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Locally finite groups with all subgroups either subnormal or nilpotent-by-Chernikov

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Abstract. Let G be a locally finite group satisfying the condition given in the title and suppose that G is not nilpotent-by-Chernikov. It is shown that G has a section S that is not nilpotent-by-Chernikov, where S is either a p-group or a semi-direct product of the additive group A of a locally finite field F by a subgroup K of the multiplicative group of F, where K acts by multiplication on A and generates F as a ring. Non-(nilpotent-by-Chernikov) extensions of this latter kind exist and are described in detail.

1. Introduction

Let $\mathfrak{NC}$ denote the class of groups that are nilpotent-by-Chernikov. By results from [1] and [5], a locally graded group with every proper subgroup in $\mathfrak{NC}$ is itself in $\mathfrak{NC}$, and from [2] it is known that a locally finite group in which every subgroup is subnormal is also in $\mathfrak{NC}$. (Examples in [7] and [9] show that one cannot remove the hypothesis of local finiteness here.) Theorem 4 of [8] states that a locally finite group G in which every subgroup is either subnormal or *nilpotent* has a subgroup of finite index in which every subgroup is subnormal, and together with [2] this shows that G belongs to $\mathfrak{NC}$. It is natural to ask next whether a locally finite group G in which every subgroup is either subnormal or in $\mathfrak{NC}$ necessarily lies in the class $\mathfrak{NC}$. It was shown in [10] that if G is a locally soluble-by-finite group in which every subgroup is either subnormal or in $\mathfrak{NC}$ then G is soluble-by-finite, and if G is not nilpotent-by-Chernikov then G is in fact soluble, and so the above question becomes a question about soluble groups. Let us denote by $\mathfrak{X}$ the class of groups G in which every subgroup is either subnormal or in $\mathfrak{NC}$. Our first result here is as follows.

Theorem 1.1. *Let G be a locally finite group in the class $\mathfrak{X}$. If G has finite exponent then G is nilpotent-by-finite.*

It turns out that there are locally finite groups in $\mathfrak{X}$ that are not in $\mathfrak{NC}$. Besides describing such counterexamples in detail we are able to present a necessary and sufficient condition for a locally finite group to lie in $\mathfrak{X}$ but not in $\mathfrak{NC}$. This condition is not quite satisfactory, on account of the fact that we have been unable to decide whether there are any locally finite p-groups in $\mathfrak{X} \setminus \mathfrak{NC}$. We have made some progress with the p-group case, having shown in particular that every Baer p-group in $\mathfrak{X}$ is also in $\mathfrak{NC}$, but we have chosen to postpone such discussion to a subsequent article. In the statement of the following result we make reference to some groups described in more detail in Section 3 of this paper. For a given group X, $\pi(X)$ denotes as usual the set of primes p such that X has an element of order p, and we recall that a field is *locally finite* if and only if it has positive characteristic and is algebraic over its prime subfield.

Theorem 1.2. *Let G be a locally finite group in $\mathfrak{X}$ and suppose that, for every prime p, all p-sections of G belong to $\mathfrak{NC}$. Then the following are equivalent.*

(a) $G \notin \mathfrak{NC}$.

(b) *For some locally finite field F, G has a section isomorphic to a group $G(F, K) := A \rtimes K$, that satisfies the following.*

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(i) *A is the additive group of F.*
(ii) *K is a subgroup of the multiplicative group F^* of F that acts on A by multiplication and generates F as a ring.*
(iii) *$\pi(K)$ is infinite but $\pi(K \cap F_1^*)$ is finite for every proper subfield F_1 of F.*

Theorem 3.4 (which is given in Section 3) shows that there do indeed exist groups $A \rtimes K$ of the kind described in part (b) of Theorem 1.2.

2. Background results and the proof of Theorem 1.1

As a preliminary remark, let us note that a group G that is an extension of an $\mathfrak{NC}$-group by a Chernikov group is again in $\mathfrak{NC}$ — a reference for this result, which will be used without further mention on several occasions, is provided in the proof of Lemma 1 of [5]. Our first lemma deals with the locally nilpotent case of Theorem 1.1.

