THE CYLINDER CONJECTURE(S) AND A REDUCTION THEREOF

Work in progress with J. De Beule (VUB),
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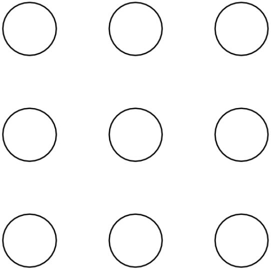

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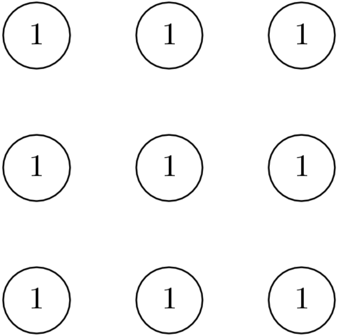

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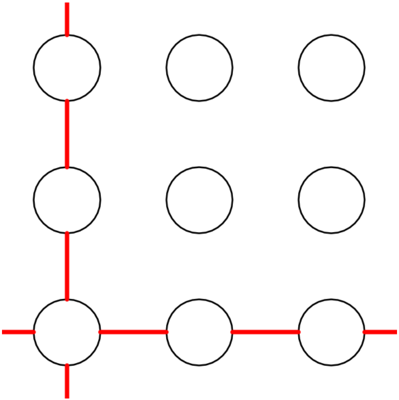

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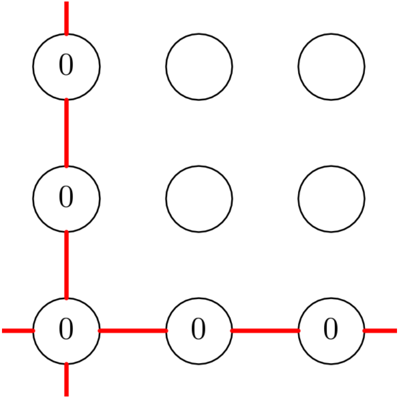

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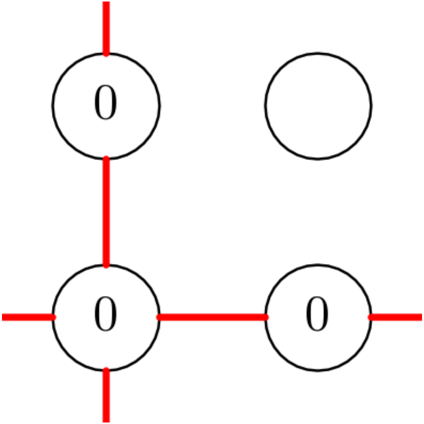
$p = 2$



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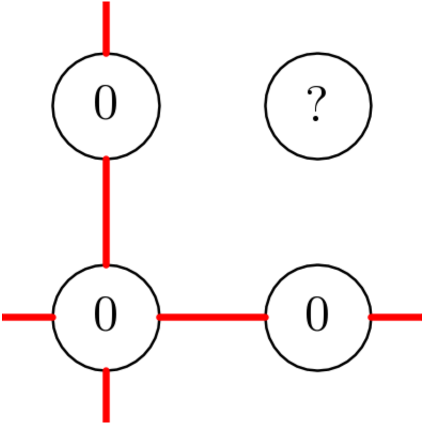

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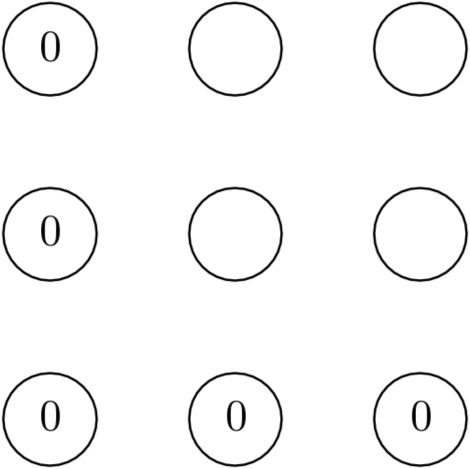

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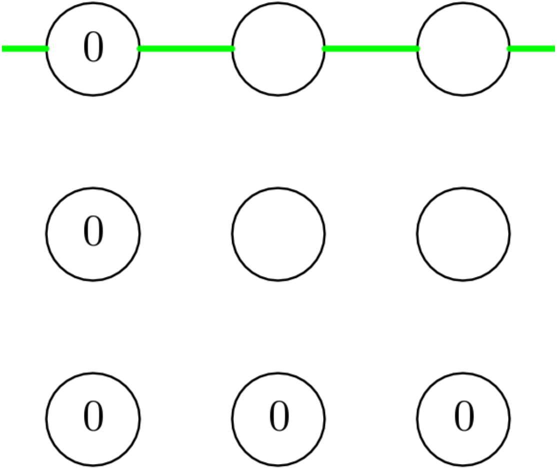

