Verbal sets and cyclic coverings

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ABSTRACT. We consider groups G such that the set of all values of a fixed word w in G is covered by a finite set of cyclic subgroups. Fernández-Alcober and Shumyatsky studied such groups in the case when w is the word $[x_1, x_2]$, and proved that in this case the corresponding verbal subgroup G' is either cyclic or finite. Answering a question asked by them, we show that this is far from being the general rule. However, we prove a weaker form of their result in the case when w is either a lower commutator word or a non-commutator word, showing that in the given hypothesis the verbal subgroup $w(G)$ must be finite-by-cyclic. Even this weaker conclusion is not universally valid: it fails for verbose words.

In [1], Gustavo Fernández-Alcober and Pavel Shumyatsky consider a special case of the following problem: given a (group) word w on n variables and a group G such that the verbal set $w\{G\} := \{w(g_1, \dots, g_n) \mid g_1, g_2, \dots, g_n \in G\}$ of all 'values' of w in G is covered by finitely many cyclic subgroups of G (in the sense that there is a finite subset F of G such that $w\{G\} \subseteq \bigcup\{\langle g\rangle \mid g \in F\}$), is it necessarily true that the verbal subgroup $w(G) := \langle w\{G\}\rangle$ is either cyclic or finite? A similar problem had been investigated earlier in [5].

When w is the word x of length one, then $w\{G\} = G$ and the question translates into: 'is it true that a group covered by finitely many cyclic subgroups must be either cyclic or finite?', the answer to which is well known to be in the positive, as was first pointed out by Baer (see [4], p. 105). On the other hand, there are cases in which the answer is negative. Indeed, let w be a word which is verbose, that is, such that there are groups G with the property that $w\{G\}$ is finite, and yet $w(G)$ is infinite—Ivanov [2] gave examples of such words. Then, for such a group G, the direct square $H = G \times G$ is such that $w\{H\} = w\{G\} \times w\{G\}$ is finite (and hence covered by finitely many cyclic subgroups) but $w(H) = w(G) \times w(G)$ is neither cyclic nor finite, actually it is not even finite-by-cyclic.

Fernández-Alcober and Shumyatsky solved the problem, by answering the question in the positive, in the case where w is the commutator word $[x_1, x_2]$: they showed that *if the set of all commutators $\{[x, y] \mid x, y \in G\}$ of a group G is covered by finitely many cyclic subgroups then the commutator subgroup G' is either cyclic or finite.* They asked the question whether the same positive answer can be obtained for an arbitrary (concise, that is: not verbose) word. We shall see that this is not the case. However, the following partial result holds in the positive. Recall that a commutator word is a word w such that $w(G) = 1$ for all abelian groups G. This is equivalent to the requirement that for each variable x appearing in w the sum of all exponents in the occurrences of x in w is zero. It is easy to show that all verbose words are commutator words (see, for instance, [4], Lemma 4.27).

Theorem A. *Let G be a group and w a word which is not a commutator word. If $w\{G\}$ is covered by finitely many cyclic subgroups then $w(G)$ is finite-by-cyclic.*

Thus the original question has a positive answer in the case of torsion-free groups and non-commutator words. That this result does not hold for mixed groups is shown in Example 1.5 below: the word w is in this case the power word x^n, where n is an arbitrary integer greater than 1 (also see Example 1.6). Moreover, even for torsion-free groups, the conclusion of Theorem A can fail to hold if w is a commutator word: the group H given above provides a counterexample.

In the final section we discuss the possibility of a direct generalization of the main result in [1], by considering lower commutator words $[x_1, \dots, x_n]$ for $n > 2$. Unlike what happens for $n = 2$, for all larger values of n the situation is the same as in the case of non-commutator words: in the usual hypothesis for the group G and the lower commutator word of weight n, the corresponding verbal subgroup, that is, the lower central term $\gamma_n(G)$, may be infinite and not cyclic (the relevant counterexamples are in Example 2.8), but we can prove that it is finite-by-cyclic.

