|}\int_0^{|T|} |(u_t u_r)\big|_{t=T}| r^2\, dr + C\operatorname{Flux}(T) \\
&\le \frac{C}{|T|}\int_0^{|T|} e(T,r) r^2\, dr + C\operatorname{Flux}(T) \\
&\le \frac{C}{|T|}\left(\int_0^{\lambda|T|} e(T,r)r^2\, dr + \int_{\lambda|T|}^{|T|} e(T,r)r^2 dr\right) + C\operatorname{Flux}(T) \\
&\le C(\lambda E_0 + E_{\text{ext}}^{\lambda}(T) + \operatorname{Flux}(T))\,.
\end{aligned}$$

Given $\epsilon > 0$ we then may choose $\lambda > 0$ such that the first term on the right is less then $\epsilon/3$. By Lemma 5.1 and by decay of the flux the second and third terms also will be less than $\epsilon/3$ for T sufficiently close to 0. $\square$

Appendix B: The Cauchy problem for wave maps

In this Appendix we recall the approach presented in [21] for showing global existence and uniqueness for the Cauchy problem for wave maps from the $(1+m)$-dimensional Minkowski space, $m \ge 4$, to any complete Riemannian manifold with bounded curvature, provided the initial data are small in the critical norm.

We study the Cauchy problem for wave maps $u \colon \mathbb{R}^{m+1} \to N$ solving the equation

$$D^\alpha \partial_\alpha u = 0 \tag{77}$$

with initial data

$$(u, u_t)_{|t=0} = (u_0, u_1) \in \dot{H}^{\frac{m}{2}} \times \dot{H}^{\frac{m}{2}-1}(\mathbb{R}^m; TN). \tag{78}$$

Also recall the equivalent extrinsic form of equation (77)

$$\Box u = -\partial^\alpha \partial_\alpha u = u_{tt} - \Delta u = B(u)(\partial_\alpha u, \partial^\alpha u). \tag{79}$$

With $L^{(2m,2)}(\mathbb{R}^m) \hookrightarrow L^{2m}(\mathbb{R}^m)$ denoting the Lorentz space, the main result from [21] may be stated, as follows.

Theorem 6.1. *Suppose N is complete, without boundary and has bounded curvature in the sense that the curvature operator R and the second fundamental form B and all their derivatives are bounded, and let $m \geq 4$. Then there is a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1) \in H^{\frac{m}{2}} \times H^{\frac{m}{2}-1}(\mathbb{R}^m; TN)$ satisfying*

$$||u_0||_{\dot{H}^{\frac{m}{2}}} + ||u_1||_{\dot{H}^{\frac{m}{2}-1}} < \varepsilon_0$$

there exists a unique global solution $u \in C^0(\mathbb{R}; H^{\frac{m}{2}}) \cap C^1(\mathbb{R}; H^{\frac{m}{2}-1})$ of (77), (78) satisfying

$$\sup_t ||du(t)||_{\dot{H}^{\frac{m}{2}-1}} + \int_{\mathbb{R}} ||du(t)||^2_{L^{(2m,2)}(\mathbb{R}^m)}\, dt \leq C\varepsilon_0 \tag{80}$$

and preserving any higher regularity of the data.

Terence Tao, and independently also Sergiu Klainerman and Igor Rodnianski pointed out that estimates similar to the crucial $L^1_t L^\infty_x$-estimate in Lemma 6.2 below can also be obtained from bilinear estimates for the wave equation obtained by Klainerman-Tataru [11]. Tristan Rivière has brought to our attention further applications of Lorentz spaces in gauge theory related to our use of Lorentz spaces here. We refer to the Introduction for a further discussion of the result.

