A birational embedding of an algebraic curve into a projective plane with two Galois points

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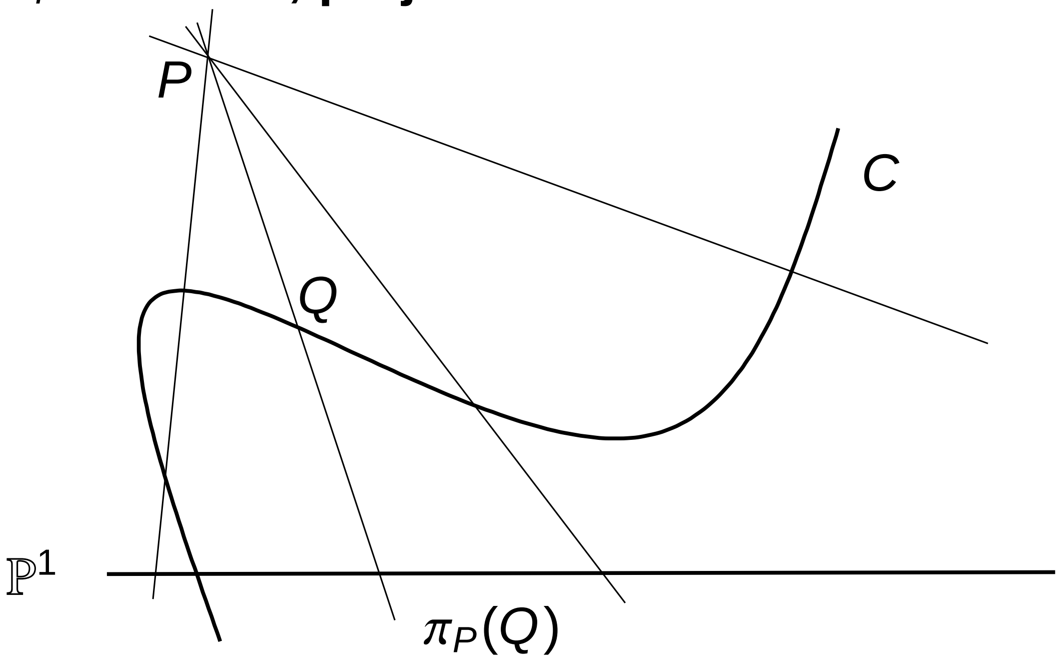
I. What is a Galois point ?

$k = \overline{k}$: **alg. closed field,** p = char $k \geq 0$

$C \subset \mathbb{P}^2$: **irred. plane alg. curve of deg.** $d \geq 4$

$P \in \mathbb{P}^2$: **point**

$\pi_P : C \dashrightarrow \mathbb{P}^1$; **projection from** P



Definition (Hisao Yoshihara, 1996)

P: **Galois point (for** C**)** $\Leftrightarrow k(C)/\pi_P^* k(\mathbb{P}^1)$: **Galois extension**

Example 1

$p \neq 2, 3$

$C \subset \mathbb{P}^2 : X^3Z + Y^4 + Z^4 = 0$

$P_1 = (1 : 0 : 0) \in C$

$P_2 = (0 : 1 : 0) \in \mathbb{P}^2 \setminus C$

Galois point

The reason that P_1 is Galois

$\pi_{P_1} = (Y : Z) = (y : 1) : C \dashrightarrow \mathbb{P}^1$

$k(C)/k(\mathbb{P}^1) = k(x, y)/k(y)$**:** $x^3 + y^4 + 1 = 0$**.**

cyclic extension

Example 2

$p \geq 3$

$H \subset \mathbb{P}^2 : X^pZ + XZ^p - Y^{p+1} = 0$

$P_1 = (1 : 0 : 0) \in H$

$P_2 = (0 : 1 : 0) \in \mathbb{P}^2 \setminus H$

Galois point

The reason that P_1 is Galois

$\pi_{P_1} = (Y : Z) = (y : 1) : H \dashrightarrow \mathbb{P}^1$

$k(H)/k(\mathbb{P}^1) = k(x, y)/k(y)$**:** $x^p + x - y^{p+1} = 0$**.**

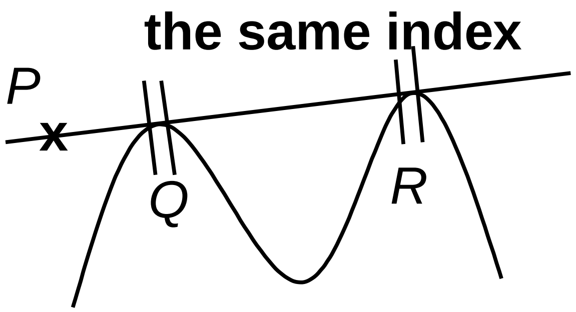
Artin-Schreier extension

Remark $\pi_P : C \dashrightarrow \mathbb{P}^1$; **projection from** P

(1) $Q \in C_{sm} \setminus \{P\} \Rightarrow e_Q = I_Q(C, \overline{PQ})$ **[rami. index = intersect. mult.]**