Lemma 2.1. *Let G be a locally nilpotent group of finite exponent and suppose that $G \in \mathfrak{X}$. Then G is nilpotent.*

Proof. If G is not nilpotent then there is a prime p for which the p-component of G is not nilpotent, and we may therefore assume that G is a p-group. By hypothesis, every non-subnormal subgroup of G is nilpotent-by-Chernikov and hence nilpotent-by-finite. By Theorem 1 of [10] (though we need only apply Proposition 2 of that paper), G is soluble, and now G is a soluble p-group of finite exponent and so G is a Baer group (see, for example, Theorem 7.17 of [6]) and hence every non-subnormal subgroup of G is nilpotent. By Theorem 3 of [8] every subgroup of G is subnormal, and since G has finite exponent it follows immediately from [4] that G is nilpotent, thus completing the proof. □

Much of our effort will be directed towards determining when a locally finite group G in the class $\mathfrak{X}$ is (locally nilpotent)-by-Chernikov, that is, in the class $(L\mathfrak{N})\mathfrak{C}$. Our next three results, which have application beyond the finite-exponent case, together show that a locally finite group in $\mathfrak{X} \setminus (L\mathfrak{N})\mathfrak{C}$ has a non-$(L\mathfrak{N})\mathfrak{C}$ section that is a split extension of an abelian group by an abelian group that is either elementary or of rank one.

Lemma 2.2. *Let G be a locally finite group in $\mathfrak{X}$ and suppose that $G \notin (L\mathfrak{N})\mathfrak{C}$. Then G has a non-$(L\mathfrak{N})\mathfrak{C}$ subgroup G_0 with a locally nilpotent normal subgroup X such that G_0/X is either elementary abelian or of rank one and with all nontrivial primary components of prime order.*

Proof. By Proposition 3 of [8], G is soluble. Let X be its locally nilpotent radical. Periodic soluble groups satisfying the the minimal condition on subnormal abelian subgroups are Chernikov (see [6], vol. 1, page 176) but G/X is not Chernikov, hence it has a subnormal abelian subgroup A/X which is not Chernikov. Then A/X has an infinite subgroup G_0/X which either is of prime exponent or has all primary components of prime order. Also, since G_0 is subnormal in G, X is the locally nilpotent radical of G_0. Therefore G_0 is a subgroup of the required type. □

Lemma 2.3. *Let G be a locally finite group in $\mathfrak{X}$, N the locally nilpotent radical of G, and suppose that G/N is an infinite elementary abelian p-group for some prime p. Then G has a non-$(L\mathfrak{N})\mathfrak{C}$ section $G_0 := A \rtimes H$, where H is an elementary abelian p-group, A is the locally nilpotent radical of G_0 and A is an abelian p'-group.*

Proof. We shall assume that every section of the type described is in $(L\mathfrak{N})\mathfrak{C}$ and proceed to obtain the contradiction that $G \in (L\mathfrak{N})\mathfrak{C}$; note that every $(L\mathfrak{N})\mathfrak{C}$-section of G belongs to the class $(L\mathfrak{N})\mathfrak{F}$ of (locally nilpotent)-by-finite groups. We have $N = M \times B$ for some p-subgroup M and p'-subgroup B, where both M and B are G-invariant. Since G/M has no nontrivial normal p-subgroups we see that N/M is the $L\mathfrak{N}$-radical of G/M, and there is no loss in factoring by M and hence assuming that N is a p'-group. Certainly every finite p-subgroup of G is contained in a larger one, and so there is an infinite abelian p-subgroup H of G and, since N is the locally nilpotent radical of NH, we may suppose that $G = N \rtimes H$. Among all such non-$(L\mathfrak{N})\mathfrak{F}$ examples G, we may choose one with the derived length of N minimal. By hypothesis N is not abelian, and so there is a G-invariant abelian subgroup A of N such that each of AH and G/A is in $(L\mathfrak{N})\mathfrak{F}$. Hence there is a subgroup K of finite index in H such that AK and NK/A are locally nilpotent. Now N is a p'-group and K is a p-group, so we have $[A, K] = 1$ and $[N, K] \leq A$, and it follows that K acts nilpotently and hence trivially on N. But then NK is nilpotent and of finite index in G, and we have our contradiction. □

Lemma 2.4. *Let G be a locally finite group in $\mathfrak{X}$, N a normal locally nilpotent subgroup of G such that G/N is an infinite abelian group with all nontrivial primary components of prime order. If $G \notin (L\mathfrak{N})\mathfrak{C}$ then G has a non-$(L\mathfrak{N})\mathfrak{C}$ section $G_0 := A_0 \rtimes H$, where A_0 is abelian and normal, H is an infinite abelian subgroup with primary components of prime order and $\pi(H) \cap \pi(A_0) = \varnothing$.*