6.1. **Uniqueness and higher regularity.** The condition (80) easily yields uniqueness when we consider the extrinsic form (79) of the wave map system. Indeed, let u and v be solutions to (79) of class $H^{\frac{m}{2}}$ with $u, v \in C^0(\mathbb{R}; H^{\frac{m}{2}}) \cap C^1(\mathbb{R}; H^{\frac{m}{2}-1})$, and suppose that

$$u_{|t=0} = v_{|t=0},\ u_{t|t=0} = v_{t|t=0}.$$

Moreover, we assume (80), that is, in particular,

$$||du||^2_{L^2_t L^{2m}_x} = \int_{\mathbb{R}} ||du(t)||^2_{L^{2m}(\mathbb{R}^m)}\, dt < \infty,$$

and similarly for v. Then $w = u - v$ satisfies

$$w_{tt} - \Delta w = [B(u) - B(v)](\partial_\alpha u, \partial^\alpha u) + B(v)(\partial_\alpha u + \partial_\alpha v, \partial^\alpha w).$$

Multiplying by w_t, we obtain

$$\frac{1}{2}\frac{d}{dt}||dw(t)||^2_{L^2} = I(t) + II(t),$$

where by Sobolev's embedding $\dot{H}^1(\mathbb{R}^m) \hookrightarrow L^{\frac{2m}{m-2}}(\mathbb{R}^m)$ we can estimate

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^m} \langle [B(u) - B(v)](\partial_\alpha u, \partial^\alpha u), w_t\rangle\, dx \leq C \int_{\mathbb{R}^m} |du|^2 |w||dw|\, dx \\ &\leq C||du||^2_{L^{2m}} ||w||_{L^{\frac{2m}{m-2}}} ||dw||_{L^2} \leq C||du||^2_{L^{2m}} ||dw||^2_{L^2}. \end{aligned}$$

In order to bound the term $II(t)$, we note that orthogonality $\langle B(u)(\cdot,\cdot), u_t\rangle = 0 = \langle B(v)(\cdot,\cdot), v_t\rangle$ implies

$$\begin{aligned} |\langle B(v)(\partial_\alpha u, \partial^\alpha w), w_t\rangle| &= |\langle B(v)(\partial_\alpha u, \partial^\alpha w), u_t\rangle| \\ &= |\langle [B(v) - B(u)](\partial_\alpha u, \partial^\alpha w), u_t\rangle| \leq C|du|^2|w||dw|, \end{aligned}$$

and similarly for the term involving $\partial_\alpha v$.

Thus also this term can be bounded

$$II(t) \leq C(||du||^2_{L^{2m}} + ||dv||^2_{L^{2m}})||dw||^2_{L^2},$$

yielding the inequality

$$\frac{d}{dt}||dw||^2_{L^2} \le C(||du||^2_{L^{2m}} + ||dv||^2_{L^{2m}})||dw||^2_{L^2}.$$

Hence we obtain the uniform estimate

$$||dw||^2_{L^\infty_t L^2_x} \le ||dw(0)||^2_{L^2} \cdot \exp(C(||du||^2_{L^2_t L^{2m}_x} + ||dv||^2_{L^2_t L^{2m}_x})).$$

Since $dw(0) = 0$, uniqueness follows.

Higher regularity estimates for (smooth) solutions u of (79) satisfying (80) for sufficiently small $\varepsilon > 0$ can be obtained in similar fashion by differentiating the intrinsic form of the wave map equation covariantly in spatial directions and using standard energy estimates; see [21] for details.

6.2. **Moving frames and Gauge condition.** Our approach requires the construction of a suitable frame for the pull-back bundle u^*TN, as pioneered by Christodoulou-Tahvildar-Zadeh [2] and Hélein [7]. With no loss of generality, we may assume that TN is parallelizable, that is, there exist smooth vector fields $\overline{e}_1, \dots, \overline{e}_k$ such that at each $p \in N$ the collection $\overline{e}_1(p), \dots, \overline{e}_k(p)$ is an orthonormal basis for T_pN; see [2], [7]. Given a (smooth) map $u\colon \mathbb{R}^{m+1} \to N$ then the vector fields $\overline{e}_a \circ u, 1 \le a \le k$, yield a smooth orthonormal frame for the pull-back bundle u^*TN. Moreover, we may freely rotate this frame at any point $z = (t, x) \in \mathbb{R}^{m+1}$ with a matrix $(R^b_a) = (R^b_a(z)) \in SO(k)$, thus obtaining the frame

