Groups whose infinite subgroups are centralizers

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Abstract. We study groups in which all infinite subgroups are centralizers. Such groups are periodic; we completely describe them in the additional hypothesis that they are locally graded.

Centralizers in a group G form a subset $\mathsf{C}(G) = \{C_G(H) \mid H \leq G\}$ of the lattice $\mathfrak{L}(G)$ of all subgroups of G; of course this subset is closed under taking (arbitrary) intersections, but usually it is not join-closed. Gaschütz [4] described those finite groups G in which all subgroups are centralizers, that is, such that $\mathsf{C}(G) = \mathfrak{L}(G)$. The corresponding result for infinite groups is due to Stonehewer and Zacher [8], Corollary 6.2: a locally graded group G satisfies $\mathsf{C}(G) = \mathfrak{L}(G)$ if and only if it is periodic and has a Hall decomposition in which each factor is nonabelian of order the product of two primes. Here, as elsewhere, by a *Hall decomposition* of a periodic group we mean a decomposition as a direct product in which elements from different factors always have coprime orders. For the sake of shortness we shall also call *groups of type* $\mathfrak{pq}$ the nonabelian groups whose order is the product of two (necessarily different) primes.

It is worth mentioning that every group in which all subgroups are centralizers is periodic (see [9], Proposition 4.1) and that Tarski groups provide more examples of groups with this property.

We use brackets to denote intervals in the subgroup lattice of a group. In this notation $\mathsf{C}(G) \subseteq [G/Z(G)] = \{H \leq G \mid Z(G) \leq H\}$ for all groups G; Reuther [5] determined the structure of those finite groups G for which $\mathsf{C}(G) = [G/Z(G)]$, also see [2]; the infinite case is discussed in [1].

A useful tool in obtaining these results is the *centralizer mapping*: $H \in \mathfrak{L}(G) \mapsto C_G(H) \in \mathfrak{L}(G)$. If H is a subgroup of a group G it is clear that $C_G\big(C_G(H)\big)$ is the smallest centralizer in G containing H. Thus $H \in \mathsf{C}(G)$ if and only if $H = C_G\big(C_G(H)\big)$. Therefore the centralizer mapping induces a duality in the poset $\mathsf{C}(G)$.

We are interested in those groups in which every infinite subgroup is a centralizer. Let us call (IC) the class of such groups. We shall see that all (IC)-groups are periodic (Corollary 1.5). As is so often the case for properties defined by restrictions on infinite subgroups, on the one hand it is necessary to impose extra hypotheses on the groups considered, to exclude some groups, difficult to handle, all whose proper subgroups are finite. On the other hand, even under fairly general solubility or finiteness assumptions, imposing some restrictions on the infinite subgroups is in many cases equivalent to imposing them on all subgroups. In our case, it turns out that for infinite locally graded groups G the property (IC) is equivalent to the condition $\mathsf{C}(G) = \mathfrak{L}(G)$ considered by Gaschütz, Stonehewer and Zacher, unless G is a finite extension of a Prüfer group. Therefore, in this paper we shall mostly deal with Prüfer-by-finite groups.

Our main result is the following classification of locally graded (IC)-groups; it turns out that they are either finite or soluble. Let us say that a finite group is *of Gaschütz type* if all subgroups of its are centralizers, that is, if it has a Hall decomposition whose factors are groups of type $\mathfrak{pq}$.

Theorem. *Let G be an infinite locally graded group. Then $G \in$ (IC) if and only if either $\mathsf{C}(G) = \mathfrak{L}(G)$ or G has a Prüfer p-subgroup P of finite index (p a prime) and one of the following holds:*

(a) *$G = P \times F$, where F is of Gaschütz type;*

(b) *$G = R \times F$, where R is a p-group and F is a p'-group of Gaschütz type, $P = Z(R) < R$, R' is cyclic and R/P has modular subgroup lattice;*

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(c) *$G = C\langle x\rangle$, where $C = C_G(P)$ has prime index q in G, also $C = P \times F$ where F has a Hall decomposition $F_1 \times F_2 \times \cdots \times F_n$ whose factors are of type $\mathfrak{pq}$ (hence F is of Gaschütz type), $[F_2F_3 \cdots F_n, x] = 1$ and one of the following holds:*
  (c1) *$[F_1, x] = 1$ and q does not divide $|F|$; moreover either $x^q = 1$ or $p = q = 2$ and x^2 is the element of order 2 in P;*
  (c2) *$x^{q^2} = 1 \neq x^q \in F_1$;*
  (c3) *$p = q = 2$ and $F_1 = \langle y\rangle \rtimes \langle h\rangle$ is dihedral of order $2r$, where r is a prime congruent to 1 modulo 4, $x^2 = uh$, where u is an element of order 4 in P, $[h, x] = u^2$ and x induces an automorphism of order 4 on $\langle y\rangle$.*

(d) *$p = 2$ and $G = G_1 \times F$, where F is of Gaschütz type and has odd order, G_1 is a 2-group, $P < C = C_{G_1}(P) < G_1$ and $P = C_{G_1}(C)$, and G_1/P is elementary abelian;*

Examples of groups of each of the types described can be constructed in a straightforward way. It can also be mentioned that there is no overlap between the cases considered in the Theorem: a group G can satisfy at most one of (a), (b), (c1), (c2), (c3), (d). For the only (possibly) doubtful case (c) see the Remark following Lemma 2.5.

1. Preliminary results

Lemma 1.1. *Let G be a group and let $U \leq G$. If $[G/U] \subseteq \mathsf{C}(G)$ then $C_V\big(C_V(U)\big) = UZ(V)$ for all $V \in [G/U]$.*

Proof. Let $U \leq V \leq G$. Since V is a centralizer $V = C_G\big(C_G(V)\big)$, hence $C_V(U) = V \cap C_G(U) = C_G\big(C_G(V)\big) \cap C_G(U) = C_G(UC_G(V))$. Then $C_G\big(C_V(U)\big) = C_G\big(C_G(UC_G(V))\big) = UC_G(V)$, because $UC_G(V)$ is a centralizer too. Therefore $C_V\big(C_V(U)\big) = V \cap UC_G(V) = U(V \cap C_G(V)) = UZ(V)$. □

In other words: if every subgroup containing U is a centralizer then U is a centralizer in a subgroup V containing it if and only if $Z(V) \leq U$. It is not hard to deduce from this a remark made by Antonov ([1], Lemma 1.1): *the class of groups in which all subgroups containing the centre are centralizers is subgroup-closed.*

Another easy lemma which will be of some use is the following.