Proof. Supposing that every such section of G is in $(L\mathfrak{N})\mathfrak{C}$ and hence in $(L\mathfrak{N})\mathfrak{F}$, we shall obtain the contradiction $G \in (L\mathfrak{N})\mathfrak{F}$. Suppose first that the set π of primes p such that N has nontrivial p-component is finite. By passing to a subgroup of finite index in G we may assume in this case that G/N is a π'-group. Since G/N is countable we may apply the Schur-Zassenhaus Theorem to obtain a π'-subgroup H of G (which is abelian and has all nontrivial primary components of prime order) such that $G = N \rtimes H$. By induction on the derived length of N we may suppose that HN' is almost locally nilpotent, that is, H_1N' is locally nilpotent for some subgroup H_1 of finite index in H. Since H is a π'-group, H_1 centralizes N' and so $H_1N/N' \notin (L\mathfrak{N})\mathfrak{F}$. But H_1N/N' is a section of the type described above, and by hypothesis it is in $(L\mathfrak{N})\mathfrak{C}$. This is a contradiction, and from now on we shall suppose that $\pi := \pi(N)$ is infinite.

There is an infinite set σ of primes and a set $S := \{g_i : i \geq 1\}$ of elements of G such that S generates G modulo N and, for each i, g_i has order a power of p_i for some $p_i \in \sigma$, where $i \neq j$ implies $p_i \neq p_j$. Since $G \notin (L\mathfrak{N})\mathfrak{F}$ we may choose an element h_1 of S such that $N\langle h_1\rangle$ is not locally nilpotent. If h_1 has order a power of the prime r_1 then there is a prime q_1, necessarily distinct from r_1, and a finite subgroup F_1 of N that has order a power of q_1 and is such that $\langle h_1, F_1\rangle \notin (L\mathfrak{N})$. In particular, there is an element x_1 of F_1 such that $[x_1, h_1] \neq 1$. Let $\pi_1 = \{r_1, q_1\}$, let N_1 be the π_1-radical of N and write $N = N_1 \times M_1$. From our earlier argument we have that $G/M_1 \in (L\mathfrak{N})\mathfrak{F}$, so all but finitely many elements of S centralize N/M_1. Thus we may choose an element h_2 of S with order a power of some $r_2 \in \sigma \setminus \pi_1$ such that $\langle M_1, h_2\rangle \notin (L\mathfrak{N})$. Arguing as before, we may choose an element x_2 of q_2-power order in M_1, where q_2 is prime, such that $[x_2, h_2] \neq 1$ and $q_2 \notin \pi_1 \cup \{r_2\}$. Set $\pi_2 = \pi(\langle x_1, x_2, h_1, h_2\rangle)$ and write $N = N_1 \times N_2 \times M_2$, where $N_1 \times N_2$ is the π_2-radical of N. We have $G/M_2 \in (L\mathfrak{N})\mathfrak{F}$, and we may choose an element h_3 of S, with order a power of some $r_3 \in \sigma \setminus \pi_2$, such that $\langle M_2, h_3\rangle \notin (L\mathfrak{N})\mathfrak{F}$. Then we choose an element x_3 of q_3-power order in M_2 such that $[x_3, h_3] \neq 1$, where q_3 is a prime not in $\pi_2 \cup \{r_3\}$. Set $\pi_3 = \pi(\langle x_1, x_2, x_3, h_1, h_2, h_3\rangle)$ and write $N = N_1 \times N_2 \times N_3 \times M_3$, where $N_1 \times N_2 \times N_3$ is the π_3-radical of N.