$$e_a = R^b_a \overline{e}_b \circ u, 1 \le a \le k.$$

Expressing du as

$$du = q^a e_a \tag{81}$$

with an $\mathbb{R}^k$-valued 1-form $q = q_\alpha \, dx^\alpha$, then we have

$$|du|^2 = |q|^2 = \sum_{\alpha=0}^m |q_\alpha|^2 .$$

In particular, for $1 \le p \le \infty$ the L^p-norm of du is well-defined, independently of the choice of "gauge" (R^b_a), and coincides with the L^p-norm of du in the extrinsic representation of u as a map $u\colon \mathbb{R}^{m+1} \to N \subset \mathbb{R}^m$. Later we will see that if the gauge R is suitably chosen, and if $\varepsilon_0 > 0$ is sufficiently small, also the norms of the derivatives of du and the derivatives of q agree up to a multiplicative constant.

Letting $D = (D_\alpha)_{0 \le \alpha \le m}$ be the pull-back covariant derivative, we have

$$De_a = A^b_a e_b, 1 \le a \le k, \tag{82}$$

for some matrix-valued 1-form $A = A_\alpha dx^\alpha$. Fix a pair of space-time indices $0 \le \alpha, \beta \le m$. The curvature of D enters in the commutation relation

$$\begin{aligned} D_\alpha D_\beta e_a - D_\beta D_\alpha e_a &= D_\alpha(A^b_{a,\beta} e_b) - D_\beta(A^b_{a,\alpha} e_b) \\ &= (\partial_\alpha A^c_{a,\beta} - \partial_\beta A^c_{a,\alpha} + A^c_{b,\alpha} A^b_{a,\beta} - A^c_{b,\beta} A^b_{a,\alpha}) e_c = F^c_{a,\alpha\beta} e_c, \end{aligned}$$

or

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = F_{\alpha\beta} = R(\partial_\alpha u, \partial_\beta u) \tag{83}$$

for short. (The comma separates the form subscript from the vector subscript and does not indicate a differential.)

Following Hélein [7] we choose the columb gauge

$$\sum_{i=1}^{m} \partial_i A_i = 0. \tag{84}$$

This results in the equation

$$\Delta A_\beta + \partial_i [A_i, A_\beta] = \partial_i F_{i\beta} = \partial_i (R(\partial_i u, \partial_\beta u)), 0 \le \beta \le m, \tag{85}$$

where we tacitly sum over $1 \le i \le m$. Given $u\colon \mathbb{R}^{m+1} \to N$ with du having sufficiently small L^m-norm, this equation admits a unique solution A which for any fixed time we may represent as

$$A_\beta = G_i * ([A_i, A_\beta] - F_{i\beta}), \tag{86}$$

where

$$G(x) = \frac{c}{|x|^{m-2}}$$

is the fundamental solution to the Laplace operator on $\mathbb{R}^m$ and $G_i = -\partial_i G$.

Indeed, from (85) and elliptic regularity theory we have the a-priori estimate

$$\begin{aligned} ||A||_{L^m} &\le C||A||_{\dot{W}^{1,\frac{m}{2}}} \le C||[A, A]||_{L^{\frac{m}{2}}} + C||F||_{L^{\frac{m}{2}}} \\ &\le C||A||^2_{L^m} + C||R||_{L^\infty} ||du||^2_{L^m}; \end{aligned}$$

confer [5], Section 4.3. For sufficiently small $||A||_{L^m}$ we may absorb the first term on the right on the left hand side of this equation to obtain at any fixed time the estimate with constants C independent of t

$$||A||_{L^m} \le C||A||_{\dot{W}^{1,\frac{m}{2}}} \le C||du||^2_{L^m} \le C||du||^2_{\dot{H}^{\frac{m}{2}-1}} \le C\varepsilon_0. \tag{87}$$

For later use we derive further estimates for the connection 1-form A and the curvature F, assuming that $\varepsilon_0 > 0$ is sufficiently small. For the sake of exposition, we indicate these estimates only in the case when $m = 4$ and refer to [21] for the general case. For $1 \le s \le \infty$ again denote as $L^{(p,s)}(\mathbb{R}^m)$ the Lorentz space.