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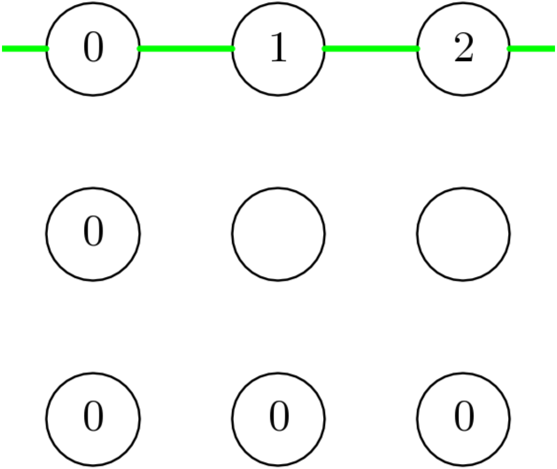

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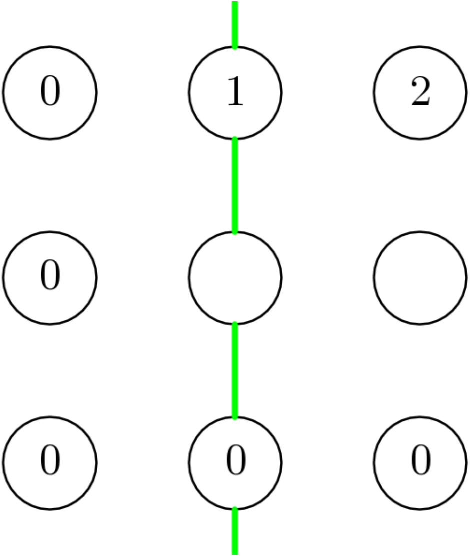

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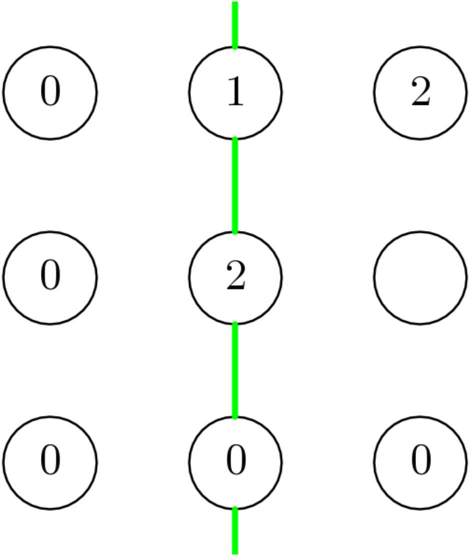

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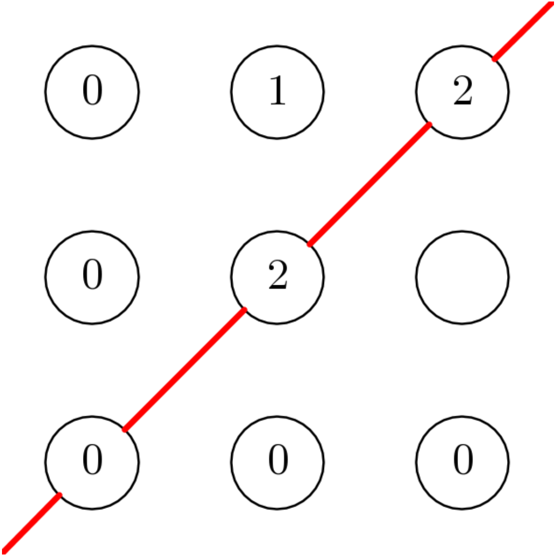

Results for small p

Theorem (M., De Beule, Demeyer, Sziklai)

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Conjecture

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Example

A plane in $\mathrm{AG}(3, p)$ is a trivial example of a cylinder.

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Weak Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3, p)$, not determining at least $p+1$ directions, then S is a cylinder.

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Let S be a set of p^2 points in $\mathrm{AG}(3, p)$ such that every plane intersects it in $0 \pmod{p}$ points, then S is a cylinder.

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Facts about S

Embed $\mathrm{AG}(3,p)$ in $\mathrm{PG}(3,p)$ with plane at infinity $W=0$. The affine point (x,y,z) then has projective coordinates $(x,y,z,1)$.