Continue in this manner and let $H = \langle h_i : i \geq 1\rangle, B = \langle x_i : i \geq 1\rangle^H$. Also, set $\lambda = \{r_i : i \geq 1\}, \mu = \{q_i : i \geq 1\}$. Then B is a μ-group and $\lambda \cap \mu = \varnothing$. Let $G_1 = BH$. If $G_1 \in (L\mathfrak{N})\mathfrak{F}$ then there is a subgroup H_0 of finite index n, say, in H such that BH_0 is locally nilpotent. But then for every element h of H that has order co-prime to n we have $B\langle h\rangle$ locally nilpotent, and so all but finitely many of the elements $[x_i, h_i]$ are trivial, contrary to construction. Thus $G_1 \notin (L\mathfrak{N})\mathfrak{F}$. Similarly, if M denotes the λ-radical of $N \cap G_1$ then we see that G_1/M is also not in $(L\mathfrak{N})\mathfrak{F}$. There is nothing lost, therefore, in factoring by M, which means that every λ-subgroup of G_1 is now of rank one, with nontrivial primary components of prime order. Since the locally finite group G_1 is countable there is, by the Schur-Zassenhaus Theorem, a λ-subgroup H_1 of G_1 that supplements $N \cap G_1$. If $G_2 := B \rtimes H_1 \in (L\mathfrak{N})\mathfrak{F}$ then there is a subgroup C of finite index in H_1 that centralizes B, and then $D := C^{G_1}$ centralizes B. But $\pi(G_1/D)$ contains just finitely many elements of λ, again contrary to construction, and so $G_2 \notin (L\mathfrak{N})\mathfrak{F}$.

Suppose that B has derived length d. By our hypothesis on sections of G, $H_1B^{(i)}/B^{(i+1)} \in (L\mathfrak{N})\mathfrak{F}$ for every $i = 0, \dots, d-1$, and so there is a subgroup H_2 of finite index in H_1 that centralizes each $B^{(i)}/B^{(i+1)}$. But then $[B,_d H_2] = 1$ and so $[B, H_2] = 1$, which gives BH_2 locally nilpotent and hence $G_2 \in (L\mathfrak{N})\mathfrak{F}$, a contradiction that establishes the result. □

Lemma 2.5. *Let G be a locally finite group in $\mathfrak{X}$, and suppose that, for every prime p, all p-sections of G belong to $\mathfrak{NC}$ but $G \notin \mathfrak{NC}$. Then, for some locally finite field F, G has a section isomorphic to a group $A \rtimes K$, where A is the additive group of F and K is an infinite subgroup of the multiplicative group F^* of F that acts on A by multiplication and generates F as a ring.*

Proof. Since every p-section of G lies in $\mathfrak{NC}$, so does the locally nilpotent radical R, say, of G, by Lemma 2 of [10]. Thus G/R is not Chernikov, and by Lemmas 2.2, 2.3 and 2.4 we may assume that $G = A \rtimes K$ for some abelian normal subgroup A and infinite abelian subgroup K that either is of prime exponent p or has nontrivial primary components of prime order and, in this latter case, $\pi(A) \cap \pi(K) = \varnothing$. Also in the former case we may assume that A is a p'-group. For if $A = A_p \times A_{p'}$, where $A_p, A_{p'}$ respectively denote the p- and p'-components of A, then $A_pK \in \mathfrak{NC}$, by hypothesis, and if $A_{p'}K$ is also in $\mathfrak{NC}$ then there is a subgroup of finite index in K that centralizes $A_{p'}$, and we easily obtain the contradiction $G \in \mathfrak{NC}$.

Now, in either case, K acts nilpotently and hence trivially on $A/[A, K, K]$ and so $[A, K, K] = [A, K]$, while if $K[A, K]$ is nilpotent-by-Chernikov then $[A, K]$ is centralized by some subgroup of finite index in K and again the result follows. Replacing A by $[A, K]$ if necessary, we may assume that $A = [A, K]$. Factoring, we may also assume that $C_K(A) = 1$.

Now let D be an arbitrary proper K-invariant subgroup of A. Since $A = [A, K] = [A, KD]$ we cannot have KD subnormal in G, so KD is nilpotent-by-finite and D is centralized by some subgroup of finite index in K. In particular, if $[A, k] < A$ for some non-trivial element k of K then we have $[A, k, H] = 1$ for some H with K/H finite, and since K is abelian we deduce from the three-subgroup lemma that $1 = [A, H, k]$ and hence that $[A, H] < A$, as $C_K(A) = 1$. This in turn gives $[A, H]K$ nilpotent-by-finite, from which it follows that there is a subgroup L of finite index in K that centralizes $[A, H]$. But then $[A, H \cap L, H \cap L] = 1$, hence $[A, H \cap L] = 1$. Since K is infinite, $H \cap L \neq 1$ and we have a contradiction, so $[A, k] = A$ for every non-trivial element k of K.