Lemma 6.2. *Let $m = 4$, and fix $r = 8/5$.*

(i) For any time t there holds

$$\|\nabla^2 A\|_{L^r} + \|\nabla \partial_0 A\|_{L^r} \le C\|\nabla F\|_{L^r} \le C\|du\|_{L^8} \|du\|_{\dot{H}^1} .$$

(ii) For any time t we have

$$\|A\|_{L^\infty} \le C\|du\|^2_{L^{(8,2)}} .$$

Proof. (i) To estimate $\nabla^2 A$, observe that equation (85) implies

$$\|\nabla^2 A\|_{L^r} \le C\|\nabla[A, A]\|_{L^r} + C\|\nabla F\|_{L^r} . \tag{88}$$

By Hölder's inequality and Sobolev's embedding we can estimate

$$\|\nabla[A, A]\|_{L^r} \le 2\|\nabla A\|_{L^{r_1}} \|A\|_{L^m} \le C\|\nabla^2 A\|_{L^r} \|A\|_{L^m} ,$$

where

$$\frac{1}{r_1} = \frac{1}{r} - \frac{1}{m} = \frac{3}{8}.$$

From (87) and (88) then, for sufficiently small $\varepsilon_0 > 0$ we obtain

$$\|\nabla^2 A\|_{L^r} \le C\|\nabla F\|_{L^r}.$$

The term ∇F only involves terms of the form $R(\nabla\partial_\alpha u, \partial_\beta u)$ and $\nabla R(\partial_\alpha u, \partial_\beta u)$ and therefore may be estimated

$$|\nabla F| \le C(|\nabla du||du| + |du|^3).$$

Letting $q = 8 = 2m$, so that $1/r = 5/8 = 1/q + 1/2$, upon estimating

$$\|\nabla F\|_{L^r} \le C(\|\nabla du\|_{L^2}\|du\|_{L^q} + \|du\|^2_{L^4}\|du\|_{L^q})\,,$$

from Sobolev's embedding $\|du\|_{L^4} \le C\|du\|_{\dot H^1} \le C$ we conclude that

$$\|\nabla^2 A\|_{L^r} \le C\|\nabla F\|_{L^r} \le C\|du\|_{L^8}\|du\|_{\dot H^1}.$$

To estimate $\nabla\partial_0 A$ we note that the equations

$$\partial_0 A_i = \partial_i A_0 + [A_i, A_0] + F_{0i}$$

and

$$\Delta\partial_0 A_0 + \partial_i\partial_0[A_i, A_0] = \partial_i\partial_0 F_{i0}\,,$$

from (85) make exchanging of time derivative by spatial derivative possible and thus imply the desired estimate.

(ii) By the Sobolev embedding into Lorentz spaces and i), we have

$$\|A\|_{L^{(8,2)}} \le C\|A\|_{L^{(8,\frac{8}{5})}} \le \|A\|_{\dot W^{2,\frac{8}{5}}} \le C\|du\|_{L^8}\,.$$

