Subgroups defining automorphisms in locally nilpotent groups

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Abstract: We investigate some situation in which automorphisms of a group G are uniquely determined by their restrictions to a proper subgroup H. Much of the paper is devoted to studying under which additional hypotheses this property forces G to be nilpotent if H is. As an application we prove that certain countably infinite locally nilpotent groups have uncountably many (outer) automorphisms.

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Homomorphisms of groups are defined by their restriction to any generating set of their domain. This property actually characterizes generating subsets of groups, for if H is a proper subgroup of the group G then there are two different homomorphisms from G to the same group K whose restrictions to H are the same: a simple construction due to Eilenberg and Moore is suggested as Exercise 3.35 in [10], p. 54.

The situation can be quite different if—rather than referring to all homomorphisms of domain a group G—we restrict attention to, say, endomorphisms of G only. For instance, if G is isomorphic to the rational group $\mathbb{Q}$ and g is any nontrivial element of G, then two endomorphisms of G coincide if they agree on g, in other words endomorphisms of G are uniquely determined by their restrictions to $\{g\}$. This suggests the following definition. Let G be a group and let Γ be a set of endomorphisms of G. We say that a subset X of G is a *Γ-basis* if and only if, for every $\alpha, \beta \in \Gamma$, it holds $\alpha = \beta$ if $\alpha_{|X} = \beta_{|X}$, where $\alpha_{|X}$ and $\beta_{|X}$ denote restrictions to X. We shall also use expressions like 'End-basis', 'Aut-basis' or 'Inn-basis of G' as synonym with $\operatorname{End} G$-, $\operatorname{Aut} G$-, or $\operatorname{Inn} G$-basis respectively. For instance, the above example can be rephrased by saying that in the rational group every one-element subset different from the identity subgroup is an End-basis. More generally, every maximal independent subset of a torsion-free abelian group A is an End-basis of A.

The Aut-bases of a group G are just the bases of $\operatorname{Aut} G$ viewed as a permutation group on G, whence the name. Indeed, it is clear that for any $\Gamma \leq \operatorname{Aut} G$ a subset X of G is a Γ-basis if and only if $C_\Gamma(X) = 1$. In particular, the Inn-bases of a group G are the subsets X of G such that $C_G(X) = Z(G)$.

Other self-evident facts about Γ-bases (for a set Γ of endomorphisms of a group G) are that if X is a Γ-basis then X_1 is a Γ_1-basis for any subset Γ_1 of Γ and any superset X_1 of X contained in G. Also, X is a Γ-basis if and only if $\langle X \rangle$ is a Γ-basis, so the property of being a Γ-basis could be equivalently defined as an embedding property for subgroups. In this paper we shall assume this point of view and discuss some instances of the general problem of when a group G inherits group theoretical properties from a subgroup of G which is a Γ-basis, for some specific $\Gamma \subseteq \operatorname{End} G$. For example, it is immediate to observe that a group is abelian if it has an abelian subgroup which is an Inn-basis (Lemma 1.7). We will be mainly concerned with the case $\Gamma = \operatorname{Aut} G$.

A drastically restrictive result is that a direct power of every centreless group can be embedded as a normal subgroup which is an Aut-basis in a group with rather arbitrary structure (see Corollary 1.5). This is the reason why we turn our attention to group classes without nontrivial centreless groups, and mainly study nilpotent (sub)groups.

Even the property of having a nilpotent subgroup as an Aut-basis seems not to be very strong by itself. We give several examples of non-nilpotent groups with a nilpotent subgroup as an Aut-basis; such groups may even be locally nilpotent and the subgroup may satisfy various embedding conditions. However, it is possible to obtain some information in the positive. The main results in the first two sections of this paper are the following. Assume that H is a subgroup and an Aut-basis of the group G. If G is nilpotent then it has the same nilpotency class as H (see Theorem 1.9). A sufficient condition for G to be nilpotent is that H is a subnormal nilpotent subgroup of the hypercentre of G (Theorem 1.13). Another sufficient condition is given in Theorem 2.6: if G is locally nilpotent and H is not only an Aut-basis of G but also an Aut-basis of every subgroup of G containing it, then G is hypercentral, or nilpotent, provided H has the same property. In this case, also, the hypercentral lengths of H and G coincide.

A condition of special interest to us is when the basis considered is finite, or, equivalently, the subgroup considered is finitely generated. Results about the existence of finite Aut-bases in groups (or other structures) have appeared in the literature. By a theorem in model theory due to D.W. Kueker [4] a countable group G has a finite Aut-basis if and only if $\operatorname{Aut} G$ is countable, and if this fails to happen then $|\operatorname{Aut} G| = 2^{\aleph_0}$ (a proof only using the language of group theory is in [1], Lemma III.1). A related result by R. Baer is also in [1], Satz III.4: every quotient of a group G has countable automorphism group if and only if G is countable and every quotient of G has a finite Aut-basis. Relevant examples of groups with a finite Aut-basis are the groups with only finitely many automorphisms. Such groups have been the subject of several investigations (see for instance [8]). Among them there exist torsion-free abelian groups of very high cardinality (more precisely, of any arbitrary infinite cardinality less than the first strongly inaccessible cardinal) whose only endomorphisms are the mappings $x \mapsto x^n$ for integers n, so that the only non-trivial automorphism is the inverting automorphism. Of course the singleton of any non-trivial element of such a group is an End-basis of it.

Locally nilpotent groups with a finite Aut-basis are the subject of the third and last section of this paper. A special case of Theorem 3.4 is that a locally nilpotent group has a finite subgroup as an Aut-basis if and only if it is finite. From the same Theorem 3.4 we also draw some consequences on the existence of outer automorphisms. It is a theorem by O. Puglisi [7] that every periodic countably infinite locally nilpotent group has $2^{\aleph_0}$ (outer) automorphisms. We extend this result to countable periodic-by-(finitely generated) locally nilpotent groups which are not finitely generated (Corollary 3.6).

1. First results and examples

For convenience of further reference we start by stating again an observation already made in the introduction.

Lemma 1.1. *A subset X of a group G is an Inn-basis if and only if $C_G(X) = Z(G)$. In particular, the latter equality holds if X is an Aut-basis.*

Lemma 1.2. *Let the subgroup H be an Aut-basis of the group G and let $N = H^G$. Then* $\operatorname{Hom}\big(G/N, Z(H)\big) = 0$.

Proof — The group of all automorphisms of G acting trivially on both N and G/N is trivial since N is an Aut-basis of G. As is well-known (see [9], p. 66, for instance), this group is isomorphic to the group $\operatorname{Der}\big(G/N, Z(N)\big)$ of all derivations from G/N to $Z(N)$. Now, Lemma 1.1 yields $\operatorname{Der}\big(G/N, Z(N)\big) = \operatorname{Hom}\big(G/N, Z(N)\big)$, so the latter group is trivial as well. Finally $Z(H) = Z(G) \cap H \leq Z(N)$ by Lemma 1.1, hence $\operatorname{Hom}\big(G/N, Z(H)\big) = 0$. □

A special case of the next lemma is the well-known fact that the property of being an Aut-basis and the property of being an Inn-basis are equivalent for a normal subset of a centreless group.

Lemma 1.3. *Let N be a normal subgroup of a group G containing $Z(G)$. Then N is an Aut-basis of G if and only if $C_G(N) = Z(G)$ and* $\mathrm{Hom}\big(G/N, Z(G)\big) = 0$.

Proof — The two previous lemmas ensure that the condition is necessary. Conversely, assuming that it holds, we have to check that $\Gamma := C_{\mathrm{Aut}\,G}(N)$ is trivial. By the Three Subgroup Lemma (see for instance [5], Lemma 3.2) $[G, \Gamma, N] = 1$, so, by hypothesis, $[G, \Gamma] \leq C_G(N) = Z(G) \leq N$. Thus Γ acts trivially on both N and G/N. Hence Γ embeds in $\mathrm{Der}\big(G/N, Z(N)\big) = \mathrm{Hom}\big(G/N, Z(G)\big) = 0$.□

Now, if $N \lhd G$ and $C_G(N) = 1$ then $C_H(N) = 1 = Z(H)$ for all subgroups H of G containing N. Hence N is an Aut-basis of every such H. This situation suggests the following definition.