If $A = D \times E$ for some nontrivial K-invariant subgroups D and E then there is a subgroup H of finite index in K that centralizes both D and E and hence A, which gives a contradiction. Thus A is a q-group for some prime q. Next, let k be an arbitrary nontrivial element of K and let $C = C_A(k), C_1/C = C_{A/C}(k)$. Then $[C_1, k, k] = 1$ and hence $[C_1, k] = 1$ and $C_1 = C$. The map $a \mapsto [a, k]$ induces an isomorphism from A/C onto A, and the pre-image of C under this map is just C_1/C, which is trivial. It follows that $C_A(k) = 1$ for all nontrivial $k \in K$. We now claim that A is simple as a K-module: if B is a proper K-invariant subgroup of A then $BK \in \mathfrak{NC}$ and so $[B, H] = 1$ for some H of finite index in K, and since $C_A(k) = 1$ for all nontrivial $k \in K$ it follows that $B = 1$ and the claim is established. In particular, $A^q = 1$.

Let R denote the group ring $\mathbb{Z}_qK$. As an R-module, A is isomorphic to the factor module $F := R/M$, where M is a maximal right ideal of R and the action of K on F is induced by right multiplication. Since R is commutative, M is a maximal ideal of R and F is a field. Now K embeds in a natural way in the multiplicative group of F, and since K generates R as a ring its image in F generates F. Finally, F has characteristic q and K is periodic, meaning that the elements of K (and hence those of F) are algebraic over the prime subfield of F. Thus F is locally finite, and this shows that G has the required structure. □

Proof of Theorem 1.1. Since G has finite exponent Lemma 2.1 shows that, for every prime p, all p-sections of G are nilpotent. If G is not nilpotent-by-finite then $G \notin \mathfrak{NC}$ and hence G has a section $A \rtimes K \notin \mathfrak{NC}$ as described in Lemma 2.5. But K has rank one, as a periodic subgroup of the multiplicative group of a field, hence K is finite because $\exp G$ is finite. This is a contradiction, which completes the proof of the theorem. □

3. Examples and the proof of Theorem 1.2

Prompted by Lemma 2.5, we shall study split extensions $A \rtimes K$ of the type described there and determine under which conditions they belong to the class $\mathfrak{X} \setminus \mathfrak{NC}$.

Let F be a locally finite field of (prime) characteristic p and let A and F^* respectively denote the additive and multiplicative groups of F. If K is a subgroup of F^* that generates F as a ring (or, equivalently, as a field) then we denote by $G(F,K)$ the split extension $A \rtimes K$, where the action of K on A is by (field) multiplication. With this setting we have, for $G = G(F,K)$, $p \notin \pi(K)$ and $C_G(A) = A$, so $A = \mathrm{Fitt}(G)$; also, if $0 \neq a \in A$ then $C_G(a) = A$ and $\langle a\rangle^K = A$. This latter equality follows from the fact that for all $b \in A$ we have $ba^{-1} = k_1 + k_2 + \cdots + k_n$ for elements k_i of K and hence, in group notation, $b = a^{k_1}a^{k_2}\cdots a^{k_n}$. Thus G is a metabelian group with monolith, derived subgroup and locally nilpotent radical all equal to A. Then $G \in \mathfrak{NC}$ if and only if K is Chernikov, that is (since K has rank one), if and only if $\pi(K)$ is finite. In what follows it will often be convenient to regard each element of G as an ordered pair (a,k), with the obvious identification of $a \in A$ with $(a, 1_F)$ and $k \in K$ with $(0_F, k)$. Then, for example, the rules for forming conjugates and commutators are given by $(a,k)^{(b,l)} = (-bl + al + blk^{-1}, k), [(a,k),(b,l)] = (ak(l-1_F) + bl(1_F - k)), 1_F)$.