Therefore, and since for any $m \ge 4$ we have $G_i \in L^{(\frac{m}{m-1},\infty)}$, the dual of $L^{(m,1)}$, using the representation of A given by (86) we obtain

$$\|A\|_{L^\infty} \le C(\|[A,A]\|_{L^{(4,1)}} + \|F\|_{L^{(4,1)}}) \le C(\|A\|^2_{L^{(8,2)}} + \|du\|^2_{L^{(8,2)}}) \le C\|du\|^2_{L^{(8,2)}}\,,$$

as claimed. □

6.3. **Equivalence of Norms.** Estimate (87) implies the equivalence of the extrinsic H^ℓ-norm of du and the H^ℓ-norm of q for any ℓ, provided $\varepsilon_0 > 0$ is sufficiently small. To see this consider a vector field W in u^*TN whose coordinates in the frame $\{e_a\}$ are given by

$$W = Q^a e_a = Qe$$

with

$$\|W\|_{L^2} = \|Q\|_{L^2}\,.$$

The extrinsic partial derivative of W can be computed from the covariant derivative and the second fundamental form B as

$$D_k W = \partial_k W + B(u)(\partial_k u, W) = (\partial_k Q + AQ)e\,;$$

that is,

$$\partial_k W = (\partial_k Q + AQ)e - B(u)(\partial_k u, Qe).$$

Therefore from (12), Sobolev embedding, and boundedness of the second fundamental form B we obtain

$$\begin{aligned}\big|\, ||\partial W||_{L^2} - ||\partial Q||_{L^2}\big| &\le C\|AQ\|_{L^2} + \|duQ\|_{L^2}\\ &\le C(\|A\|_{L^m} + \|du\|_{L^m})\|\partial Q\|_{L^2} \le C\varepsilon_0\|\partial Q\|_{L^2}\,.\end{aligned}$$

By linearity of the map $Q \mapsto W$ and interpolation we conclude the equivalence of the H^s-norms of Q and W for all $0 \le s \le 1$. The same argument establishes the equivalence of the covariant and extrinsic H^s-norms of W for $0 \le s \le 1$. By applying this argument iteratively to $W = \nabla^\ell du$ for $\ell = 0, 1, \ldots,$ we then obtain the equivalence of the H^s-norm of du and H^s-norm of q for any $s \geq 0$, provided $\varepsilon_0 > 0$ is sufficiently small.

6.4. **A priori bounds.** In order to obtain the a-priori bounds from which we may derive existence, we represent a local smooth solution u of (77), (78) in terms of the 1-form q given by (81), where the frame (e_a) is in Coulomb gauge.

From (82) then we have the equations

$$0 = D_\alpha \partial_\beta u - D_\beta \partial_\alpha u = (D_\alpha q_\beta - D_\beta q_\alpha) e,$$

where we denote

$$D_\alpha q_\beta = (\partial_\alpha + A_\alpha) q_\beta; \tag{89}$$

in components, this is

$$D_\alpha (q^a_\beta e_a) = (\partial_\alpha q^c_\beta + A^c_{a,\alpha} q^a_\beta) e_c.$$

Again the comma separates the form subscript from the vector subscript and does not indicate a differential.

That is, we have

$$D_\alpha q_\beta - D_\beta q_\alpha = 0. \tag{90}$$

Moreover, the wave map equation (77) yields the equation

$$D^\alpha q_\alpha = 0. \tag{91}$$

Differentiating (91) with respect to x^β and using (83), (90), we derive the covariant wave equation

$$0 = D_\beta D^\alpha q_\alpha = D^\alpha D_\beta q_\alpha + F^\alpha_\beta q_\alpha = D^\alpha D_\alpha q_\beta + F^\alpha_\beta q_\alpha.$$

Expanding this identity using (89), we obtain

$$(\partial_t^2 - \Delta) q_\beta = 2A^\alpha \partial_\alpha q_\beta + (\partial^\alpha A_\alpha) q_\beta + A^\alpha A_\alpha q_\beta + F^\alpha_\beta q_\alpha =: h_\beta. \tag{92}$$