Lemma 1.2. *Let G be a group, $A \in \mathsf{C}(G)$ and suppose that A is abelian and $B = C_G(A)$. Then $A = Z(B)$ and, for all $H \in [B/A]$, H is a centralizer in G if and only if it is a centralizer in B.*

Proof. Since A is a centralizer then $A = C_G\big(C_G(A)\big) = C_G(B) = Z(B)$. Now, if $H \in [B/A]$ then also $C_G(H)$ and $C_G\big(C_G(H)\big)$ are in $[B/A]$. Hence $C_G\big(C_G(H)\big) = C_B\big(C_B(H)\big)$ and the result follows. □

Lemma 1.3. *Let the periodic group G have a Hall decomposition $G = R \times F$ where R is infinite and F is finite. Then $G \in$ (IC) if and only if $R \in$ (IC) and every subgroup of F is a centralizer.*

Proof. Every $H \leq G$ can be written as $H = (H \cap R) \times (H \cap F)$, and $C_G\big(C_G(H)\big) = C_R\big(C_R(H \cap R)\big) \times C_F\big(C_F(H \cap F)\big)$. Hence $H \in \mathsf{C}(G)$, that is $H = C_G\big(C_G(H)\big)$, if and only if $H \cap R$ is a centralizer in R and $H \cap F$ is a centralizer in F. Moreover, H is infinite if and only if $H \cap R$ is infinite. The lemma follows. □

Let $G \in$ (IC). If every finite subgroup of G is the intersection of some infinite subgroups then every subgroup of G is a centralizer. Thus we can restrict our investigation to groups not having the former property. This excludes residually finite groups and, more generally, all groups in which every finite subgroup is contained in an infinite residually finite subgroup. The presence of an abelian subgroup not satisfying the minimal condition triggers this latter property in (IC)-groups.

Lemma 1.4. *Let $G \in$ (IC). If G has an abelian subgroup not satisfying* Min *then $\mathsf{C}(G) = \mathfrak{L}(G)$.*

Proof. If G is not periodic let A be an infinite cyclic subgroup of G. If, instead, G is periodic and B is an abelian subgroup of G not satisfying Min let A be the socle of B. In either case A is residually finite and there exists a strictly decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of (infinite) subgroups of A such that $\bigcap_{n\in\mathbb{N}} A_n = 1$. For all $n \in \mathbb{N}$ let $C_n = C_G(A_n)$. Since $A_n \in \mathsf{C}(G)$ for all $n \in \mathbb{N}$, the centralizer map induces an auto-duality in $\mathsf{C}(G)$ and $\bigcap_{n\in\mathbb{N}} A_n = 1$, we have that $\bigcup_{n\in\mathbb{N}} C_n = \langle C_n \mid n \in \mathbb{N}\rangle = G$. Let H be a finite subgroup of G. Then $H \leq C_n$ for some $n \in \mathbb{N}$, and so $\langle H, A_n\rangle = A_nH$ is an infinite residually finite group. Therefore H is intersection of infinite subgroups of A_nH (of finite index in A_nH) and so $H \in \mathsf{C}(G)$. This proves the lemma. □

Corollary 1.5. *Every* (IC)*-group is periodic.*

Proof. Let G be a nonperiodic (IC)-group. Then $\mathsf{C}(G) = \mathcal{L}(G)$, by Lemma 1.4. This is impossible for a nonperiodic group, as we observed in the introduction. □

Lemma 1.6. *Let S be an infinite subgroup of the* (IC)*-group G, and let $C = C_G(S)$. Then $\mathcal{L}(C/Z(C))$ is self-dual. If C is locally graded then it is soluble.*

Proof. Property (IC) yields $S = C_G(C)$, hence $S \cap C = Z(C)$, furthermore $[CS/S] \subseteq \mathsf{C}(G)$. For all $H \in [C/Z(C)]$ we have $H = SH \cap C \in \mathsf{C}(G)$, thus $[C/Z(C)] \subseteq \mathsf{C}(G)$. Moreover $C_G(CS) = S \cap C = Z(C)$, hence the centralizer map induces an anti-isomorphism $[CS/S] \to [C/Z(C)]$. But $CS/S \simeq C/Z(C)$, hence $\mathcal{L}(C/Z(C))$ is self-dual. It is straightforward to check that the duality of $\mathcal{L}(C/Z(C))$ obtained by composing the natural isomorphism $[C/Z(C)] \to [CS/S]$ with the centralizer mapping $[CS/S] \to [C/Z(C)]$ is the mapping $H \in [C/Z(C)] \mapsto C_C(H) \in [C/Z(C)]$. If C is locally graded then $C/Z(C)$ is locally graded (see [7]). Now the lemma follows from [8], Theorem E. □

Lemma 1.7. *Let G be an infinite, locally graded* (IC)*-group. Then either $\mathsf{C}(G) = \mathcal{L}(G)$ or G is a finite extension of a Prüfer group.*