Let us say that a subset X of a group G is a *hereditary Aut-basis* of G if it is an Aut-basis of every subgroup of G containing it. (This is still a property that generalizes the property of being a generating set.)

Thus Lemma 1.3 shows that a normal subgroup of a group G whose centralizer in G is trivial is a hereditary Aut-basis. This is the statement we will make use of, however we note that from Lemma 8.1.1 of [6] the following stronger result follows:

Lemma 1.4. *Let H be an ascendant subgroup of a group G such that $C_G(H) = 1$. Then H is a hereditary Aut-basis of G.*

Therefore every centreless group embeds as an ascendant subgroup and hereditary Aut-basis in a complete group $\hat{H}$, by means of the automorphism tower construction. If H is finite then $\hat{H}$ is also finite and H is subnormal in $\hat{H}$. Another embedding, of a group closely related to a given centreless group, as a normal hereditary Aut-basis in a much more arbitrary group is given by the following corollary.

Corollary 1.5. *Let H and K be groups and assume $Z(H) = 1$. Then there exists a group G containing a normal subgroup N such that:*
- (*i*) *N is isomorphic to the direct product of κ copies of H, where $\kappa = \max\{\aleph_0, |K|\}$;*
- (*ii*) *$G/N \simeq K$;*
- (*iii*) *N is a hereditary Aut-basis of G.*

Proof — If K is finite let K^* be a countably infinite group in which K is embedded, otherwise let $K^* = K$. Hence $\kappa = |K^*|$. Now let G^* be the standard wreath product of H by K^* and let N be its base subgroup. In view of Lemma 1.4 the required conditions are satisfied if we set $G = NK$. □

Now, we are looking for statements of the following kind: "if the group G has a subgroup H which is an Aut-basis and satisfies some group theoretical condition $\mathfrak{X}$ then G itself satisfies $\mathfrak{X}$". Possibly we could have to strengthen the hypothesis by also requiring some embedding property for H, or that G satisfies some condition weaker than $\mathfrak{X}$. Corollary 1.5 suggests that reasonable candidates for $\mathfrak{X}$ in order that such a statement be true—at least under the hypothesis that H is normal in G—are finiteness conditions and conditions, like nilpotency or hypercentrality, which exclude nontrivial centreless groups.

Rather than on finiteness conditions our interest will be focused on nilpotent subgroups as Aut-bases. With regard to finiteness conditions we only recall how well-known results on groups of automorphism of groups satisfying finiteness conditions translate into analogous results about normal Inn-bases, and observe that normality is necessary here. Indeed, if the normal subgroup N of a group G is an Inn-basis then $G/Z(G) = G/C_G(N)$ embeds in $\mathrm{Aut}\,N$. For instance, if N is polycyclic and G is a radical group, then G is centre-by-polycyclic. In particular, *every locally nilpotent group with a finitely generated normal subgroup as an Inn-basis is nilpotent*. As said, normality of N plays a relevant role in this context, even if the condition of being an Inn-basis is replaced by the stronger requirement that N is an Aut-basis. Indeed, let A be one of the torsion-free abelian groups of high cardinality referred to in the introduction, whose only non-trivial automorphism α is the inversion, and let $G = A \rtimes \langle\alpha\rangle$. For any $a \in A \smallsetminus 1$ then $H = \langle a, \alpha\rangle$ is an Aut-basis of G. Clearly H is isomorphic to the infinite dihedral group, while G is a metabelian centreless group of the same cardinality as A.

The next example shows that a group does not need to be locally nilpotent, or soluble, even if it has a normal nilpotent subgroup which is a finitely generated Aut-basis.

Example 1.6. Let N be the free group on two elements x and y in the variety of nilpotent groups of class at most 2. Then $Z(N) = N'$ is generated by $c = [x, y]$. Let γ be the involutory automorphism of N defined by $x \mapsto x^{-1}$ and $y \mapsto y^{-1}$, and let $G = N \rtimes \langle\gamma\rangle$. Then $C_G(N) = \langle c\rangle = Z(G)$ and $\mathrm{Hom}\big(G/N, Z(G)\big) = 0$. Thus we may apply Lemma 1.3 to obtain that N is an Aut-basis of the non-nilpotent polycyclic group G. As N is maximal in G, it is even a hereditary Aut-basis.

An insoluble example can be constructed as follows. The action of $\mathrm{Aut}\,N$ on $N_{\mathrm{ab}} = N/N'$ gives rise to an epimorphism $\varphi : \mathrm{Aut}\,N \longrightarrow\!\!\!\!\rightarrow \mathrm{GL}_2(\mathbb{Z})$, since every automorphism of N_{ab} is induced by an automorphism of N. The kernel of φ is $I := \mathrm{Inn}\,N$. Consider the group $H = N \rtimes \mathit{\Gamma}$, where $\mathit{\Gamma}$ is the centralizer of c in $\mathrm{Aut}\,N$. As immediately checked, $\mathit{\Gamma}$ is the preimage of $\mathrm{SL}_2(\mathbb{Z})$ under φ, thus $\mathit{\Gamma}/I \simeq \mathrm{SL}_2(\mathbb{Z})$. Let $C = \{a^{-1}\tilde{a} \mid a \in N\}$, where $\tilde{a}$ denotes the inner automorphism of N determined by a. Then $C = C_H(N) \simeq N$ and $C_H(C) = N$. Clearly $NC = NI$ is nilpotent of class 2. Now, NC is an Aut-basis of H: this follows again from Lemma 1.3, since $C_H(NC) = C \cap N = \langle c\rangle = Z(H)$ and $\mathrm{Hom}\big(H/NC, Z(H)\big) \simeq \mathrm{Hom}\big(\mathit{\Gamma}/I, \langle c\rangle\big) = 0$. □

Despite the above example there are some cases in which the presence of a nilpotent subgroup that is an Aut-basis in a group forces the latter to be nilpotent. We first record a simple remark.

Lemma 1.7. *Let the subgroup H be an Inn-basis of the group G. Then, for any $n \in \mathbb{N}$:*
(*i*) *G is nilpotent of class at most n if and only if $H \le Z_n(G)$;*
(*ii*) *if H is abelian then G is abelian;*
(*iii*) *if $H \le G'$ and G is hypercentral then G is abelian.*

Proof — Assume $H \le Z_n(G)$. Then $Z(G) = C_G(H) \ge C_G\big(Z_n(G)\big) \ge \gamma_n(G)$, so $\gamma_{n+1}(G) = 1$. This proves (*i*). To prove (*ii*) suppose that H is abelian. Then $H \le C_G(H) = Z(G)$ and (*i*) shows that G is abelian. Finally, assume $H \le G'$. Then $Z_2(G) \le C_G(G') = Z(G)$. If G is hypercentral this shows that G is abelian. □

Proposition 1.8. *Let G be a metabelian group and let H be a subnormal subgroup of defect d in G. If H is an Inn-basis of G then $Z_i(H) \le Z_{i+d}(G)$ for every positive integer i. In particular, if H is nilpotent of class c then G is nilpotent of class at most $c + d$.*

Proof — If X is any metabelian group, then $[Y, a, b] = [Y, b, a]$ for all $a, b \in X$ and $Y \le X'$, as the automorphisms induced by conjugation on X' by a and b generate a commutative subring of $\mathrm{End}\,X'$. Therefore $[U, V, L_1, L_2, \dots, L_t] = [U, V, L_{1\sigma}, L_{2\sigma}, \dots, L_{t\sigma}]$ for any $t \in \mathbb{N}$, $U, V, L_1, \dots, L_t \le X$ and $\sigma \in \mathbb{S}_t$.