We can estimate q in terms of the initial data and h by using the Strichartz estimate for the linear wave equation

$$\Box v = h, v_{|_{t=0}} = f, v_{t|_{t=0}} = g. \tag{93}$$

Again denoting as $\dot{H}^\gamma = (\sqrt{-\Delta})^{-\gamma} L^2(\mathbb{R}^m)$ the homogeneous Sobolev space, and as $L^{(p,r)}(\mathbb{R}^m)$ the Lorentz space, from Keel-Tao [9], Corollary 1.3, if $h = 0$ for any $T > 0$ we have

$$\begin{aligned} ||v||_{L^2([0,T];L^{\frac{2(m-1)}{m-3}}(\mathbb{R}^m))} + ||v||_{C^0([0,T];\dot{H}^\gamma(\mathbb{R}^m))} + ||v_t||_{C^0([0,T];\dot{H}^{\gamma-1}(\mathbb{R}^m))} \\ \leq C(||f||_{\dot{H}^\gamma(\mathbb{R}^m)} + ||g||_{\dot{H}^{\gamma-1}(\mathbb{R}^m)}). \end{aligned}$$

where $\gamma = \frac{m+1}{2(m-1)}$. If $m = 4$, we have $\gamma = \frac{5}{6}$ and the preceding becomes

$$\begin{aligned} ||v||_{L^2([0,T];L^6(\mathbb{R}^4))} + ||v||_{C^0([0,T];\dot{H}^{5/6}(\mathbb{R}^4))} + ||v_t||_{C^0([0,T];\dot{H}^{-1/6}(\mathbb{R}^4))} \\ \leq C(||f||_{\dot{H}^{5/6}(\mathbb{R}^4)} + ||g||_{\dot{H}^{-1/6}(\mathbb{R}^4)}). \end{aligned} \tag{94}$$

By real interpolation between this estimate and the analogous estimate for derivatives of v, and using the embedding (in the notation of [9])

$$(L^2_t L^6_x, L^2_t \dot{W}^{1,6}_x)_{\frac{1}{6},2} \hookrightarrow L^2_t L^{(8,2)}_x,$$

we obtain

$$||v||_{L^2_t L^{(8,2)}_x} + ||dv||_{C^0([0,T];L^2)} \le C(||f||_{\dot H^1} + ||g||_{L^2}). \tag{95}$$

By Duhamel's principle, for general h it then follows that

$$||v||_{L^2_t L^{(8,2)}_x} + ||dv||_{C^0_t L^2_x} \le C(||f||_{\dot H^1} + ||g||_{L^2} + ||h||_{L^1_t L^2_x}). \tag{96}$$

(The crucial gain of the Lorentz exponent by real interpolation was already observed by Keel and Tao [9] but was omitted in the final statement of their theorem.)

We will apply estimate (96) to equation (92) on any time interval $[0,T]$ such that $||du||_{\dot H^1}$ remains sufficiently small, uniformly for $0 < t < T$. Also using the equivalence of the H^s-norms of du and q for $s \le 1$ on any such time interval, we obtain

$$\begin{aligned} ||du||_{C^0_t \dot H^1_x} + ||du||_{L^2_t L^{(8,2)}_x} &\le C(||dq||_{C^0_t L^2_x} + ||q||_{L^2_t L^{(8,2)}_x}) \\ &\le C(||dq(0)||_{L^2} + ||h||_{L^1_t L^2_x}) \le C(||du(0)||_{\dot H^1} + ||h||_{L^1_t L^2_x}) \\ &\le C(||u_0||_{\dot H^2} + ||u_1||_{\dot H^1} + ||h||_{L^1_t L^2_x})\,. \end{aligned}$$

To estimate the various terms in h we observe that by Lemma 6.2 at any time t with $r_1 = 8/3$ we have

$$\begin{aligned} ||h||_{L^2} &\le 2\|A\partial q\|_{L^2} + \|\partial A q\|_{L^2} + \|A^2 q\|_{L^2} + \|Fq\|_{L^2} \\ &\le 2\|A\|_{L^\infty}\|q\|_{\dot H^1} + \big(\|\nabla A\|_{L^{r_1}} + \|A^2\|_{L^{r_1}} + \|F\|_{L^{r_1}}\big)\|q\|_{L^8}\,. \end{aligned}$$