Proof. By Corollary 1.5, G is periodic. In view of Lemma 1.4 and a well-known theorem by Šunkov (see, e.g., [6], vol I, pag. 98), we may assume that every locally finite subgroup of G is a Černikov group. Assume that G is not Černikov. Then G has some finitely generated, infinite subgroups. We claim that every such subgroup has finite centralizer. Let H be such a subgroup and let $S = C_G(H)$. Then $H = C_G(S)$, hence, if S is infinite, Lemma 1.6 yields that H is finite, a contradiction. Thus the claim is established. For the same H, the centralizer map induces a bijection from $[G/H]$ to $\mathsf{C}(G) \cap \mathcal{L}(S)$, hence $[G/H]$ is finite. Therefore G is finitely generated. Since G is locally graded it has a proper normal subgroup N_1 of finite index; since N_1 is finitely generated as well it also has a proper G-invariant subgroup of finite index. This suggests how to define a strictly decreasing sequence $(N_i)_{i\in\mathbb{N}}$ of normal subgroups of finite index in G. For all $i \in \mathbb{N}$, the centralizer $C_i := C_G(N_i)$ is finite, because N_i is infinite and finitely generated, and $C_i < C_{i+1}$, because the centralizer map restricted to the set of infinite subgroups of G is injective. Therefore $K := \bigcup_{i\in\mathbb{N}} C_i$ is an infinite, locally finite subgroup of G, hence it is Černikov. Let J be the finite residual of K. Then $J \lhd G$. There exists $n \in \mathbb{N}$ such that C_n contains the set X of all elements of J whose order is cube-free. By a theorem of Baer (see [6], Lemma 3.28), the only periodic automorphism of J fixing X elementwise is the identity, therefore $[N_n, J] = 1$. This is a contradiction, because $C_n = C_G(N_n)$ is finite. Therefore G is Černikov; let R be its finite residual. If R is not a Prüfer group then the set $\mathcal{S}$ of its infinite subgroups is infinite. The centralizer of each element of $\mathcal{S}$ lies in the finite interval $[G/R]$. However $\mathcal{S} \subseteq \mathsf{C}(G)$, hence the restriction of the centralizer map to $\mathcal{S}$ is injective. This is a contradiction, so R is a Prüfer group and G is Prüfer-by-finite. □

Thus from now on we need to consider Prüfer-by-finite groups only. Note that Prüfer-by-finite groups necessarily have finite subgroups which are not centralizers, as follows from an argument similar to that in the final part of the proof of Lemma 1.7. As a matter of fact, at most one of the proper subgroups of the Prüfer subgroup of a Prüfer-by-finite (IC)-group is a centralizer, as will be seen in Lemma 2.3.

In the special case when G is a 2-group the following easy remark will be of some use.

Lemma 1.8. *Let G be a 2-group, and suppose that G is a finite, noncentral extension of a normal subgroup $P \simeq \mathcal{C}_{2^\infty}$. Let $C = C_G(P)$ and $x \in G \setminus C$. Then $C_{C/P}(xP) = C_C(x)P/P$.*

Proof. Let $c \in C$ be such that $[c, x] \in P$. Then $[c, x] = b^2$ for some $b \in P$. It is clear that x acts on P like the inverting automorphism, thus $b^x = b^{-1}$. Hence $[bc, x] = b^{-2}[c, x] = 1$, so $bc \in C_C(x)$ and $c \in C_C(x)P$. □

2. Necessity

We fix the following notation that will be in use throughout this section. G is an (IC)-group with a (normal) subgroup $P \simeq \mathcal{C}_{p^\infty}$ of finite index (p a prime). We also let $C = C_G(P)$. We shall show that G satisfies one of the conditions (a–d) of the Theorem in the introduction. For a start, it is clear that G/C is cyclic and $|G/C| \leq 2$ if $p = 2$, while $|G/C|$ divides $p - 1$ otherwise.

The infinite subgroups of G are precisely those containing P. Also, P is a centralizer, hence $Z(G) \leq P$; actually $P = C_G\big(C_G(P)\big) = C_G(C) = Z(C)$. Of course, in the case when $Z(G) = P$ then the property that $G \in$ (IC) amounts to saying that G has the property that $\mathsf{C}(G) = [G/Z(G)]$ considered in [5, 1].

From the descriptions that will follow it is easy to see that the class (IC) is not closed under taking subgroups; however the following weaker property holds.

Lemma 2.1. *Let V be an infinite subgroup of G. Then $V \in$ (IC) if and only if $Z(V) \leq P$.*

Proof. Since V is infinite $P \leq V$. If $V \in$ (IC) then P is a centralizer in V, hence $Z(V) \leq P$. Conversely, if $Z(V) \leq P$ and U is an infinite subgroup of V then $Z(V) \leq P \leq U$ and so Lemma 1.1 yields $C_V\big(C_V(U)\big) = U$. Hence $V \in$ (IC). □

Lemma 2.2. *$C \in$ (IC). Moreover $C = R \times F$ where*

(i) *R is a p-group such that $P = Z(R)$, R/P has modular subgroup lattice and R' is cyclic;*
(ii) *F is of Gaschütz type;*
(iii) *if $P < R$ then p does not divide $|F|$.*

Proof. By Lemma 2.1, or by Lemma 1.2, $C \in$ (IC). Since PC' is a direct factor of C modulo C' there exists $K \leq C$ such that $C = PK$ and $P \cap K = P \cap C'$. As C' is finite, because C is centre-by-finite, K is finite as well; also note that $Z(K) = P \cap K$. Now $H \in [C/P] \mapsto H \cap K \in [K/P \cap K]$ is a bijection and any given $H \in [C/P]$ is a centralizer in C if and only if $H \cap K$ is a centralizer in K. Thus the fact that $C \in$ (IC) is equivalent to the property that every subgroup of K containing $Z(K) = P \cap K$ is a centralizer in K. Hence Reuther's theorem in [5] applies, thus $K = A \times B$ where A is abelian and B has a Hall decomposition