To prove the proposition it will be clearly enough to assume $d = 1$, that is: $H \lhd G$. Let $K = Z_i(H)$. Then $[K, G] \le K$ and $[K, G, {}_iH] = 1$. Hence $[K, G, {}_{i-1}H] \le C_G(H) = Z(G)$, so $[K, G, {}_{i-1}H, G] = 1$. If $i > 1$ we can apply the property of metabelian groups just recalled to get $[K, G, {}_{i-2}H, G, H] = 1$. This again gives $[K, G, {}_{i-2}H, G, G] = 1$, because H is an Inn-basis. By carrying on this procedure we eventually obtain $[K, H, {}_iH] = 1$. Therefore $K \le Z_{i+1}(G)$, as required. If H is nilpotent the last clause in the statement follows now from Lemma 1.7 (*i*). □

Some remarks about Proposition 1.8 are in order. Firstly, every nonabelian subgroup of a dihedral group is an Inn-basis. This makes easy to find examples showing that nonnilpotent metabelian groups may have (non-subnormal) nilpotent subgroups as Inn-bases and that the bound $c+d$ for the nilpotency class of G found in the proposition is the best possible (unless $c = 1$, of course, in which case G is abelian by Lemma 1.7). Secondly, the hypothesis that the group be metabelian is essential. It is in fact possible to construct centre-by-metabelian groups with a nilpotent normal subgroup of class 2 as an Inn-basis and which are either non-nilpotent (see Example 1.6, or also Examples 1.14 and 1.15) or nilpotent of arbitrarily high class. Indeed, let $n \in \mathbb{N}$ and let W be the standard wreath product of the quaternion group of order 8 and a cyclic group of order 2^n. Let B be the base group of W and $C = [W, Z(B)]$. Then B/C is an Inn-basis of $G = W/C$ and has class 2, while the class of G is at least 2^n+1. Further examples with the same properties can be read off from Example 1.15, in particular they can be chosen to be p-groups for any prime p.

In contrast to these last examples, nilpotency classes are preserved when we deal with Aut-bases instead of Inn-bases. Let us denote by $\mathrm{CAut}\, G$ the group of all automorphisms of a group G acting trivially on both $Z(G)$ and $G/Z(G)$. We stick to our terminological convention and call any $\mathrm{CAut}\, G$-basis a CAut-basis of G. Now $\mathrm{CAut}\, G = \{1+\varepsilon \mid \varepsilon \in \mathrm{CEnd}\, G\}$, where $\mathrm{CEnd}\, G$ is the set of all endomorphisms ε of G such that $\mathrm{im}\,\varepsilon \leq Z(G) \leq \ker\varepsilon$. Thus the property of being a CAut-basis of G is the same as being a CEnd-basis. One can reword Lemma 1.3 as follows: a normal subgroup containing the centre is an Aut-basis if and only if it is simultaneously an Inn- and a CAut-basis.

Theorem 1.9. *Let the subgroup H be a CAut-basis of the nilpotent group G. Then $Z_n(H) = Z_n(G)\cap H$ for any $n \in \mathbb{N}_0$. In particular, H and G have the same nilpotency class.*

Proof — Let c be the nilpotency class of G. It will be enough to prove $Z_n(H) \leq Z_n(G)$ for every non-negative integer $n \leq c$. We will argue by induction on $c-n$, our claim being obvious for $n = c$. Thus, suppose $0 \leq n < c$ and $Z_{n+1}(H) \leq Z_{n+1}(G)$. For every integer i such that $0 \leq i \leq n$ let $X_i = [Z_n(H), {}_iG, {}_{n-i}H]$. Our aim is to prove $X_n = 1$. Assume false. Then we can pick the least integer i between 0 and n such that $X_i \neq 1$ (of course $i > 0$) and elements $g_1, g_2, \ldots, g_{i-1} \in G$, $h_{i+1}, h_{i+2}, \ldots, h_n \in H$ and $a \in Z_n(H)$ such that $[a, g_1, \ldots, g_{i-1}, G, h_{i+1}, \ldots, h_n] \neq 1$. The mapping

$$\varepsilon : x \in G \mapsto [a, g_1, g_2, \ldots, g_{i-1}, x, h_{i+1}, h_{i+2}, \ldots, h_n] \in G$$

belongs to $\mathrm{CEnd}\, G$, since $a \in Z_{n+1}(H) \leq Z_{n+1}(G)$. Moreover $H \leq \ker\varepsilon$, because

$$H^\varepsilon = [a, g_1, \ldots, g_{i-1}, H, h_{i+1}, \ldots, h_n] \leq X_{i-1} = 1.$$

Since H is a CEnd-basis of G it follows $\varepsilon = 0$, that is $[a, g_1, \ldots, g_{i-1}, G, h_{i+1}, \ldots, h_n] = 1$, contradicting our choice above. □

Neither the analogue of Lemma 1.7 (i) nor that of Theorem 1.9 holds for hypercentral groups: a term of the upper central series of a hypercentral group may be an Aut-basis even if its hypercentral length is less than that of the group. The examples we exhibit to support this statement also have the feature that the Aut-basis is periodic but the group is not. For the sake of later reference the construction is slightly more general than what is required here.

Example 1.10. Let D be a locally nilpotent periodic group with infinitely many non-trivial primary components, none of which is abelian. Further assume that $Z(D)$ is reduced. Let $\pi = \pi(D)$, the set of all primes dividing the order of some element of D, and let $C = \mathrm{Cr}_{p\in\pi} D_p$, the cartesian product of the primary components of D. For any $p \in \pi$ let a_p be a noncentral element of D_p. Consider the subgroup $A = \mathrm{Cr}_{p\in\pi}\langle a_p\rangle$ of C and its element $a = (a_p)_{p\in\pi}$. Then $D = \mathrm{tor}\, C$ and AD/D is a divisible torsion-free abelian group. Let $T/D\langle a\rangle = \mathrm{tor}(AD/D\langle a\rangle)$. Clearly $T/D \simeq \mathbb{Q}$ and T is locally nilpotent. Now let G/D be any non-trivial π-divisible subgroup of T/D (a possible choice is $G = T$). Then $C_G(D) = Z(D) = Z(G)$, otherwise we would have $[a^n, D] = 1$ for some $n \in \mathbb{N}$, while $[a^n, D_p] \neq 1$ for any p in π not dividing n. As $\mathrm{Hom}\big(G/D, Z(D)\big) = 0$ we are now in position to apply Lemma 1.3 and conclude that D is an Aut-basis of G.

Now, for each $p \in \pi$ assume that D_p is nilpotent of class greater than n_p and $a_p \notin Z_{n_p}(D_p)$, where $\{n_p \mid p \in \pi\}$ is unbounded. Then $D = Z_\omega(D) = Z_\omega(G)$ and $G = Z_{\omega+1}(G)$. □

Next we discuss subnormal nilpotent subgroups as Aut-bases.

Lemma 1.11. *Let H be a normal subgroup of the group K. Assume that H is contained in the hypercentre $\bar{Z}(K)$ of K. Then one of the following holds: either*
(a) $\mathrm{Hom}\big(K/H, Z(K)\cap H\big) \neq 0$; *or*
(b) $Z_i(H) \leq Z_i(K)$ *for all* $i \in \mathbb{N}_0$.