(2) P**: Galois point,** $Q, R \in C_{sm} \setminus \{P\}$ **s.t.** $\pi_P(Q) = \pi_P(R)$

$\Rightarrow e_Q = e_R$



P**: Galois** $\Rightarrow$ P**: intersect. point of multiple tangents (roughly)**

Problem **How many Galois points?**

Notation

- **Galois points on $C_{\rm sm}$: inner**

  $\delta(C)$ = # **of inner Galois points**

- **Galois points in $\mathbb{P}^2 \setminus C$: outer**

  $\delta'(C)$ = # **of outer Galois points**

When C is smooth, the answer is as follows.

Thm (Yoshihara, Miura, Homma, F)

Assume C: smooth. If $\delta(C) \geq 2$ or $\delta'(C) \geq 2$, then $C \sim$ one of the following.

	$\delta(C)$	**char.** p	**deg.**	**curve**
(1)	q^3+1	>0	$q+1$	**Fermat (Hermitian)**
(2)	$q+1$	2	$q+1$	$\prod_{\alpha\in\mathbb{F}_q}(x+\alpha y+\alpha^2)+cy^{q+1}=0$ **(**$c\neq 0,1$**)**
(3)	4	$\neq 2,3$	4	$x^3+y^4+1=0$

	$\delta'(C)$	**char.** p	**deg.**	**curve**
(1)	$q^4-q^3+q^2$	>0	$q+1$	**Fermat (Hermitian)**
(2)	7	2	4	**Klein quartic**
(3)	3	≥ 0	$\not\equiv 0$ **mod** p, $\neq q+1$	**Fermat**
(4)	3	2	**4**	$(x^2+x)^2+(x^2+x)(y^2+y)+(y^2+y)^2+c=0$ **(**$c\neq 0,1$**)**

Table of plane curves with $\delta(C) \geq 2$ (November, 2015)

	$\delta(C)$	**char.** p	**deg.**	**curve**	**group**
(1)	∞	> 0	q	$x - y^q = 0$	**cyclic**
(2)	$q^3 + 1$	> 0	$q + 1$	**Hermitian**	**elem.** p
(3)	$q + 1$	> 0	$q + 1$	**Ballico-Hefez**	**elem.** p
(4)	$q + 1$	2	$q + 1$	$\prod_{\alpha \in \mathbb{F}_q}(x + \alpha y + \alpha^2) + cy^{q+1} = 0$ **(**$c \neq 0, 1$**)**	**elem.** p
(5)	4	$\neq 2, 3$	4	$x^3 + y^4 + 1 = 0$	**cyclic**
(6)	3 **or** 2	$\neq 3$	4	$((t + \alpha)^3 : t(t + \beta)^3 : t(t + 1)^3)$ **(**$\beta^4 \neq \beta$**,** $\alpha = (\beta^2 + \beta + 1)/3$**)**	**cyclic**
(7)	2	$\neq 2, 3$	4	$(1 : (1 + t)^3 : t^4)$	**cyclic**

Problem **Find new examples of** C **with** $\delta(C) \geq 2$**.**

Main Thm (Criterion)

C**: smooth projective curve**

$G_1, G_2 \subset \mathrm{Aut}(C)$**: finite subgroups with** $G_1 \neq G_2$

Then:

$\exists$ **birational embedding** $\varphi : C \to \mathbb{P}^2$ **&** $\exists$ **inner Galois points** $\varphi(P_1) \neq \varphi(P_2)$

s.t. $G_{\varphi(P_1)} = G_1$ & $G_{\varphi(P_2)} = G_2$

$\Leftrightarrow$

(a) $C/G_1 \cong \mathbb{P}^1$**,** $C/G_2 \cong \mathbb{P}^1$

(b) $G_1 \cap G_2 = \{1\}$

(c) $\exists P_1, P_2 \in C$ **(**$P_1 \neq P_2$**) s.t.**

$$P_1 + \sum_{\sigma \in G_1} \sigma(P_2) = P_2 + \sum_{\tau \in G_2} \tau(P_1)$$

(as divisors)