$$B = B_1 \times B_2 \times \cdots \times B_n, \tag{$*$}$$

where each of the subgroups B_i either has prime-power order (with B_i' cyclic and $B_i/Z(B_i)$ modular) or is a semidirect product of two cyclic groups of prime-power order and such that $B_i/Z(B_i)$ is of type $\mathfrak{pq}$. Recall that $Z(K) = P \cap K$ is a cyclic p-group, hence $A \leq P$ and either $A = 1$ or $Z(B) = 1$. In the latter case all factors B_i in $(*)$ are of type $\mathfrak{pq}$; if this happens then (i–iii) are satisfied if we let $R = P$ and $F = B$. Thus we may assume $Z(B) \neq 1$, hence $A = 1$. There is no loss of generality in further assuming $Z(B_1) \neq 1$. Since $Z(K)$ is a p-group this yields that p divides $|B_1|$. Therefore all factors B_i in $(*)$ with $i > 1$ are p'-groups, so they have trivial centres and hence are of type $\mathfrak{pq}$. If B_1 is a p-group let $R = PB_1$ and $F = B_2 \times \cdots \times B_n$; also in this case (i–iii) are satisfied, because $R/P \simeq B_1/Z(B_1)$ has modular subgroup lattice. The remaining possibility is that $B_1/Z(B_1)$ is of type $\mathfrak{pq}$, of order pq for some prime $q \neq p$. Since $Z(B_1)$ is a p-group any Sylow q-subgroup of B_1 has order q, hence $B_1 = \langle a, b\rangle$, where a has p-power order, $a^p \in Z(B_1) \leq Z(K) \leq P$ and $b^q = 1$. If $q < p$ then $\langle a\rangle \lhd B_1$. Since b centralizes $\langle a^p\rangle \neq 1$ this implies that b acts nilpotently on $\langle a\rangle$, which is a contradiction. Thus $q > p$, hence $\langle b\rangle \lhd B_1$. There exists $c \in P$ such that $c^{-p} = a^p$. Hence ca has order p and so $B_0 := \langle ca, b\rangle$ is of type $\mathfrak{pq}$. If we let $R = P$ and $F = B_0 \times B_2 \times \cdots \times B_n$, once again (i–iii) are satisfied. Now the proof is complete. □

Lemma 2.2 shows that if our group G is such that $C = G$, that is, $P = Z(G)$ then it satisfies either (a) or (b) of the Theorem. Therefore from now on we shall assume that $C < G$ or, equivalently, $Z(G) < P$.

Lemma 2.3. *If $C < G$ then $|G/C|$ is prime.*

Proof. If $p = 2$ then $|G/C| = 2$, so we may assume $p > 2$. The centralizer map induces an anti-isomorphism from $[G/C]$ to the set $\mathcal{S} = \mathsf{C}(G) \cap \mathcal{L}(P)$ of all subgroups of P which are centralizers in G. Let $X < P$ and suppose that $X \in \mathsf{C}(G)$. Then $Y := C_G(X) > C_G(P) = C$. But every element in $G \smallsetminus C$ acts fixed point freely on P, hence $X = C_G(Y) = C_P(Y) = 1$. Thus $\mathcal{S} = \{1, P\}$ and the result follows. □

For the rest of this section we let $q = |G/C| > 1$ and fix a q-element $x \in G \smallsetminus C$, so that $G = C\langle x\rangle$; further we let $L = C_C(x)$. Note that L is finite since $P \nleq L$. Also note that $C_P(x) = \Omega_1(P) = Z(G)$ has order 2 if $p = 2$, because in this case $b^x = b^{-1}$ for all $b \in P$, while $C_P(x) = Z(G) = 1$ if $p > 2$.

Lemma 2.4. *In the notation just established,*

(i) *$L = C_G(P\langle x\rangle)$ and $C_G(L) = P\langle x\rangle$;*
(ii) *$P\langle x^q\rangle = C_G(PL)$ and $C_G(P\langle x^q\rangle) = PL$;*
(iii) *$Z(L) = Z(G)\langle x^q\rangle = C_G(PL\langle x\rangle)$ and $C_G(Z(L)) = C_G(x^q) = PL\langle x\rangle$;*
(iv) *the centralizer map induces an anti-isomorphism from $[P\langle x^q\rangle/P]$ to $[C/PL]$.*

Proof. Clearly $C_G(P\langle x\rangle) = C_G(P) \cap C_G(x) = C_C(x) = L$. Moreover $P\langle x\rangle = C_G\big(C_G(P\langle x\rangle)\big) = C_G(L)$. Thus (i) is proved. Next, $C_G(PL) = C_G(P) \cap C_G(L) = C \cap P\langle x\rangle = P(C \cap \langle x\rangle) = P\langle x^q\rangle$, and (ii) follows. Now $C_G(PL\langle x\rangle) = C_C(L\langle x\rangle) = C_L(L) = Z(L)$. On the other hand $Z(L) = L \cap P\langle x\rangle$ by (i). Since $C \cap P\langle x\rangle = P\langle x^q\rangle$ we have $Z(L) = L \cap P\langle x^q\rangle = (L \cap P)\langle x^q\rangle = Z(G)\langle x^q\rangle$. Thus $PL\langle x\rangle = C_G\big(C_G(PL\langle x\rangle)\big) = C_G(Z(G)\langle x^q\rangle) = C_G(x^q)$ and also (iii) is proved. Finally, the intervals $[P\langle x^q\rangle/P]$ and $[C/PL]$ are contained in $\mathsf{C}(G)$, hence (ii) shows that the former is mapped onto the latter by the the centralizer map, and (iv) follows. □

We fix yet another piece of notation: we let R be the (uniquely determined) subgroup defined in Lemma 2.2. We shall consider the two cases $R = P$ (that is, $Z(C/P) = 1$) and $R > P$ (that is, $Z(C/P) \neq 1$) separately. The former is settled by the next lemma, showing that in this case G has the structure described in (c) of the Theorem.

Lemma 2.5. *Let $C < G$ and suppose that $P = R$. Then $C = P \times F$ where F has a Hall decomposition $F_1 \times F_2 \times \cdots \times F_n$ whose factors are of type $\mathfrak{pq}$, and one of the following holds:*

(1) *$[F, x] = 1$ and q does not divide $|F|$; moreover either $x^q = 1$ or $p = q = 2$ and x^2 is the element of order 2 in P;*
(2) *$1 \neq x^q \in F_1$ and $[F_2F_3 \cdots F_n, x] = 1$;*
(3) *$p = q = 2$ and $F_1 = \langle y\rangle \rtimes \langle h\rangle$ is dihedral of order $2r$, where r is a prime congruent to 1 modulo 4, $x^2 = uh$, where u is an element of order 4 in P, and x acts on F as follows:*
   - *$[h, x] = u^2$;*
   - *x induces an automorphism of order 4 on $\langle y\rangle$;*
   - *$[F_2F_3 \cdots F_n, x] = 1$.*