Proof — For all $j \in \mathbb{N}_0$ set $C_j = Z_j(H)$. Assume that (b) is false. Then the factor C_{i+1}/C_i is not central in K for some $i \in \mathbb{N}_0$. Since $H \leq \bar{Z}(K)$ then $C_i < A < B \leq C_{i+1}$, where $A/C_i =$

$Z(K/C_i) \cap C_{i+1}/C_i$ and $B/A = Z(K/A) \cap C_{i+1}/A$. Let $b \in B \smallsetminus A$. The mapping $\varepsilon : k \mapsto [k,b]C_i$ from K to A/C_i is a non-zero homomorphism whose kernel contains H. For each $j \in \mathbb{N}_0$ consider the property

$$\mathcal{P}_j : \quad \begin{cases} \text{there exists a non-zero homomorphism from } K/H \\ \text{to } C_{j+1}/C_j \text{ whose image is a central factor of } K. \end{cases}$$

We have just shown that $\mathcal{P}_i$ holds. Thus we can pick the minimal $j \in \mathbb{N}_0$ such that $\mathcal{P}_j$ is true. Suppose $j > 0$. As $\mathcal{P}_{j-1}$ is false the factor C_j/C_{j-1} is central in K. Let η be a non-zero homomorphism from K/H to C_{j+1}/C_j such that $X/C_j := \operatorname{im}\eta$ is central in K. There exists $g \in H$ such that $[X,g] \not\leq C_{j-1}$. The mapping

$$xC_j \in X/C_j \mapsto [x,g]C_{j-1} \in C_j/C_{j-1}$$

is a non-zero homomorphism whose image is central in K. Since X/C_j is an epimorphic image of K/H this yields $\mathcal{P}_{j-1}$, against our choice of j. Thus $j = 0$ and $\mathcal{P}_0$ holds. But $\mathcal{P}_0$ is precisely condition (a) in our statement, so the proof is complete. □

For the sake of simplifying the next lemma we introduce a piece of notation. If H is a subgroup of the group K, we say that K has property (χ) with respect to H if and only if $\operatorname{Hom}\big(K/H^K, Z(H)\big) = 0$.

Lemma 1.12. *Let K be a group, and let $H \leq \bar{Z}(K)$. If K has property (χ) with respect to H then also H^K satisfies (χ) with respect to H.*

Proof — Assume that $N := H^K$ does not satisfy (χ), that is: $\operatorname{Hom}\big(N/H^N, Z(H)\big) \neq 0$. Then there exists a proper normal subgroup S of N such that $H \leq S$ and N/S embeds in $Z(H)$. Let $V = S_K$ and $C/V = (N/V) \cap Z(K/V)$. Obviously $C \cap S = V$, thus C/V embeds in N/S and so in $Z(H)$. Now $S \ntrianglelefteq K$ as $N = H^K$, hence $C < N$. As $N \leq \bar{Z}(K)$ there exists $d \in N \smallsetminus C$ such that dC is central in K/C. Let D/V be the centralizer of dV in K/V. Then $N \leq D$ (indeed, N/V is abelian as N/S is abelian) and K/D is isomorphic to $[d,K]V/V$, a non-trivial subgroup of C/V. Since the latter embeds in $Z(H)$, this gives rise to a non-zero homomorphism from K/N to $Z(H)$, in contradiction to the assumption that K satisfies (χ). □

Theorem 1.13. *Let the group G have a nilpotent subnormal subgroup H contained in the hypercentre and which is an Aut-basis. Then G is nilpotent.*

Proof — Argue by induction on the subnormal defect d of H in G. We may of course assume $d > 0$. By Lemma 1.2 we know that G has property (χ) with respect to H. Lemma 1.12 shows that all terms of the standard normal closure series of H in G satisfy (χ). Let K be the term of this series whose defect is $d-1$. Then $H \triangleleft K$ and Lemma 1.11 gives $H \leq Z_c(K)$ where c is the nilpotency class of H. Now Lemma 1.7 (i) proves that K is nilpotent; by the induction hypothesis also G is nilpotent. □

Thus a hypercentral group having a nilpotent subnormal subgroup as an Aut-basis is itself nilpotent. The same does not necessarily happen with more general locally nilpotent groups, even if the subgroup is normal. Also the hypothesis that H is an Aut-basis in Theorem 1.13 cannot be replaced by H being just an Inn-basis. This is shown in the next two examples, that can also be compared with Proposition1.8. We leave open the question whether a hypercentral group with a nilpotent subgroup as an Aut-basis must necessarily be nilpotent.

Example 1.14. For every prime number p there exists a centre-by-metabelian Fitting p-group which is not nilpotent and has a normal nilpotent subgroup of class 2 (extraspecial, as a matter of fact) as an Aut-basis.

Indeed, let P be a non-abelian group of order p^3. Let $A \simeq \mathcal{C}_{p^\infty}$, a Prüfer p-group. Build the standard wreath product $W = P \wr A$ and let $B = \mathrm{Dr}_{a \in A} P^a$ be its base group. For every finite subgroup A_0 of A the subgroup BA_0 is nilpotent (of the same class as $P \wr A_0$), hence W is a Fitting group. Set $Z(P) = \langle c \rangle$ and let $Z = Z(B)$. The elements of Z are of the form $c^{n_1 a_1} c^{n_2 a_2} \cdots c^{n_t a_t}$, where $t \in \mathbb{N}_0$ and $n_i \in \mathbb{Z}$ and $a_i \in A$ for each i. Then $K := [Z, A]$ is the kernel of the epimorphism $Z \twoheadrightarrow \mathbb{Z}_p$

which maps any such element to $\sum_{i=1}^{t} n_i \in \mathbb{Z}_p$. Let $G = W/K$ and $N = B/K$. Since $C_A(B/Z) = 1$ it is clear that $C_G(N) = Z(N)$. Moreover $Z(N) = Z(B/K) = Z/K$. Indeed, let $b \in B \smallsetminus Z$. Then there exists $a \in A$ such that the image b_a of b under the natural projection $B \twoheadrightarrow P^a$ is not central in P^a. Let $g \in P^a \smallsetminus C_{P^a}(b_a)$. Then $[b, g] = [b_a, g] \notin K$, as $P^a \cap K = 1$. Hence $bK \notin Z(N)$. Therefore $C_G(N) = Z/K = Z(G)$. As $\mathrm{Hom}(G/N, Z(G)) \simeq \mathrm{Hom}(\mathcal{C}_{p^\infty}, \mathcal{C}_p) = 0$, Lemma 1.3 shows that N is an Aut-basis of G. That N is extraspecial and G satisfies the requirements of our claim is now clear. □

Example 1.15. For every prime number p there exists a centre-by-metabelian hypercentral p-group of length $\omega + 1$ with a class-2 nilpotent subgroup of index p as an Inn-basis.

Consider first the case $p = 2$. For every $n \in \mathbb{N}$ let F_n be a free group on two generators x_n and y_n in the variety of class-2 nilpotent groups of exponent at most 2^{n+1}. Thus $F_n = (\langle x_n\rangle \times \langle c_n\rangle) \rtimes \langle y_n\rangle$ where x_n and y_n have order 2^{n+1} and $c_n = [x_n, y_n]$ has order 2^n. Also, $Z(F_n) = \langle c_n, x_n^{2^n}, y_n^{2^n}\rangle$. Now let $N = \mathrm{Dr}_{n\in\mathbb{N}} F_n$. Clearly N has an automorphism α of order 2 defined by $x_n^\alpha = x_n^{-1}$ and $y_n^\alpha = y_n^{-1}$ for all $n \in \mathbb{N}$. Let $G = N \rtimes \langle\alpha\rangle$. Of course, G is a hypercentral 2-group of length $\omega + 1$. The equality $C_G(N) = Z(N)$ is easily checked. Now α centralizes $Z(N)$, thus N is an Inn-basis of G.

The construction for odd primes p is rather similar. For all $n \in \mathbb{N}$ let F_n be a free group on generators $x_1,\ x_2, \ldots, x_{p-1}$ in the variety of all nilpotent groups of class at most 2 and exponent dividing p^n. The automorphism α_n of F_n defined by $x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{p-1} \mapsto x_1^{-1}x_2^{-1}\cdots x_{p-1}^{-1}$ has order p. Let $P_n = F_n\big/[F_n', \alpha_n]$. Now let $N = \mathrm{Dr}_{n\in\mathbb{N}} P_n$ and $G = N \rtimes \langle\alpha\rangle$, where α has order p and acts on each direct factor P_i like α_i. Then N is an Inn-basis of G, and G is hypercentral, but not nilpotent. □

We close this section with two examples in the class of finite groups. The Sylow 2-subgroup of $\mathrm{SL}(2, 3)$ (normal and isomorphic to the quaternion group of order 8) is an Aut-basis, obviously hereditary since it is a maximal subgroup. More interesting is that the Sylow 2-subgroups of $\mathrm{SL}(4, 3)$ are End-bases, which proves that (even among finite groups) the existence of a nilpotent End-basis does not necessarily imply nilpotency.