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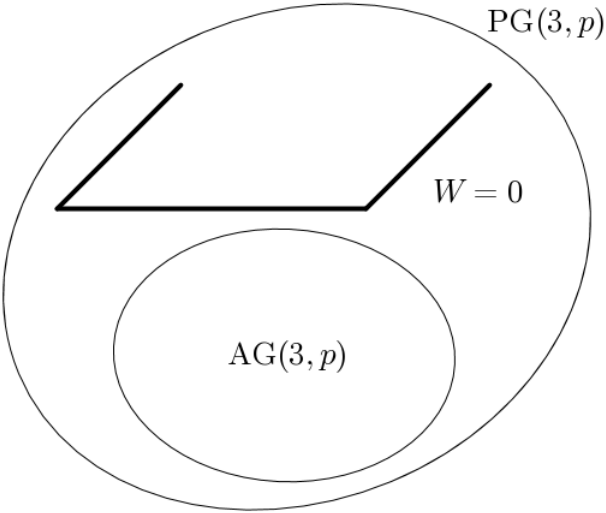

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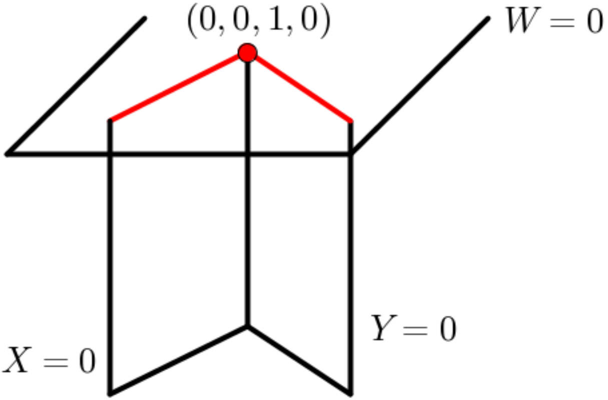

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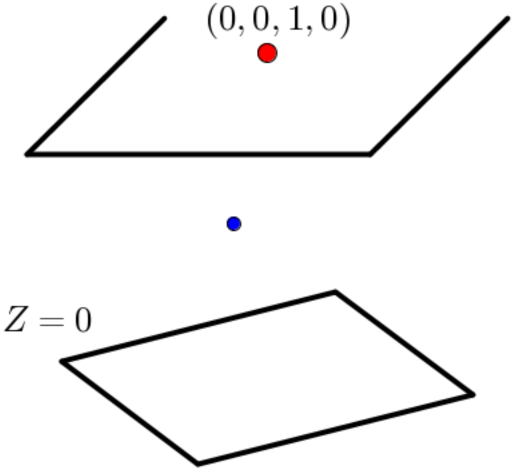

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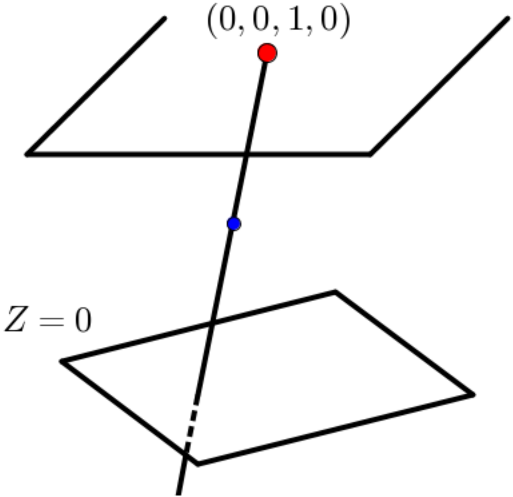

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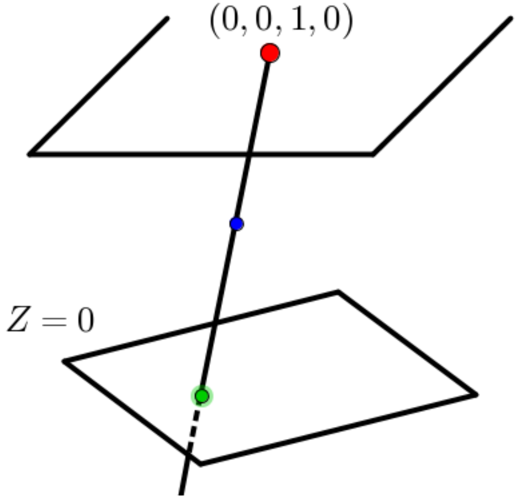

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Corollary

The Weak Cylinder Conjecture is true for all primes $p \leq 13$.

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What about the Strong Cylinder Conjecture?

Counterexample

For all $p \geq 5$,

$$f(t) = 1 - \frac{t^p - t}{t^2 - t},$$

$$w(X, Y) = f(X) + f(Y) - f(X + Y).$$

Thank you for your attention!

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