Proof. Let F and its factors F_1, F_2, …, F_n be defined as in Lemma 2.2, so that $C = P \times F$. Recall that $\langle x^q\rangle = C \cap \langle x\rangle$. Since x is a q-element and the factors F_i have pairwise coprime orders, either $x^q \in P$ (and hence $[F, x^q] = 1$) or $x^q \in PF_j \setminus P$ (hence $F_j \cap P\langle x\rangle \neq 1$) for exactly one $j \in \{1, 2, \dots, n\}$, in which case $F_i \cap P\langle x\rangle = 1$ and $[F_i, x^q] = 1$ for all $i \neq j$. We also know, from Lemma 2.4(ii), that $C_G(P\langle x^q\rangle) = C_C(x^q) = PL$. Thus $F_i \leq PL$ for all $i \in \{1, 2, \dots, n\}$ such that $F_i \cap P\langle x\rangle = 1$. Our first aim is showing that, at the expense of redefining one of the factors F_i, if needed, the following stronger statement holds:

$$\text{for all } i \in \{1, 2, \dots, n\}, \quad \text{if } F_i \cap P\langle x\rangle = 1 \text{ then } q \text{ does not divide } |F_i| \text{ and } F_i \leq L. \qquad (*)$$

Let $i \in \{1, 2, \dots, n\}$ be such that $F_i \cap P\langle x\rangle = 1$. Then, as already remarked, $F_i \leq PL$. By the same remark, $[C, x] \leq PJ$, where J is the (only) factor F_j such that $F_j \cap P\langle x\rangle \neq 1$, if there is such a factor, and $J = 1$ otherwise (it is clear that PJ is x-invariant); hence $F_i \cap [C, x] = 1$. Suppose that q divides $|F_i|$, so that F_i has an element g of order q. Let $H = P\langle xg\rangle$. If $x \in H$ then $g \in H$ and so $g \in P\langle x\rangle$, because H/P is a cyclic q-group, $x \notin P$ and $g^q = 1$. This is impossible, as $F_i \cap P\langle x\rangle = 1$, hence $x \notin H$. But $H = C_G\big(C_G(H)\big)$, since $G \in$ (IC), hence there exists $c \in C_G(H)$ such that $[c, x] \neq 1$; clearly $C_G(H) \leq C$, so $c \in C$. Now $[c, xg] = 1$, hence $1 \neq [c, x] = [c, g]^{-g^{-1}}$. This is a contradiction, as $[c, g] \in F_i$ and $F_i \cap [C, x] = 1$. Therefore q does not divide $|F_i|$. Next, if $F_i \nleq L$ there exists $f \in F_i \setminus L$, but $f = bl$ for suitable $b \in P$ and $l \in L$. Now $f \notin \langle b^{-1}f\rangle \leq L$; since $[b^{-1}, f] = 1$ and f has prime order it follows that f is a p-element, hence $f^p = 1$ and $b^{-p} = (b^{-1}f)^p \in P \cap L$. Also, since q does not divide $|F_i|$ it follows that $p \neq q$, hence $p > 2$, so that $P \cap L = 1$ and $b^p = 1$. Now $|F_i| = pt$ for some prime $t \neq p$. If $\langle f\rangle \lhd F_i$ then F_i is generated by t-elements, since it is of type $\mathfrak{pq}$. In this case $F_i = \langle g \in C \mid g^t = 1\rangle \lhd G$, so $[F_i, x] \leq F_i \cap [C, x] = 1$ and $F_i \leq L$. Otherwise, the proper nontrivial normal subgroup $\langle k\rangle$ of F_i has order t, hence $F_i = \langle k\rangle \rtimes \langle f\rangle$. Then $F_i \simeq F_i^* := \langle k, b^{-1}f\rangle$ and $PF_i = PF_i^*$. Moreover $\langle k\rangle \lhd G$, because $\langle k\rangle$ is a normal Sylow subgroup of C, hence $[k, x] \in \langle k\rangle \cap P = 1$ (recall that $F_i \leq PL$, hence $[F_i, x] \leq P$), thus $k \in L$ and $F_i^* \leq L$. Therefore we may substitute F_i^* for F_i (and change F accordingly) to establish $(*)$.

Now, $\langle x\rangle \cap P = 1$ if $p > 2$, as $q \neq p$ in this case, and $|\langle x\rangle \cap P| \leq 2$ if $p = 2$, since x induces the inverting automorphism on P in this second case. Therefore $(*)$ yields (1) if $x^q \in P$. Suppose that $x^q \notin P$. Up to relabelling the subgroups F_i we may assume that $x^q \in PF_1$, hence $[F_2F_3 \cdots F_n, x] = 1$ by $(*)$. If $p > 2$ then $x^q \in F_1$, because all q-elements of PF_1 are in F_1. In this case, and also if $p = 2$ and $x^2 \in F_1$, (2) is satisfied. In the remaining case $p = q = 2$ and F_1 is a dihedral group, $x^2 = uh$ where $1 \neq u \in P$ and h is an element of order 2 in F_1. We have $F_1 = \langle y\rangle \rtimes \langle h\rangle$ for some y. If $u^2 = 1$ then $x^4 = 1$ and $F_1^* := \langle y, x^2\rangle \simeq F_1$; as above we may replace F_1 with F_1^* so that we are reduced to the previous case and (2) holds. Finally, suppose $u^2 \neq 1$. As $x^2 = uh$ we have $1 = [uh, x] = u^{-2}[h, x]$, because $u^x = u^{-1}$ and $[u, h] = 1$. Hence $[h, x] = u^2$ and $1 = [h^2, x] = [h, x]^2 = u^4$. Thus u has order 4. Also, if r is the order of y then $\langle y \rangle$ is the only Sylow r-subgroup of C, hence $\langle y \rangle \lhd G$, and x induces on $\langle y \rangle$ an automorphism, of order 4 because $[x^2, y] = [h, y] \neq 1$ and $x^4 = u^2 \in Z(G)$; hence $r \equiv 1 \mod 4$. Then (3) holds in this case. The proof is complete. □

Remark. Cases (1–3) of Lemma 2.5 are mutually exclusive. Indeed, in case (1) all q-elements $g \in G \smallsetminus C$ satisfy $g^q \in P$, which is false in cases (2) and (3). Also, $X := \Omega_1(P)F \lhd G$, as X is the set of all elements of square-free order in C. To exclude that G can satisfy (2) and (3) at same time note that, if $q = 2$ and $g = xbf \in G \smallsetminus C$, where $b \in P$ and $f \in F$, then $g^2 \equiv (xb)^2 = x^2$ modulo X. Thus $g^2 \in X$ in case (2) but $g^2 \notin X$ in case (3).