2. Hereditary Aut-bases in locally nilpotent groups

The examples above show that a locally nilpotent group G with a nilpotent subgroup H as an Aut-basis does not need to be nilpotent. We prove in this section that this is the case if we strengthen the condition and require that H is a hereditary Aut-basis: in this case G is bound to be nilpotent (of the same class as H, by Theorem 1.9). The hypothesis that G be locally nilpotent cannot be easily relaxed, as is shown by the polycyclic group in Example 1.6.

Let us start with a couple of simple lemmas. If G is a group and H is a subgroup of G, we call a *section between H and G* a section U/V of G such that $H \leq V$. If H is a hereditary Aut-basis of G and X is a section between H and G then $\mathrm{Hom}\big(X, Z(H)\big) = 0$ by Lemma 1.2. If $Z(H) \neq 1$ (resp. if $Z(H)$ has elements of prime order p) then this shows that X cannot be infinite cyclic (resp. of order p). Thus:

Lemma 2.1. *Let the subgroup H be a hereditary Aut-basis of the group G. If $Z(H) \neq 1$ then:*
(i) there is no infinite cyclic section between H and G;
(ii) for every prime $p \in \pi\big(Z(H)\big)$ there is no section of order p between H and G.

Lemma 2.2. *Let $G = N \rtimes K$ be a semidirect product. If K is a hypercentral Aut-basis of G, then $G = N \times K$ and $|N| \leq 2$.*

Proof — Let $C = C_K(N)$. Argue by means of contradiction and assume $C < K$. Then K/C has a nontrivial central element xC. The conjugation automorphism induced by x on N commutes with those induced by elements of K, hence the mapping defined by $ak \mapsto a^x k$ (for $a \in N$ and $k \in K$) is a non-trivial automorphism of G which acts trivially on K. This is a contradiction, since K is an Aut-basis of G. Therefore $C = K$, that is: $G = N \times K$. Again the fact that K is an Aut-basis yields $\mathrm{Aut}\, N = 1$, hence $|N| \leq 2$. □

The next lemma and the corollary following it are doubtless well known; we include proofs since we could not find suitable references to them.

Let H be a subgroup of the locally nilpotent group G and let π be a set of primes. We write $I_{G,\pi}(H)$ for the *π-isolator* of H in G, i.e., the subgroup consisting of all elements x of G such that $x^n \in H$ for some π-number n. According to standard terminology H is *π-isolated* in G if and only if $H = I_{G,\pi}(H)$. Also, π' denotes the set of all primes not in π. If π is the set of all primes we write $I_G(H)$ for $I_{G,\pi}(H)$, the isolator of H in G. We shall make use of the following two well-known facts:

(1) if H is hypercentral of length α and $G = I_{G,\pi}(H)$ has trivial π-component, then also G is hypercentral of length α (indeed, by adapting the proof of [5], Lemma 4.8, for instance, one shows that $Z_\beta(G)$ is π-isolated in G and contains $Z_\beta(H)$ for every ordinal β. Hence $Z := Z_\alpha(G) \geq H$ and so $Z = I_{G,\pi}(Z) \geq I_{G,\pi}(H) = G$);

(2) if G is a finitely generated nilpotent group and $H \leq G$, then $|G : H|$ is finite if and only if G/HG' is finite.

Lemma 2.3. *Let x be an element of infinite order in the locally nilpotent group G. Let H be maximal among the subgroups of G subject to the condition $H \cap \langle x\rangle = 1$. Then x normalizes H.*

Proof — Let $\mathcal{S}$ be a local system of finitely generated subgroups of H. For every $K \in \mathcal{S}$ let $K^* = K\langle K, x\rangle'$. Since $\big|\langle K, x\rangle : K\big|$ is infinite the remark immediately preceding this lemma shows that the cyclic factor $\langle K, x\rangle/K^*$ is infinite. Let $\mathcal{S}^* = \{K^* \mid K \in \mathcal{S}\}$. The mapping $K \in \mathcal{S} \mapsto K^* \in \mathcal{S}^*$ preserves inclusion, hence $\mathcal{S}^*$ is a local system of subgroups of $H^* := \langle K^* \mid K \in \mathcal{S}\rangle$. Of course x normalizes H^* and $H \leq H^*$. Suppose $H < H^*$. Then, by maximality of H, there exists $n \in \mathbb{N}$ such that $x^n \in H^*$. But then $x^n \in K^*$ for some $K^* \in \mathcal{S}^*$. This is a contradiction, since $\langle K, x\rangle/K^*$ is infinite cyclic. Therefore $H = H^*$ is normalized by x. □

Corollary 2.4. *Let H be a subgroup of the locally nilpotent group G. Then either $G = I_G(H)$ or there is an infinite cyclic section between H and G.*

Proof — If G is not the isolator of H in G there exists in G an element x of infinite order such that $H \cap \langle x\rangle = 1$. By Zorn's Lemma there is a subgroup M of G maximal with respect to $H \leq M$ and $M \cap \langle x\rangle = 1$. It follows from Lemma 2.3 that $\langle M, x\rangle/M$ is an infinite cyclic section between H and G. □

Lemma 2.5. *Let H be a hypercentral subgroup of the locally nilpotent group G. Let $\pi = \pi(H)$. If H is a hereditary Aut-basis of G, then:*

(i) *if U/V is a periodic section of G such that $\pi(U/V) \subseteq \pi$, then $\langle H, U\rangle = \langle H, V\rangle$;*

(ii) *$G = I_{G,\pi'}(H)$;*

(iii) *$\operatorname{tor} H$ is the π-component of $\operatorname{tor} G$; the π'-component of $\operatorname{tor} G$ has order at most 2.*

Proof — (i) Assume $X := \langle H, V\rangle < \langle H, U\rangle$ and let $x \in U \setminus X$. There is a maximal subgroup M of $Y := \langle X, x\rangle$ containing X. Since G is locally nilpotent and $x^n \in V \leq X$ for some π-number n, the order of Y/M is a prime belonging to π. But Y/M is a section between H and G. This and Lemma 2.1 (ii) yield a contradiction.

(ii) Equality $G = I_G(H)$ immediately follows from Lemma 2.1 (i) and Corollary 2.4. On the other hand, what we have just proved as part (i) ensures that H is π-isolated in G. It is then clear that $G = I_{G,\pi'}(H)$ holds.

(iii) Let P be the π-component of $\operatorname{tor} G$. Then $\langle H, P\rangle = H$ by part (i), hence $P \leq H$. This shows that $\operatorname{tor} H$ is the π-component of $\operatorname{tor} G$. Finally, if N is the π'-component of $\operatorname{tor} G$, then H is an Aut-basis of $\langle H, N\rangle = N \rtimes H$, so $|N| \leq 2$ by Lemma 2.2. □

Now we are in position to prove the main result of this section.

Theorem 2.6. *Let the locally nilpotent group G have a nontrivial hypercentral subgroup H as a hereditary Aut-basis. Then G is hypercentral of the same hypercentral length as H. In particular, if H is nilpotent then also G is nilpotent.*

Proof — Let H be hypercentral of length α. At the expense of replacing H by $(\operatorname{tor} G)H$—which is still hypercentral of length α by Lemma 2.5 (iii)—we may assume $\operatorname{tor} G \leq H$. Set $\pi = \pi(H)$. Then G has trivial π'-component and $I_{G,\pi'}(H) = G$ by Lemma 2.5 (ii). Thus G is hypercentral of length α, as required. □

We recall from Example 1.10 that even if G is hypercentral but the Aut-basis H is not hereditary it may happen that the hypercentral lengths of H and G differ.