Theorem B. *Let n be an integer greater than 1 and G be a group in which the set of all lower commutators $[x_1, x_2, \dots, x_n]$ of weight n on elements of G is covered by a finite set of cyclic subgroups. Then $\gamma_n(G)$ is finite-by-cyclic.*

However, in the extra hypothesis that the group is (locally) finite-by-nilpotent the same, stronger conclusion as in [1] can be drawn:

Theorem C. *Let n and G be as in Theorem B, and further assume that G is locally finite-by-nilpotent. Then $\gamma_n(G)$ is either finite or cyclic. Moreover, if $\gamma_n(G)$ is infinite then G is nilpotent of class n.*

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1. General results and proof of Theorem A

Let us introduce some terminology. For a word w and group G we say that the set $\mathcal{F}$ of cyclic subgroups of G is a w-covering if and only if $w\{G\} \subseteq \bigcup \mathcal{F}$; in this case $w(G) \leq \langle \bigcup \mathcal{F} \rangle$, as is obvious. Further, we say that $\mathcal{F}$ is a sharp w-covering in G if it is a w-covering with the extra property that every $K \in \mathcal{F}$ is generated by elements of $w\{G\}$, that is: $K = \langle K \cap w\{G\} \rangle$. If $\mathcal{F}$ is an arbitrary (finite) w-covering, by replacing each $K \in \mathcal{F}$ with $\langle K \cap w\{G\} \rangle$ we obtain a (finite) sharp w-covering of G. We also make use of the equivalence relations defined as follows on sets of elements of infinite order or on sets of infinite cyclic subgroups of G (and all denoted by $\sim$): any two such elements x and y (or the infinite cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$) are dependent, and we write $x \sim y$ (or $\langle x \rangle \sim \langle y \rangle$) in this case, if and only if $\langle x \rangle \cap \langle y \rangle \neq 1$.

The argument for the following lemma is essentially taken from a special case proved in [1], see Propositions 2.3 and 2.5.

Lemma 1.1. *Suppose that w is a group word and G is a group with a finite w-covering $\mathcal{F}$. Then:*

(i) *$w(G)$ is finitely generated and $G/C_G(w(G))$ is finite;*
(ii) *the periodic elements of $w(G)$ form a finite subgroup. If $w(G)$ is torsion-free then it is abelian;*
(iii) *if $\mathcal{F}$ is sharp, for all $K \in \mathcal{F}$ the subgroup $\langle L \in \mathcal{F} \mid K \cap L \neq 1 \rangle$ is cyclic-by-finite.*

Proof. That $w(G)$ is finitely generated follows at once from the fact that G has a finite w-covering. Hence, by elementary properties of FC-groups, to prove (i) and (ii) it will be enough to show that $w(G)$ is contained in the FC-centre F of G. Without loss of generality, we may assume that $\mathcal{F}$ is sharp. The set of all finite elements in $\mathcal{F}$ covers the (characteristic) set of all periodic elements in $w\{G\}$. Thus this latter set is finite, hence it is contained in F. Now consider the set W of all nonperiodic elements in $w\{G\}$, which also is a characteristic set. Since $\mathcal{F}$ is finite it is clear that W is partitioned into finitely many $\sim$-classes. So, these classes form a finite $(\operatorname{Aut} G)$-invariant set. Let $x \in W$ and let C be the $\sim$-class of x, then $H := \langle C \rangle$ has finitely many conjugates and $N := N_G(H)$ has finite index in G. Also let $\mathcal{K} = \{K \in \mathcal{F} \mid K \sim \langle x \rangle\}$, so $H = \langle \bigcup \mathcal{K} \rangle$, because $\mathcal{F}$ is sharp, and $\mathcal{K}$ is finite. Then $1 \neq \bigcap \mathcal{K} := Z \leq Z(H)$ and C is the set of all elements in W which are periodic modulo Z. This implies that C is a normal subset of H. Since $\mathcal{K}$ is finite $\{cZ \mid c \in C\}$ is finite and so H/Z is finite by Dietzmann's Lemma (see for instance [4], p. 45, Corollary 2). Therefore H is a finite, central extension of a cyclic group. Hence $\operatorname{Aut} H$ is finite and so $N/C_G(H)$ is finite. This shows that $H \leq F$. Therefore $W \subseteq F$, hence $w(G) \leq F$. Thus (i) and (ii) are proved.