Now we turn our attention to the case in which $P < R$. The structure of (IC)-groups of this type is much more restricted than in the previous case. Our next Lemma (together with Lemma 1.3) shows that their study reduces to the case of 2-groups.

Lemma 2.6. *Let $C < G$ and suppose $P < R$. Then $p = 2$ and $G = R\langle x \rangle \times F$, where $R\langle x \rangle$ is a 2-group and F is a group of Gaschütz type and odd order.*

Proof. Let R and F be as in Lemma 2.2, so $R \times F$ is a Hall decomposition of C. It follows that $L = (R \cap L) \times (F \cap L)$, hence $Z(R \cap L) \leq Z(L)$. If $p > 2$ then $Z(G) = 1$ and $Z(L) \leq \langle x \rangle$ by Lemma 2.4(iii). Thus $Z(R \cap L) \leq \langle x \rangle$, but $q \neq p$, hence $Z(R \cap L) = 1$. Since R is nilpotent $R \cap L = 1$ and $L \leq F$. Thus $R \leq C_G(L) = P\langle x \rangle$ by Lemma 2.4(i). This is a contradiction, because $P < R$ and p does not divide $|P\langle x \rangle / P|$. Therefore $p = 2$, hence $q = 2$, so $R\langle x \rangle$ is a 2-group while $|F|$ is odd. Then $C_F(L) = F \cap P\langle x \rangle = 1$ (see Lemma 2.4(i)). As $L = (L \cap R) \times (L \cap F)$ it follows that $C_F(L \cap F) = 1$. But $\mathcal{L}(F) = \mathsf{C}(F)$, hence $L \cap F = C_F\big(C_F(L \cap F)\big) = F$, that is, $[F, x] = 1$. The lemma follows. □

So we only need to consider the case when G is a 2-group. In this case $R = C$ and C/P has modular subgroup lattice, by Lemma 2.2. Our first aim is showing that this quotient actually is abelian.

Lemma 2.7. *Suppose that G is a 2-group and $C < G$. Then C/P is abelian.*

Proof. By Lemma 2.2, C' is cyclic. Suppose that $C' \not\leq P$. Let bars denote images modulo P. The centralizer map induces an anti-isomorphism of $[C/P]$ onto itself, hence Proposition 5.2 of [8] shows that there exist $u, v \in C$ such that, after setting $K = P\langle u, v \rangle$, we have $\bar{K} = \langle \bar{u} \rangle \rtimes \langle \bar{v} \rangle$, where $\bar{u}$ and $\bar{v}$ have the same order 2^n and $[\bar{u}, \bar{v}] = \bar{u}^{2^s}$ for some integer s such that $2 \leq s < n$; moreover $\bar{C} = \bar{K} \times \bar{T}$ where $T = C_G(K)$, and $\bar{T}$ is abelian of exponent at most 2^s. We may actually assume $K' \leq \langle u \rangle$. For, K' is cyclic and $K'P = \langle u^{2^s} \rangle P$, hence $K' = \langle bu^{2^s} \rangle$ for some $b \in P$. There exists $z \in P$ such that $z^{2^s} = b$, hence $\langle (zu)^{2^s} \rangle = K'$ and we may substitute zu for u. Now $\langle u \rangle \cap P = \langle u \rangle \cap Z(C) = C_{\langle u \rangle}(v)$, because $[u, PT] = 1$. Since $\langle [u, v] \rangle = \langle u^{2^s} \rangle$ it follows that $|P \cap \langle u \rangle| = 2^s$. As $K' \leq \langle u \rangle$ and $K' \not\leq P$ we have $P \cap \langle u \rangle < K'$, hence $P \cap \langle u \rangle = P \cap K'$. Since C' is cyclic and $K' \leq C'$ a similar argument yields $P \cap C' = P \cap K'$. Hence $P \cap C' = P \cap \langle u \rangle$. Then $|P \cap C'| > 2$, as $s > 1$. Now $L' \leq C'$ and $|L \cap P| = 2$. Therefore $P \cap C' \not\leq L'$ and so $L' \leq P \cap C'$, hence $|L'| \leq 2$; as a consequence $L^2 \leq Z(L) \leq P\langle x^2 \rangle$—see Lemma 2.4(iii).

Let 2^m be the order of x^2 modulo P. Then $m \leq n$, because $2^n = \exp(\bar{C})$. Also, from Lemma 2.4(iv) it follows that $2^m = |C : PL|$. Let $X = PL \cap K$ and $Y = P\langle x^2 \rangle \cap K$. Then $|\bar{Y}| \leq 2^m$ and $|K : X| = |KL : PL| \leq 2^m$. Next, $X^2 \leq Y$, because $L^2 \leq P\langle x^2 \rangle$, hence $|X/Y| \leq 4$, because K/P is metacyclic. Thus $2^{2n} = |\bar{K}| \leq 2^{2m+2}$ and $n \leq m + 1$. Moreover, if $n = m + 1$ then $|X/Y| = 4$ and $Y = P\langle x^2 \rangle$, hence $X^2 = P\langle x^2 \rangle$ and $x^2 \in g^2 P$ for some $g \in L$ (note that X/P is abelian, as $L' \leq P$). In this case $x_1 := xg^{-1} \in G \smallsetminus C$ and $x_1^2 \in P$. By repeating the argument in this paragraph with x_1 in place of x (and hence 0 in place of m) we obtain $n \leq 1$. But then $|K/P| \leq 4$ and $\bar{K}$ is abelian, a contradiction. Hence $m = n$. We have proved that all elements of $G \smallsetminus C$ have the same order, namely 2^{n+1}, modulo P.