As a final remark we note that Lemma 2.5 shows that, apart from trivial instances, no hypercentral subgroup may be a hereditary Aut-basis in a periodic locally nilpotent group. It is obvious (see Lemma 2.1 (i)) that a locally nilpotent group is periodic if it has a periodic subgroup with non-trivial centre which is a hereditary Aut-basis. So we have:

Corollary 2.7. *Let G be a locally nilpotent group. Then a periodic hypercentral subgroup H of G is a hereditary Aut-basis if and only if either $G = H$ or $2 \notin \pi(H)$ and $G = H \times K$ with $|K| = 2$.*

Proof — The necessity of the condition follows from Lemma 2.5 (iii) and the last observation. Its sufficiency is immediate, as K is characteristic in G and $\operatorname{Aut} K = 1$, if $H \neq G$. □

If the hypotheses on H are relaxed, it is possible to obtain less obvious examples of hereditary Aut-bases in locally nilpotent groups. Indeed, in a torsion-free abelian group every maximal independent subset is a hereditary Aut-basis. Also, if D is a locally nilpotent periodic group with trivial centre and infinitely many nontrivial primary components, the construction in Example 1.10 embeds D as a hereditary Aut-basis (and the torsion subgroup) of a non-periodic locally nilpotent group; this follows by a direct application of Lemma 1.4.

3. Finite Aut-bases and outer automorphisms

If G is a periodic countably infinite locally nilpotent group then $|\operatorname{Aut} G| = 2^{\aleph_0}$. This was proved by O. Puglisi in [7]. According to Kueker's theorem quoted in the introduction this result is equivalent to the following: a countable periodic locally nilpotent group has a finite Aut-basis if and only if it is finite. After rewording Puglisi's theorem in this form, inspection of the proofs in [7] reveals that the countability hypothesis can be disposed of. Here we aim at further extensions of this theorem, by relaxing the periodicity hypothesis and also by weakening the requirement of being an Aut-basis.

Let G be a group. With respect to subgroups Γ of $\operatorname{Aut} G$ it can be of interest to consider $C_{\operatorname{Aut} G}(\Gamma)$-bases. This is because of the following: assume $G \lhd E$ for some group E and $E = GK$ for some $K \leq E$. Then there exists no non-trivial automorphism of E which acts trivially on K and normalizes G if and only if $H := K \cap G$ is a $C_{\operatorname{Aut} G}(\Gamma)$-basis, where Γ is the group of all automorphisms induced on G by conjugation by elements of K. Hence H is a $C_{\operatorname{Aut} G}(\Gamma)$-basis if K is an Aut-basis of E, and the two properties are equivalent if G is characteristic in E.

We will make use of the following variation of Lemma 4 of [7].

Lemma 3.1. *Let G be a locally nilpotent group, π a finite set of primes, n the product of the primes in π. Assume that a π-subgroup H of $G'G^n$ is an Inn-basis of G. If either G is hypercentral or H is finite, then G is abelian.*

Proof — Assume first that G is hypercentral. Suppose that G is not abelian. Then $H \not\leq Z(G)$, as H is an Inn-basis of G. Therefore $Z_2(G)/Z(G)$ has non-trivial p-component for some $p \in \pi$. Let $xZ(G)$ be an element of order p in $Z_2(G)/Z(G)$. Then $G/C_G(x) \simeq [G,x]$ is an elementary abelian p-group, so that $G'G^p \leq C_G(x)$. Since $G'G^p$ is an Inn-basis of G this yields $x \in Z(G)$, a contradiction. Thus G is abelian in this case.

Assume now that H is finite. Suppose again that G is not abelian. Then from $H \leq G'G^n$ it follows that G has a finitely generated non-abelian subgroup X such that $H \leq X'X^n$. Since X is nilpotent and H is an Inn-basis of X this is impossible by what was proved in the previous case. □

Lemma 3.2. *Let Γ be a group of automorphisms of the group G such that the external semidirect product $E = G \rtimes \Gamma$ is locally nilpotent. Let the Γ-invariant subgroup H of G be a $C_{\operatorname{Aut} G}(\Gamma)$-basis. If either Γ is finitely generated or E is hypercentral, then $C_E(H) = C_E(G)$; in particular H is an Inn-basis of G.*

Proof — Let Δ be the group of those automorphisms of G induced by conjugation by elements of $C_E(H)$. Since H is Γ-invariant Δ is normalized by Γ, and $\Delta\Gamma \simeq C_E(H)\Gamma/C_E(G)$ is locally nilpotent. Now $C_\Delta(\Gamma) = 1$, because H is a $C_{\operatorname{Aut} G}(\Gamma)$-basis and $[\Delta, H] = 1$. By hypothesis either Γ is finitely generated or $\Delta\Gamma$ is hypercentral. In both cases from $C_\Delta(\Gamma) = 1$ it follows $\Delta = 1$, which amounts to saying $C_E(H) = C_E(G)$. □

The special case dealt with in the next lemma is of great relevance for the proofs that follow.

Lemma 3.3. *Let π be a set of primes, and let Γ be a periodic hypercentral group of automorphisms of the π-divisible abelian group G such that the external semidirect product $E = G \rtimes \Gamma$ is locally nilpotent. Assume that G has a π-subgroup H of finite exponent which is a $C_{\operatorname{Aut} G}(\Gamma)$-basis. Then both Γ and the π-component of G are trivial, and $|G| \leq 2$.*

Proof — Let $e = \exp H$ and $\psi = \pi(H)$. At the expense of replacing H by $G[e] = \{g \in G \mid g^e = 1\}$ we may assume that H is Γ-invariant. The ψ-component P of G is divisible, so $G = P \times Q$ for some subgroup Q.

If $\Gamma = 1$ then H is an Aut-basis of G. For any $\alpha \in \operatorname{Aut} Q$ we can define an automorphism of G which acts on Q like α and maps every element of P to its $(e+1)$-th power. This automorphism centralizes H, hence it is the identity. Thus $\operatorname{Aut} Q = 1$ and $P^e = 1$. It follows $|Q| \leq 2$ and $P = 1$, since P is divisible; by the same reason the π-component of $G = Q$ is trivial. Thus all we have to prove is that Γ is trivial.

Assume $\Gamma \neq 1$. Let $C = C_\Gamma(H)$. If $C \neq 1$ then $C_C(\Gamma) \neq 1$, because Γ is hypercentral and $C \lhd \Gamma$. This is impossible, as H is a $C_{\operatorname{Aut} G}(\Gamma)$-basis. Thus $C = 1$, so that Γ is isomorphic to a periodic group of automorphisms of H. As $H \rtimes \Gamma$ is locally nilpotent it follows that Γ is a ψ-group. This also gives $[G, \Gamma] \leq P$ (indeed, let X be a finitely generated subgroup of G/P, and let $\gamma \in \Gamma$. Then $X^{\langle\gamma\rangle}$ is a finitely generated abelian group with trivial ψ-component on which the ψ-group $\langle\gamma\rangle$ acts nilpotently, hence $[X^{\langle\gamma\rangle}, \gamma] = 1$). With respect to the decomposition $G = P \times Q$ any automorphism γ of G acting trivially on G/P (hence any element of Γ) can be represented by the matrix $\left(\begin{smallmatrix} \alpha & 0 \\ \varepsilon & 1 \end{smallmatrix}\right)$, where α is the automorphism induced by γ on P and $\varepsilon \in \operatorname{Hom}(Q, P)$ is defined by $x^\gamma = x^\varepsilon x$ for every $x \in Q$. If δ is another such automorphism of G, represented by $\left(\begin{smallmatrix} \beta & 0 \\ \eta & 1 \end{smallmatrix}\right)$, then γ and δ commute if and only if the following conditions hold:

$$\alpha\beta = \beta\alpha \qquad \text{and} \qquad \varepsilon(\beta - 1) = \eta(\alpha - 1). \tag{$*$}$$