Proof of the if-part

- D**: the divisor in (c)**
- **Take generators** f, g**:** $k(f) = k(C/G_1)$**,** $k(g) = k(C/G_2)$
  with $(f)_\infty = D - P_1$**,** $(g)_\infty = D - P_2$ **(by (a)(c))**
- $f, g \in \mathcal{L}(D)$
- **Take a rational map** $\varphi = (f : g : 1) : C \to \mathbb{P}^2$
- $\varphi : C \to \varphi(C)$**: birational by (b)**
- **The sublinear system (corresp. to)** $\langle f, g, 1 \rangle \subset |D|$**: base-point-free**
  $\Rightarrow \deg \varphi(C) = \deg D$
- **The morphism** $(f : 1)$ = **the projection from** $\varphi(P_1)$

Application to $\mathbb{P}^1$ ($p \neq 2$)

For $\mathbb{P}^1$, we can drop condition (a) (by Lüroth's Thm)

To construct rational curves with two inner Galois points, we reduce to the following

Problem

$G_1, G_2 \subset \mathrm{PGL}(2, K)$**: finite subgroups with** $G_1 \neq G_2$

We consider the following:

(b) $G_1 \cap G_2 = \{1\}$

(c) $\exists P_1, P_2 \in \mathbb{P}^1$ **(**$P_1 \neq P_2$**) s.t.**

$$\{\sigma(P_2) \mid \sigma \in G_1 \setminus \{1\}\} = \{\tau(P_1) \mid \tau \in G_2 \setminus \{1\}\} \quad \textbf{(with multiplicities)}$$

When does a pair (G_1, G_2) **with (b)(c) exist?**

Thm

The following plane rational curves C with two inner Galois points P_1, P_2 exist:

(1) $\deg C = 5$ **&** $G_{P_1}, G_{P_2} \cong \mathbb{Z}/4\mathbb{Z}$ **(**$p \neq 3$**)**

(2) $\deg C = 5$ **&** $G_{P_1}, G_{P_2} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$

(3) $\deg C = 5$ **&** $G_{P_1} \cong \mathbb{Z}/4\mathbb{Z}$**,** $G_{P_2} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$

(4) $\deg C = 6$ **&** $G_{P_1}, G_{P_2} \cong \mathbb{Z}/5\mathbb{Z}$

Thm (F-Waki)

$p \geq 3$**,** q**: a power of** p

The following plane rational curves C with two inner Galois points P_1, P_2 exist:

(1) $\deg C = q$ **&** $G_{P_1}, G_{P_2} \cong D_{q-1}$ **(dihedral group)**

(2) $\deg C = q$ **&** $G_{P_1} \cong D_{q-1}$**,** $G_{P_2} \cong \mathbb{Z}/(q-1)\mathbb{Z}$

Application to the elliptic curve $F_3 : X^3 + Y^3 + Z^3 = 0$ ($p \neq 3$)

Thm

$\exists$ **birational embedding** $\varphi : F_3 \longrightarrow \mathbb{P}^2$ **s.t.**

- $\deg \varphi(F_3) = 4$
- $\delta(\varphi(F_3)) \geq 2$

	$\delta(C)$	**char.** p	**deg.**	**curve**
(1)	∞	> 0	q	$x - y^q = 0$
(2)	$q^3 + 1$	> 0	$q + 1$	**Hermitian**
(3)	$q + 1$	> 0	$q + 1$	**Ballico-Hefez**
(4)	$q + 1$	2	$q + 1$	$\prod_{\alpha \in \mathbb{F}_q}(x + \alpha y + \alpha^2) + cy^{q+1} = 0$ **(**$c \neq 0, 1$**)**
(5)	4	$\neq 2, 3$	4	$x^3 + y^4 + 1 = 0$
(6)	3 **or** 2	$\neq 3$	4	$((t + \alpha)^3 : t(t + \beta)^3 : t(t + 1)^3)$ **(**$\beta^4 \neq \beta$**,** $\alpha = (\beta^2 + \beta + 1)/3$**)**
(7)	≥ 2	$\neq 3$	4	**birational embedding of Fermat cubic**
(8)	≥ 2	$\neq 2, 3$	5	**birational embedding of** $\mathbb{P}^1$
(9)	2	$\neq 2, 5$	6	**birational embedding of** $\mathbb{P}^1$
(10)	2	$\neq 2, 3$	4	$(1 : (1 + t)^3 : t^4)$
(11)	2	≥ 3	q	$(t^{\frac{q+1}{2}} : (t - 1)^{\frac{q+1}{2}} : t^q - t)$
(12)	2	≥ 3	q	$(t^{\frac{q+1}{2}} : t - 1 : t^q - t)$