In particular, $x^2 P$ has order 2^n, hence $x^2 = u^i v^j t$, where $t \in T$ and the integers i, j are not both even. If i is odd let $u_1 = u^i t$ and $K_1 = P\langle u_1, v \rangle$. Then $\bar{u}_1$ has order 2^n and $[\bar{u}_1, \bar{v}] = \bar{u}^{i2^s} = \bar{u}_1^{2^s}$, hence $\bar{K}_1 \simeq \bar{K}$. Moreover $C_K(K_1) = P$ because $[t, K] = 1$ and $Z(K) = K \cap T = P$, but $\Omega_1(\bar{K}_1) = \Omega_1(\bar{K})$; it follows that $Z(K_1) = P$. Hence $K_1 \cap T_1 = P$, where $T_1 = C_G(K_1)$. As the centralizer map induces a duality in $[C/P]$ we have $|C : T_1| = |K_1/P|$, hence $\bar{C} = \bar{K}_1 \times \bar{T}_1$. Also, $\bar{C}'\bar{C}^{2^s} = \langle \bar{u}^{2^s}, \bar{v}^{2^s} \rangle \leq \bar{K}_1$, hence $\bar{T}_1'\bar{T}_1^{2^s} = 1$. This shows that we may replace u with u_1 in our argument. After this substitution $x^2 = uv^j \in K$. In the other case, when i is even, j is odd. In this case we let $v_1 = vt^{j^*}$, where $jj^* \equiv 1$

(mod 2^s), so that $x^2 = u^i v_1^j$. Arguing as for the previous case we see that there is no loss of generality in substituting v_1 for v. Thus, in either case, we may assume $x^2 \in K$.

Now we have $PL = C_C(x^2) \geq C_C(K) = T$, by Lemma 2.4(ii). Hence $[T,x] \leq [PL,x] = P \leq T$, so $T \lhd G$ and $K = C_G(T) \lhd G$. Also, $C_G(K\langle x\rangle) \leq T$ and $K\langle x\rangle \cap C = K\langle x^2\rangle = K$, hence $Z(K\langle x\rangle) \leq T \cap K = P$. Therefore $K\langle x\rangle \in$ (IC) by Lemma 2.1, so we may assume that $G = K\langle x\rangle$ and so $C = K$. Then $L \leq P\langle x^2\rangle$, because $|K/P| = 2^{2n}$ and $2^n = |P\langle x^2\rangle/P| = |C : PL|$, as we already observed. All elements of $G \smallsetminus K$ have order 2^{n+1} modulo P, and $\exp \bar{K}^2 = 2^{n-1}$. Thus G/K^2 is a group of order 8 in which the elements of order at most 2 form the subgroup K/K^2 of order 4. It follows that G/K^2 is abelian. Since $s > 1$ the factor K/K^4 is abelian—and likewise $K^{2^i}/K^{2^{i+2}}$ for all $i \in \mathbb{N}$. By an easy induction (or by a standard property of powerful p-groups, see for instance [3], Lemma 2.4(ii)) it follows that $[K^{2^i}, x] \leq K^{2^{i+1}}$ for all $i \in \mathbb{N}$. In particular x centralizes $K^{2^{n-1}}/K^{2^n} = \Omega_1(\bar{K})$, which is not cyclic. But $\bar{L} = \langle \bar{x}^2\rangle$, so this is a contradiction by Lemma 1.8. □

Lemma 2.8. *Suppose that G is a 2-group and $C < G$. Then G/P is elementary abelian.*

Proof. Let bars denote images modulo P. Then $\bar{C}$ is abelian by Lemma 2.7 and the mapping $\bar{c} \in \bar{C} \mapsto [\bar{c}, \bar{x}] \in \bar{G}'$ is an epimorphism with kernel $\bar{L} = C_{\bar{C}}(\bar{x})$ (see Lemma 1.8), hence $\bar{G}' = [\bar{C}, \bar{x}] \simeq C/PL$. Now, Lemma 2.4(iv) shows that C/PL is cyclic of order 2^λ, where $2^{\lambda+1}$ is the order of $\bar{x}$. Thus $\bar{G}' \simeq \mathcal{C}_{2^\lambda}$. This argument applies to all elements of $G \smallsetminus C$ in place of x, thus showing that they all have order $2^{\lambda+1} = 2|\bar{G}'|$ modulo P.