As $\Gamma \neq 1$ there exists a nontrivial element $\zeta \in Z(\Gamma)$. Assume that ζ is represented by $\left(\begin{smallmatrix} \alpha & 0 \\ \varepsilon & 1 \end{smallmatrix}\right)$. Let θ be the automorphism of G represented by $\left(\begin{smallmatrix} \sigma & 0 \\ \tau & 1 \end{smallmatrix}\right)$, where $\sigma = 1 + e(\alpha - 1)$ and $\tau = e\varepsilon$. To make sure that this definition is consistent we have to check that such σ is indeed an automorphism. It is clear that σ is a monomorphism, since it acts trivially on the socle of P. We have to prove that σ is epic. Every $x \in P$ belongs to a finite α-invariant subgroup X of P, as α is periodic. Certainly $X^\sigma \leq X$, hence $X^\sigma = X$ because σ is a monomorphism, so $x \in P^\sigma$. Therefore $\sigma \in \operatorname{Aut} P$. It is obvious that θ acts trivially on H; we claim that θ centralizes Γ. To prove this claim, let $\gamma \in \Gamma$ and assume that γ is represented by $\left(\begin{smallmatrix} \beta & 0 \\ \eta & 1 \end{smallmatrix}\right)$. Since $[\gamma, \zeta] = 1$ equalities as in $(*)$ hold. These yield $\sigma\beta = \beta\sigma$ and

$$\tau(\beta - 1) = e\varepsilon(\beta - 1) = e\eta(\alpha - 1) = \eta e(\alpha - 1) = \eta(\sigma - 1).$$

Therefore $[\theta, \gamma] = 1$ and our claim is established. It follows $\theta = 1$, as H is a $C_{\operatorname{Aut} G}(\Gamma)$-basis. Hence $\sigma = 1$ and $\tau = 0$, which in turn give $\alpha = 1$ and $\varepsilon = 0$, because P and Q are ψ-divisible. This means $\zeta = 1$, against our choice. This contradiction shows $\Gamma = 1$, so the proof is complete. □

Theorem 3.4. *Let Γ be a finitely generated group of automorphisms of the group G such that the external semidirect product $E = G \rtimes \Gamma$ is locally nilpotent. Assume that G has a finite subgroup H which is a $C_{\mathrm{Aut}\, G}(\Gamma)$-basis. Then E is finite.*

Proof — It is clear that the subgroup H^Γ is still finite (and a $C_{\mathrm{Aut}\, G}(\Gamma)$-basis). Hence—by substituting H^Γ for H if necessary—we may assume that H is Γ-invariant. Then Γ is finite since H is finite and $C_\Gamma(H) = C_\Gamma(G) = 1$ by Lemma 3.2.

Suppose that the theorem is false and choose a counterexample in which H is Γ-invariant of the minimal possible order. If $H = 1$ then $\Gamma = 1$ and $|G| \le 2$, hence $H \ne 1$.

Assume that G has no maximal subgroups whose index belongs to $\pi := \pi(H)$. Then $G = G'G^n$, where n is the product of all primes in π, hence G is abelian by Lemma 3.1 and Lemma 3.2. Moreover G is also π-divisible, thus Lemma 3.3 shows that E is finite.

Therefore G has some subgroup of index p, for some $p \in \pi$. Since Γ is finite G must then have some E-invariant proper subgroup of finite p-power index, so it has a maximal E-invariant proper subgroup, say M. As E is locally nilpotent $[G, \Gamma] \le M$ and $\big|G/M\big| = p$. Suppose first $H \le M$. Since $H \rtimes \Gamma$ is nilpotent the centralizer A of Γ in the p-component of $Z(H)$ is nontrivial, hence there exists a non-zero homomorphism $\varepsilon : G \to A$ with kernel M. Also, $A \le Z(G)$ by Lemma 3.2. Then $g \mapsto gg^\varepsilon$ defines a non-trivial automorphism of G centralizing Γ and H. This is impossible, as H is a $C_{\mathrm{Aut}\, G}(\Gamma)$-basis. Hence $H \not\le M$, so $G = MH$. Let $\bar\Gamma$ and Δ be the groups of the automorphisms of M induced by Γ and (by conjugation) by H respectively. Clearly $\bar\Gamma$ normalizes Δ, the product $\bar\Gamma\Delta$ is finite and the external semidirect product $M \rtimes \bar\Gamma\Delta$ is locally nilpotent. Every automorphism of M centralizing $\bar\Gamma\Delta$ and acting trivially on $H \cap M$ extends to an automorphism of G centralizing Γ and acting trivially on H. Since H is a $C_{\mathrm{Aut}\, G}(\Gamma)$-basis the latter automorphism is bound to be the identity. This proves that $H \cap M$ is a $C_{\mathrm{Aut}\, M}(\bar\Gamma\Delta)$-basis. Also, it is $\bar\Gamma\Delta$-invariant. By minimality of $|H|$ we conclude that M is finite, hence the same holds for E, as wished. □

The statement of Theorem 3.4 can be further improved by showing that the subgroup H^Γ is close to being equal to G. This is not needed for the next two corollaries and will be proved as a special case of Theorem 3.7.

Corollary 3.5. *A periodic-by-(finitely generated) locally nilpotent group has a finite Aut-basis if and only if it is finitely generated.*

Proof — Let G be a locally nilpotent group, and let $T = \operatorname{tor} G$. Assume that G/T is finitely generated and G has a finite Aut-basis X. There exists a finite subset Y of G such that $G = \langle T, Y\rangle$. Let $H = \langle X, Y\rangle$. Then H is a finitely generated Aut-basis of $G = TH$. Because of the argument sketched at the beginning of this section this means that $T \cap H$ is a $C_{\mathrm{Aut}\, T}(\Gamma)$-basis, where Γ is the group of the automorphisms induced by conjugation by H on T. Thus Theorem 3.4 proves that T is finite, hence G is finitely generated. □

Corollary 3.6. *Let G be a countably infinite locally nilpotent group. If $\operatorname{tor} G$ is infinite and $G/\operatorname{tor} G$ is finitely generated then $|\mathrm{Aut}\, G| = 2^{\aleph_0}$.*

Proof — Suppose $|\mathrm{Aut}\, G| \ne 2^{\aleph_0}$. Then G has a finite Aut-basis by Kueker's theorem. By Corollary 3.5 then G is finitely generated, i.e., $\operatorname{tor} G$ is finite. □

Thus periodic-by-finitely generated locally nilpotent groups which are countable but not finitely generated have (uncountably many) outer automorphisms. It appears to be not known whether all infinite finitely generated nilpotent groups have outer automorphisms—of course the automorphism group of any such group is countable anyway. A detailed discussion on this and related topics is in [2]. How much the countability hypotesis is decisive for the existence of outer automorphisms, even in the periodic case, has been shown by S. Thomas in [11] and by Dugas and Göbel in [3]. Indeed, the latter prove that for every infinite cardinal λ such that $\lambda = \lambda^{\aleph_0}$ and for every prime p there exists a complete locally nilpotent p-group of cardinality λ^+.

A better version of Theorem 3.4, more consistent with the general setting of this paper, would be obtained if the following question could be answered in the positive: is it possible to replace the hypothesis that H is finite with the hypothesis that H is finitely generated still getting that E is nilpotent? (Of course, as already remarked even for the rational group, one cannot hope to prove that E is finitely generated.) We have been unable to settle this conjecture. If true this would give as a special case, for $\Gamma = 1$, that every locally nilpotent group with a finite Aut-basis is nilpotent. Again by Kueker's theorem, this would further imply that every non-nilpotent countable locally nilpotent group would have $2^{\aleph_0}$ (outer) automorphisms. The well-known example by A.E. Zaleskiĭ [12] of a countably infinite nilpotent group with no outer automorphisms shows that the analogous statement fails to hold for nilpotent groups. On the opposite side, as remarked by S. Thomas ([11], Proposition 3), non-trivial centreless locally nilpotent groups clearly have no finite Aut-basis (actually, no finite Inn-basis), hence if such a group G is countable then $|\mathrm{Aut}\, G| = 2^{\aleph_0}$.

In connection with Theorem 3.4, it is worth mentioning that a non-periodic nilpotent (or even abelian) group may well have a periodic subgroup as an Aut-basis. Probably the most obvious example is a cartesian product C of infinitely many groups of pairwise different prime orders, whose torsion subgroup D (i.e., the corresponding direct product) is easily seen to be an End-basis, as $\mathrm{Hom}(C/D, C) = 0$. Other examples are those constructed in Example 1.10: in the notation used there, if D is nilpotent then also G is nilpotent.