Lemma 3.1. *Suppose that F_1 is a subfield of F that is generated by a nontrivial subgroup K_1 of K. Then the normalizer of $G_1 := G(F_1, K_1)$ in $G := G(F,K)$ is $G(F_1, K \cap F_1^*)$, where F_1^* is the multiplicative group of F_1.*

Proof. First note that $G(F_1, K \cap F_1^*)$ is well-defined since $K \cap F_1^*$ contains K_1 and hence generates F_1. We have $G_1 = A_1K_1$, where A_1 is the additive group of F_1, and if $a, b \in A_1, k \in K_1$ and $l \in K \cap F_1^*$, then $(a,k)^{(b,l)} \in G_1$ and so $G(F_1, K \cap F_1^*) \leq N := N_G(G_1)$. It is enough now to show that $N \leq A_1F_1^*$. Let $u = (a,k) \in N$, where $a \in A, k \in K$. Since $(1_F, 1_F) \in A_1$ it follows that $(k, 1_F) = (1_F, 1_F)^u \in A_1$ and hence $k \in F_1^*$. It follows too that $(a, 1_F) \in N \cap A$, since $(a, 1_F) = (a,k)(0_F, k^{-1})$ and $k^{-1} \in K \cap F_1^* \leq N$. Since $K_1 \neq 1$ we may choose $x \in K_1$ with $x \neq 1_F$. We have $[(a, 1_F), (0_F, x)] = (ax - a, 1_F)$, which therefore belongs to $G_1 \cap A = A_1$; hence $ax - a = a(x - 1_F) \in F_1$. Since $0_F \neq x - 1_F$ we deduce that $a \in F_1$ and hence that $u \in G(F_1, K \cap F_1^*)$, as required. □

Lemma 3.2. *With the notation of Lemma 3.1, if F_1 is a proper subfield of F then G_1 is not subnormal in G.*

Proof. Suppose that $G_1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n \leq G$, for some positive integer n, and let $N = N_G(H_0)$. By Lemma 3.1, $N = G(F_1, K \cap F_1^*) = A_1(K \cap F_1^*)$, and so $H_1 = H_1 \cap N = A_1K_2$, where $K_2 = K \cap F_1^* \cap H_1$. Since $H_1 \geq H_0 \geq K_1$ we see that $H_1 = G(F_1, K_2)$. Again by Lemma 3.1, the normalizer in G of H_1 is $G(F_1, K \cap F_1^*)$, which is just N, and repeating the argument we have that $N_G(H_i) = N$ for every $i \geq 0$. It follows that $H_n \leq N$, and since $N < G$ the lemma is proved. □

We are now in a position to provide a necessary and sufficient condition for a group $G(F,K)$ to be in $\mathfrak{X} \setminus \mathfrak{NC}$. Our result is as follows, where F, K and $G(F,K)$ are as described in the opening paragraph of this section.

Lemma 3.3. *The following are equivalent.*

(*a*) *$G := G(F,K)$ is in $\mathfrak{X} \setminus \mathfrak{NC}$.*
(*b*) *$G \notin \mathfrak{NC}$ but every subgroup of G is either normal or abelian-by-Chernikov.*
(*c*) *$\pi(K)$ is infinite but $\pi(K \cap F_1^*)$ is finite for every proper subfield F_1 of F, where F_1^* is the multiplicative group of F_1.*

Proof. Firstly suppose that (*a*) holds, and consider the statement (*c*). Since K has rank one but is not Chernikov, $\pi(K)$ must be infinite. Let $K_1 = K \cap F_1^*$, where F_1 and F_1^* are as stated, and suppose as we may that K_1 is nontrivial (which it is in any case except possibly when $p = 2$, since F_1 contains the prime subfield of F). Then $G_1 := G(F_1, K_1)$ is not subnormal in G, by

Lemma 3.2, hence $G_1 \in \mathfrak{NC}$. Since $\mathrm{Fitt}(G_1)$ is the additive group of F_1, $K_1 \cong G_1/\mathrm{Fitt}(G_1)$ is Chernikov, which amounts to saying that $\pi(K_1)$ is finite. Thus (a) implies (c). Since (b) implies (a) it suffices now to show that (c) implies (b).

Assuming that (c) holds, let A be the additive group of F and let $H \leq G$. If $H \cap A = 1$ then H is abelian. Otherwise, let a be a nontrivial element of $H \cap A$ and let $H^* = AH \cap K$. We have $H \geq \langle a \rangle^{H^*} = \langle a \rangle^{F_1}$, where F_1 is the ring (equivalently, the field) generated by H^* in F. If $F_1 = F$ then $\langle a \rangle^{F_1} = A$ and so $A \leq H \lhd G$. If $F_1 \subset F$ then by hypothesis $\pi(H^*)$ is finite, hence $HA/A \cong H^*$ is Chernikov and H is abelian-by-Chernikov. We have thus established that the arbitrary subgroup H of G is either normal in G or abelian-by-Chernikov, and this completes the proof. □

In view of Lemma 2.5, Lemma 3.3 completes the proof of Theorem 1.2.