If $\lambda = 0$ then $\bar{G}$ is abelian and all elements in $\bar{G} \smallsetminus \bar{C}$ have order 2, hence $\bar{G}$ is elementary abelian. Then we may assume that $\lambda > 0$; we shall derive a contradiction. Since $\bar{G}$ is not abelian, $\bar{x}^2 \in \bar{L} = C_{\bar{C}}(\bar{x}) = Z(\bar{G})$. This holds for every element of $G \smallsetminus C$ in place of x, hence we have $g^2 \in PL$ for all $g \in G \smallsetminus C$. Since C/PL is cyclic it follows that G/PL is dihedral, of order $2^{\lambda+1}$. Also, $L' \leq L \cap P$, hence $|L'| \leq 2$. Thus $L^2 \leq Z(L) = (L \cap P)\langle x^2\rangle$ (see Lemma 2.4(iii)). If $\bar{x}^2 \in \bar{L}^2$ then $\bar{x}^2 = \bar{l}^2$ for some $l \in L$ and $(l^{-1}x)^2 \in P$. This is false, as $l^{-1}x \notin C$ and $\lambda > 0$. Therefore $\bar{x}^2 \notin \bar{L}^2$ and $\exp \bar{L} = 2^\lambda$. Let c be such that $C = PL\langle c\rangle$. Then $C' = L'[L,c] \leq P$. But $[L,c,L] = 1$, hence $1 = [L^{2^\lambda}, c] = [L,c]^{2^\lambda}$ and $|C'| \leq 2^\lambda$. On the other hand $\exp(C/P) = \exp(C/Z(C)) = |C'|$. Thus $\exp \bar{C} = 2^\lambda$. Now $c^x = c^{-1}y$ for some $y \in PL$, because G/PL is dihedral. Moreover, $\bar{G}' = \langle [\bar{c}, \bar{x}]\rangle$ has order 2^λ. Since $[c,x] = c^{-2}y$ and $\bar{c}^{2^\lambda} = 1$ also $\bar{y}$ has order 2^λ. We have $\bar{L} = \langle \bar{x}^2\rangle \times \bar{E}$, where $\bar{E}^2 = 2$, because $\bar{x}^2$ has order $2^\lambda = \exp \bar{L}$ and $\bar{L}^2 \leq \langle \bar{x}^2\rangle$. Hence $\bar{y} = \bar{x}^{2t}\bar{e}$ for some odd integer t and $e \in E$. Then $(\bar{x}\bar{c})^2 = \bar{x}^2\bar{c}^{\bar{x}}\bar{c} = \bar{x}^2\bar{c}^{-1}\bar{y}\bar{c} = \bar{x}^{2(t+1)}\bar{e} \in \bar{C}$. Now $\bar{x}\bar{c}$ has order $2^{\lambda+1}$, since $xc \notin C$. Hence $1 \neq (\bar{x}\bar{c})^{2^\lambda} = (\bar{x}^{2(t+1)}\bar{e})^{2^{\lambda-1}} = \bar{e}^{2^{\lambda-1}}$, because $t+1$ is even. It follows that $\lambda = 1$. Then $\bar{C}^2 = 1$ and $|\bar{G}'| = 2$. Hence $\bar{G}$ is a central product of an abelian group and an extraspecial group. But the elements of order 2 in $\bar{G}$ generate the elementary abelian group $\bar{C}$, hence $\bar{G}$ has no dihedral subgroup of order 8; moreover, the product of any two elements of order 4 in $\bar{G}$ lies in $\bar{C}$, hence it has order at most 2, thus $\bar{G}$ has no quaternion subgroup of order 8. This is a contradiction, hence the proof is complete. □

Together with Lemmas 1.3 and 2.6 the previous lemma shows that in the case when $P < R$ our group G has the structure described under (d) in the Theorem in the introduction. At this point, thanks to the previous results in this section and to Lemma 1.7 the proof of the necessity part of the Theorem is complete. The proof of the sufficiency part will be the content of the next section.

3. Sufficiency

In this section we shall complete the proof of our main result by showing that the groups described in the Theorem are in fact (IC)-groups.

Let G be a group with a (normal) subgroup $P \simeq \mathcal{C}_{p^\infty}$ such that G/P is finite. Consider first the case when G has the structure described in case (a) or (b) of the Theorem. Then $G = PK$ for some finite $K \leq G$, and $P = Z(G)$. Now $K' = G'$ is cyclic and $Z(K) = P \cap K$, hence $K/Z(K) \simeq G/P$ has modular subgroup lattice. By the main result of [5], $[K/Z(K)] = \mathsf{C}(K)$. Now the argument in the first paragraph of the proof of Lemma 2.2 shows that $G \in$ (IC).

If G satisfies (c) then $C = P \times F$ has the structure described under (a), hence $C \in$ (IC) by the previous paragraph. Moreover $P = Z(C) = C_G(C)$ and it follows from Lemma 1.2 that $[C/P] \subseteq \mathsf{C}(G)$. Hence, to prove that $G \in$ (IC) we only have to consider the subgroups H containing P but not contained in C. The nontrivial Sylow subgroups of G/P have prime order, with the possible exception of the Sylow q-subgroups, which are cyclic anyway. If follows that if $P \leq H \leq G$ but $H \nleq C$ then H/P contains a Sylow q-subgroup of G/P and Hall theory shows that H is intersection of maximal subgroups of G. Thus it is enough to show that every maximal subgroup M of G different from C is a centralizer. To this end we only have to show that $Z(M) \nleq Z(G)$: in this case $M \leq C_G\big(C_G(M)\big) < G$ and so $M = C_G\big(C_G(M)\big)$. Every such maximal subgroup M has prime index $t \neq q$; it is conjugate to a subgroup of the form $P\langle x\rangle K$, where K is a maximal subgroup of F, so we may assume that $M = P\langle x\rangle K$ for such a K. There exists exactly one $i \in \{1, 2, \dots, n\}$ such that t divides $|F_i|$; then $|F_i : F_i \cap K| = t$ and $|F_i \cap K|$ is a prime. Let $\langle g\rangle = F_i \cap K$. Then $K = \langle g\rangle \times \langle F_j \mid i \neq j \in \{1, 2, \dots, n\}\rangle$, so $[g, PK] = 1$. If $i > 1$, or also in case (c1), where $[x, F] = 1$, we have $[g, x] = 1$, hence $g \in Z(M)$. Since $g \notin Z(G) \leq P$ we see that $Z(M) \nleq Z(G)$ in these cases. Otherwise, $i = 1$ and either of (c2) and (c3) holds; then $x^q \in Z(M) \smallsetminus Z(G)$. Hence, in any case, $Z(M) \nleq Z(G)$ and M is a centralizer in G. Therefore $G \in$ (IC).

Finally, suppose that (d) holds. By Lemma 1.3 we may assume that $G = G_1$. Then every infinite subgroup of G is intersection of maximal subgroups, so it will be enough to show that every maximal subgroup M of G is a centralizer. As above, this amounts to saying that $Z(M) \nleq Z(G)$. Since $P \simeq \mathcal{C}_{2^\infty}$ we have that $|G/C| = 2$. We may assume that $M \neq C$, hence $M = K\langle x\rangle$ where K is maximal in C and $x \in G \smallsetminus C$. Lemma 1.8 shows that $C = PL$, where $L = C_C(x)$. Since $P = Z(C)$ it follows that $Z(L) = P \cap L$. Moreover $L' \leq P \cap L$ and $|P \cap L| = 2$, therefore L is extraspecial. Now $K = P(L \cap K)$ and since L is extraspecial $|Z(L \cap K)| > 2$, so $Z(L \cap K) \nleq P$. But $Z(L \cap K) \leq Z(M)$ and $Z(G) \leq P$, hence $Z(M) \nleq Z(G)$. This shows that $G \in$ (IC) also in this latter case. Now the proof of the Theorem is complete.

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