One can ask whether it is possible to obtain similar examples where the Aut-basis has finite exponent. At least for nilpotent groups, a proof similar to that of Theorem 3.4 shows that the answer is negative. To state this fact in a more general setting we point at the following two properties relative to groups G:

$(\mathcal{M})$: every proper subgroup of G is contained in a maximal subgroup of G;

$(\mathcal{N})$: if H is a proper subgroup of G then $HG' < G$.

As is well known, property $(\mathcal{N})$ is a form of generalized nilpotency, in the sense that nilpotency implies $(\mathcal{N})$ and it is equivalent to $(\mathcal{N})$ for finite groups. Nilpotent groups of finite exponent satisfy $(\mathcal{M})$. Indeed, $(\mathcal{M})$ and $(\mathcal{N})$ are equivalent conditions for a locally nilpotent group of finite exponent.

Theorem 3.7. *Let G be a hypercentral group all whose quotients of finite exponent satisfy $(\mathcal{N})$. Let Γ be a finite group of automorphisms of G such that the external semidirect product $E = G \rtimes \Gamma$ is locally nilpotent. Assume that G has a subgroup H of finite exponent e which is a $C_{\mathrm{Aut}\, G}(\Gamma)$-basis. Then:*
(i) if G is nilpotent of class c then $\exp G$ is finite and divides the least common multiple of 2 and e^c;
(ii) if H is Γ-invariant then either $G = H$ or e is odd and $G = H \times N$ with $|N| = 2$.

Proof — If G is nilpotent of class c then H^Γ has finite exponent dividing e^c. Therefore (i) is a consequence of (ii) and it will be enough to prove the latter.

So, suppose $H^\Gamma = H$. Let $N = G^{e^2}$. Assume $NH < G$. Since G/N satisfies $(\mathcal{M})$ there is a maximal subgroup of G containing NH. The index of this subgroup in G is a prime p belonging to $\pi := \pi(H)$. As NH is Γ-invariant and Γ is finite, it follows that NH—and hence H— is contained in an E-invariant subgroup of index p. Since $H \rtimes \Gamma$ is hypercentral (actually, E is hypercentral, as a hypercentral-by-finite locally nilpotent group) this can be excluded by the same argument as in the proof for Theorem 3.4. Thus $NH = G$. Hence $G/N \simeq H/H \cap N$ has exponent at most e. Therefore $N = G^e$, so $N = N^e$. Now $H \cap N$ is a $C_{\mathrm{Aut}\, N}(\bar{\Gamma}\Delta)$-basis, where $\bar{\Gamma}$ and Δ are the groups of automorphisms of N induced by Γ and (by conjugation) by H respectively. Also, $N \rtimes \bar{\Gamma}\Delta$ is hypercentral, thus we may apply Lemma 3.2 and Lemma 3.1 to obtain that N is abelian and π-divisible. Now Lemma 3.3 implies that both the π-component of N and Γ are trivial, and $|N| \leq 2$. The conclusion is now clear. □

As for Corollary 2.7, a converse statement trivially holds: if $G = H \times N$ as in the last part of Theorem 3.7 then H (the odd component of G) is a characteristic Aut-basis of G.

Two further remarks about Theorem 3.7 are in order. Firstly, in the hypotheses of the theorem it is possible that the subgroup H differs greatly from G if it is not Γ-invariant. Indeed, let $G = \langle x \rangle \rtimes \langle a \rangle$

be a dihedral 2-group of order at least 8. Let $\varGamma = \operatorname{Aut} G$. Then $G \rtimes \varGamma$ is a 2-group, and it is easily seen that $\langle a \rangle$ is a $Z(\varGamma)$-basis, that is a $C_{\operatorname{Aut} G}(\varGamma)$-basis.

Secondly, if $\varGamma = 1$ the $\varGamma$-invariance condition becomes trivial, so Theorem 3.7 gives an explicit description of the Aut-bases of G, showing that only the obvious cases arise. Thus we have:

Corollary 3.8. *Let G be a periodic locally nilpotent group all of whose primary components are nilpotent, and let $H \leq G$. Assume that every primary component of H has finite exponent. Then H is an Aut-basis of G if and only if either $G = H$ or H is the $2'$-component of G and the 2-component of G has order 2.*

If p is a prime and G is a direct product of cyclic p-groups of unbounded orders then G has a subgroup H such that G/H is a Prüfer group. Now $\operatorname{Hom}(G/H, G) = 0$, so H is an End-basis of G. This shows that the hypothesis that the primary components of H have finite exponent is crucial in Corollary 3.8. Also, the group $G = N \rtimes A$ in Example 1.14 has the normal subgroup N of finite exponent as an Aut-basis, yet it is of infinite exponent. It is also possible to modify this example in order to make it into a non-periodic (but still centre-by-metabelian and Fitting) group with the same Aut-basis. To this aim let A_1 be a group isomorphic to $\mathbb{Q}_p = \{np^m \mid n, m \in \mathbb{Z}\} \leq \mathbb{Q}$ and let $\varepsilon : A_1 \twoheadrightarrow A$. Let $G_1 = N \rtimes A_1$, where A_1 acts on N via ε and the action of A. Then $C_{G_1}(N) = Z(N) \ker \varepsilon = Z(G_1)$. Let $\varGamma = C_{\operatorname{Aut} G_1}(N)$. By the Three Subgroup Lemma $[G_1, \varGamma] \leq Z(G_1)$. The only automorphism of G_1/N that acts trivially on $G_1/NZ(G_1)$ is the identity. Hence $\varGamma$ centralizes (N and) G_1/N and can be embedded in $\operatorname{Hom}\big(G_1/N, Z(N)\big) \simeq \operatorname{Hom}\big(\mathbb{Q}_p, \mathcal{C}_p\big) = 0$. Therefore N is an Aut-basis of G_1, as required.

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References

[1] R. Baer, Gruppen mit abzälbaren Automorphismengruppen, *Hamburger Math. Einzelschriften (Neue Folge)*, Heft **2** (1970) 1–122.

[2] M.R. Dixon, Automorphisms of nilpotent and related groups, in 'Groups '93 Galway/St. Andrews', vol. 1 (eds. C.M. Campbell et al.), London Math. Soc. Lecture Note Ser. **211**, Cambridge Univ. Press, 1995, pp. 156–171.

[3] M. Dugas and R. Göbel, On locally finite p-groups and a problem of Philip Hall's, *J. Algebra* **159** (1993) 115–138.

[4] D.W. Kueker, Definability, automorphisms, and infinitary languages, in 'The Syntax and Semantics of Infinitary Languages' (ed. J. Barwise), Lecture Notes in Math. **72**, Springer, Berlin, 1968, 152–165.

[5] P. Hall, 'The Edmonton Notes on Nilpotent Groups', Queen Mary College Math. Notes, 1969; also in 'The collected works of Philip Hall' (eds. K.W. Gruenberg and J.E. Roseblade), Oxford Science Publications, 1988, pp. 415–462.

[6] J.A. Hulse, Automorphism towers of polycyclic groups, *J. Algebra* **16** (1970) 347–398.

[7] O. Puglisi, A note on the automorphism group of a locally finite p-group, *Bull. London Math. Soc.* **24** (1992) 437–441.

[8] D.J.S. Robinson, A contribution to the theory of groups with finitely many automorphisms, *Proc. London Math. Soc. (3)* **35** (1977) 34–54.

[9] D.J.S. Robinson, Applications of cohomology to the theory of groups, in 'Groups—St. Andrews 1981' (eds. C.M. Campbell and E.F. Robertson), London Math. Soc. Lecture Note Ser. **71**, Cambridge Univ. Press, 1982, pp. 46–80.

[10] J.J. Rotman, 'An Introduction to the Theory of Groups' (4th. edn.), Springer, Berlin, 1995.

[11] S. Thomas, Complete existentially closed locally finite groups, *Arch. Math. (Basel)* **44** (1985) 97–109.

[12] A.E. Zaleskiĭ, An example of torsion-free nilpotent group having no outer automorphisms, *Mat. Zametki* **11** (1972) 221–226 = *Math. Notes* **11** (1972) 16–19.

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