We have not yet addressed the question of the existence of locally finite groups in $\mathfrak{X} \setminus \mathfrak{NC}$. We shall provide such examples with the help of Lemma 3.3. Note that if K and F are such that condition (c) in this lemma is satisfied, and K_1 is a subgroup of K such that $\pi(K_1)$ is infinite, then K_1 generates F and still has the property required for K, hence $G(F, K_1)$ is again in $\mathfrak{X} \setminus \mathfrak{NC}$. This holds, for instance, when K_1 is any infinite subgroup of the socle of K; in this case $G(F, K_1)$ is a group of the type arising in Lemma 2.4, and all of its subgroups are either normal or abelian-by-finite.

The key tool for our construction is Zsigmondy's Theorem (see, for instance, Theorem 1.16 of [3]), a special case of which shows that if p is a prime and n is a positive integer then, with the exception of the case when $p = 2$ and n is 1 or 6, there exists a prime q such that the multiplicative order of p modulo q is exactly n. We shall call such a q a Zsigmondy prime for p^n. This special case of Zsigmondy's Theorem can be stated equivalently as follows: if F is a finite field then, unless $|F|$ is 2 or 2^6, F is generated (as a field) by some element x of prime (multiplicative) order. For, if $0 \neq x \in F$ and $|F| = p^n$, then the order q of x divides $p^n - 1$ and the requirement that x is in no proper subfield of F means exactly that q does not divide $p^m - 1$ for any $m < n$. We shall call such a generator of prime order a Zsigmondy generator of F. We stress that the order of a Zsigmondy generator of F is a Zsigmondy prime for $|F|$, and that if F_1 is another finite field, of the same characteristic as F but different order, then Zsigmondy generators of F and F_1 also have different orders.

Theorem 3.4. *Let F be an infinite locally finite field. Then the multiplicative group of F has a subgroup K such that K generates F as a field and $G(F, K) \in \mathfrak{X} \setminus \mathfrak{NC}$*

Proof. Since F is countable, $F = \bigcup_{i \in \mathbb{N}} F_i$ for some chain $F_1 \subset F_2 \subset F_3 \subset \cdots$ of finite fields, where we may assume that F_1 has order greater than 2^6. For each of the fields F_i choose a Zsigmondy generator x_i. Let K be the subgroup of the multiplicative group of F generated by all elements x_i. Then K is the direct product of the subgroups $\langle x_i \rangle$, which have (pairwise different) prime orders, so that $\pi(K)$ is infinite. It is also clear that K generates F as a field, and that if π is an infinite subset of $\pi(K)$ then the subfield generated by the π-component of K contains infinitely many of the subfields F_i, and hence is F. It follows that K satisfies condition (c) of Lemma 3.3. Therefore $G(F, K) \in \mathfrak{X} \setminus \mathfrak{NC}$, as required. □

We make two further remarks concerning the groups $G(F, K)$ in the class $\mathfrak{X}$. Firstly, while Theorem 3.4 does not provide an explicit characterization of such groups, nevertheless it is not far away from doing so, since the construction described in its proof has a sort of converse, as follows.

Proposition 3.5. *Let F be a locally finite field and H a subgroup of its multiplicative group. If $G(F, H) \in \mathfrak{X} \setminus \mathfrak{NC}$ then the socle of H has a subgroup K such that $G(F, K) \in \mathfrak{X} \setminus \mathfrak{NC}$ and K can be obtained by the construction outlined in the proof of Theorem 3.4.*