But Lemma 6.2 with $r = 8/5$ implies

$$\begin{aligned} \|\nabla A\|_{L^{r_1}} + \|A^2\|_{L^{r_1}} + \|F\|_{L^{r_1}} &\le C(\|\nabla^2 A\|_{L^r} + \|\nabla(A^2)\|_{L^r} + \|\nabla F\|_{L^r}) \\ &\le C\|du\|_{L^8}\|du\|_{\dot H^1}. \end{aligned}$$

Here we also used Sobolev's embedding and (87) to bound

$$\|\nabla(A^2)\|_{L^r} \le C\|\nabla A\|_{L^{r_1}}\|A\|_{L^4} \le C\|\nabla^2 A\|_{L^r}.$$

From Lemma 6.2 we then obtain

$$||h||_{L^2} \le C\|q\|_{L^8}\|du\|_{L^8}\|du\|_{\dot H^1} + 2\|A\|_{L^\infty}\|q\|_{\dot H^1} \le C\|du\|^2_{L^{(8,2)}}\|du\|_{\dot H^1}\,.$$

Using these estimates, we can bound h by

$$||h||_{L^1_t L^2_x} \le C||du||^2_{L^2_t L^{(8,2)}_x}\|du\|_{L^\infty_t \dot H^1_x}$$

and we conclude that

$$||du||_{L^\infty_t \dot H^1} + ||du||_{L^2_t L^{(8,2)}_x} \le C(||u_0||_{\dot H^2} + ||u_1||_{\dot H^1} + ||du||^2_{L^2_t L^{(8,2)}_x}\|du\|_{L^\infty_t \dot H^1_x})\,.$$

A global priori bound on $||du||_{L^\infty_t \dot H^1_x} + ||du||_{L^2_t L^8_x}$ thus follows, provided $||u_0||_{\dot H^2} + ||u_1||_{\dot H^1}$ is sufficiently small.

6.5. **Existence.** Recall that $C^\infty \times C^\infty(\mathbb{R}^m;TN)$ is dense in $H^{\frac{m}{2}} \times H^{\frac{m}{2}-1}(\mathbb{R}^m;TN)$. We can thus find smooth data $(u_0^{(k)}, u_1^{(k)}) \to (u_0, u_1)$ in $H^{\frac{m}{2}} \times H^{\frac{m}{2}-1}(\mathbb{R}^m;TN)$. The local solutions $u^{(k)}$ to the Cauchy problem for (77) with data $(u_0^{(k)}, u_1^{(k)})$ by our a-priori bounds and regularity results for sufficiently small energy

$$||u_0||^2_{\dot{H}^{\frac{m}{2}}} + ||u_1||^2_{\dot{H}^{\frac{m}{2}-1}} < \varepsilon_0$$

then may be extended as smooth solutions to (77), (78) for all time and will satisfy the uniform estimates

$$||du^{(k)}||_{C^0_t \dot{H}_x^{\frac{m}{2}-1}} + ||du^{(k)}||_{L^2_t L^{2m}_x} \leq C(||u_0^{(k)}||_{\dot{H}^{\frac{m}{2}}} + ||u_1^{(k)}||_{\dot{H}^{\frac{m}{2}-1}}) < C\varepsilon_0$$

for sufficiently large k.

Hence as $k \to \infty$ a subsequence $u^{(k)} \rightharpoonup u$ weakly in $H^{\frac{m}{2}}_{loc}(\mathbb{R}^{m+1})$, where

$$||du||_{C^0_t \dot{H}^{\frac{m}{2}-1}} + ||du||_{L^2_t L^{2m}_x} \leq C(||u_0||_{\dot{H}^{\frac{m}{2}}} + ||u_1||_{\dot{H}^{\frac{m}{2}-1}})\,.$$

Since $\frac{m}{2} \geq 2$, by Rellich's theorem for a further subsequence $du^{(k)} \to du$ converges pointwise almost everywhere, and u solves (77), (78), as claimed.

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