Proof. Let S be the socle of H. We have already noted that $G(F, S) \in \mathfrak{X} \setminus \mathfrak{NC}$. We define a strict, partial order relation on the infinite set $\pi := \pi(S)$ as follows: if $p, q \in \pi$ we let $p \prec q$

if and only if the subfield of F generated by the p-component of S is strictly contained in the subfield generated by the q-component of S. We claim that, with respect to this ordering, π has no maximal elements. For, let P be the prime subfield of F and assume, for a contradiction, that q is maximal in $(\pi, \prec)$. Let x be an element of order q in S and $n = [P(x) : P]$, the degree of x over P. Let r be a prime divisor of n, $W(r)$ the set of all positive integers m such that the greatest common divisor of n and m divides n/r. Then $W(r)$ is closed under taking divisors and forming lowest common multiples, hence the set F_r of all elements of F whose degree over P is in $W(r)$ is a subfield of F. Now $x \notin F_r$, hence $F_r \neq F$ and Lemma 3.3 shows that $S \cap F_r$ is finite. It follows that π has an infinite subset ψ with the property that the ψ-component of S is disjoint from the union of the subfields F_r, where r ranges over the prime divisors of n. Let $s \in \psi$ and let y be an element of order s in S. If $m = [P(y) : P]$ then $m \notin W(r)$ for any prime divisor r of n (as $y \notin F_r$), hence n divides m and $P(x) \subseteq P(y)$. Equality may hold for finitely many values of s only, so we can choose s such that $P(x) \subset P(y)$, meaning that $q \prec s$. This contradicts the maximality of q and establishes our claim.

Now let ξ be a maximal chain in π with respect to $\prec$. Since π has no maximal elements ξ is infinite. Arrange its elements in a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ and, for all i, let x_i be an element of order q_i in S and $F_i = P(x_i)$. For all $i \in \mathbb{N}$ we have $q_i \prec q_{i+1}$ and hence $F_i \subset F_{i+1}$. Let $K = \langle x_i \mid i \in \mathbb{N}\rangle$ in the multiplicative group of F. Since $\pi(K) = \xi$ is infinite K generates F (see the remarks following Lemma 3.3) and so $F = \bigcup_{i\in\mathbb{N}} F_i$. Finally, for each i, x_i is a Zsigmondy generator for F_i, thus K is a group that can be obtained by means of the construction given in Theorem 3.4. □

Our final remark is that only in a special case do we have $G(F, F^*) \in \mathfrak{X} \setminus \mathfrak{NC}$, where F^* is the full multiplicative group of F. Let P be a field of prime order p and q a prime (not necessarily different from p). Fix a normal closure $\bar{P}$ of P and let $\bar{P}(q)$ be the subfield of $\bar{P}$ consisting of all elements whose degrees over P are powers of q. If, for all nonnegative integers i, we denote by P_i the only subfield of $\bar{P}$ with degree q^i over P, then $\bar{P}(q)$ is the union of the chain $\{P_i \mid i \in \mathbb{N}_0\}$ and the P_i account for all proper subfields of $\bar{P}(q)$.

Proposition 3.6. *Let F be a locally finite field and let F^* be its multiplicative field. Then $G(F, F^*) \in \mathfrak{X} \setminus \mathfrak{NC}$ if and only if F is isomorphic to one of the fields $\bar{P}(q)$ just defined.*

Proof. If $F \cong \bar{P}(q)$ then $\pi(F^*)$ is infinite by Zsigmondy's Theorem, and all proper subfields of F are finite. Then $G(F, F^*) \in \mathfrak{X} \setminus \mathfrak{NC}$ by Lemma 3.3. Conversely, assume $G(F, F^*) \in \mathfrak{X} \setminus \mathfrak{NC}$ and let P be the prime subfield of F. Let π be the set of all primes r such that F has an element of degree r over P. For every subset ψ of π let F_ψ be the subfield of F consisting of all elements whose degrees over P are ψ-numbers. If π is infinite, choose a strictly decreasing sequence $(\pi_i)_{i\in\mathbb{N}}$ of proper subsets of π. If K_i is the multiplicative group of F_{π_i} for all i then $(K_i)_{i\in\mathbb{N}}$ is a strictly decreasing sequence of subgroups of K_1. In that case K_1 is not Chernikov and hence $\pi(K_1)$ is infinite; since $F_{\pi_1} \neq F$ this is a contradiction. Hence π is finite.

For each $q \in \pi$, consider the set S_q of all positive integers n such that F has a subfield of degree q^n over P. If every S_q is finite then there is an upper bound on the degrees over P of the elements of F, but this gives the contradiction that F is finite, and so S_q is infinite for some $q \in \pi$. This implies that $F_{\{q\}} \cong \bar{P}(q)$. The multiplicative group of $\bar{P}(q)$ involves infinitely many primes, and we deduce from Lemma 3.3 that $F_{\{q\}} = F$, which is what we were required to prove. □

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