Finally, still in the hypothesis that $\mathcal{F}$ is sharp, let $K \in \mathcal{F}$ and $K^* = \langle L \in \mathcal{F} \mid K \cap L \neq 1 \rangle$. If K is finite then K^* is generated by finitely many periodic elements of F, hence it is finite. If K is infinite and $1 \neq x \in K \cap w\{G\}$ then $x \in W$ and K^* is the subgroup H defined by x as in the previous paragraph. Now also part (iii) follows. □

Part (ii) of the previous lemma suggests that it can be useful to consider the case when $w(G)$ is torsion-free separately.

Lemma 1.2. *Let G be a group and w a word, and let $\mathcal{F}$ be a finite w-covering of minimal size in G. If $w(G)$ is torsion-free then $K \cap L = 1$ for all distinct $K, L \in \mathcal{F}$.*

Proof. By earlier remarks, we may assume that $\mathcal{F}$ is sharp. Let $K, L \in \mathcal{F}$ and let $H = \langle K, L \rangle$. Assume $K \neq L$ and $K \cap L \neq 1$. Parts (ii) and (iii) of Lemma 1.1 show that H is torsion-free abelian and cyclic-by-finite, hence cyclic. Then, by replacing L and K with H, we obtain a w-covering of size smaller than $|\mathcal{F}|$, and this is a contradiction. The lemma follows. □

Proposition 1.3. *Let G be a group and w a word. If G has a finite w-covering and $w(G)$ is torsion-free then G has exactly one sharp w-covering of minimal size.*

Proof. By hypothesis and by our earlier remarks G has a finite sharp w-covering $\mathcal{F}$ which has minimal size among all w-coverings of G. Let $\mathcal{F}^*$ be another finite sharp w-covering in G of the same size. Lemma 1.2 shows that $K \cap L = 1$ for all distinct $K, L \in \mathcal{F}$, and $\mathcal{F}^*$ has the same property. Let $K \in \mathcal{F}$ and $Y = K \cap w\{G\} \setminus 1$. Choose $y \in Y$. Then $y \in K^*$ for some $K^* \in \mathcal{F}^*$, because $\mathcal{F}^*$ is a w-covering. By the same reason, for all $g \in Y$ there exists $L^* \in \mathcal{F}^*$ such that $g \in L^*$. But $\langle g \rangle \cap \langle y \rangle \neq 1$, hence $L^* \cap K^* \neq 1$ and so $L^* = K^*$ by Lemma 1.2. Thus $Y \subseteq K^*$; as $K = \langle Y \rangle$ we have $K \leq K^*$. By reversing the argument, in view of Lemma 1.2 we obtain $K = K^*$. By iterating this argument we easily obtain $\mathcal{F} = \mathcal{F}^*$, and the proposition is proved. □

We are almost ready for the proof of Theorem A, which is the main result of this section. Before that, we record an obvious property of groups with a finite w-covering for a word w. If H is a subgroup of such a group G and $H \subseteq w\{G\}$ then H is covered by finitely many cyclic subgroups and so it is either finite or cyclic, by the result of Baer's cited in the introduction. When G is abelian $w\{G\} = w(G) \leq G$, hence the original problem has positive solution in this case.

Lemma 1.4. *Let G be an abelian group and let w be a word. If G has a finite w-covering then $w(G)$ is either cyclic or finite.*

Proof of Theorem A. The hypothesis clearly is inherited by quotients of G. The torsion subgroup T of $w(G)$ is finite, and $w(G/T) = w(G)/T$ is torsion-free, hence abelian, by Lemma 1.1 (ii). So we may factor out T and assume that $A := w(G)$ is a torsion-free (finitely generated, abelian) group. Since w is a not a commutator word, there is a variable x appearing in w such that the sum n of the exponents in all occurrences of x in w is not zero. Then $w(A) \geq A^n$. It follows from Lemma 1.4 that A^n is cyclic, and therefore A is cyclic. This completes the proof. □

Finally, we see that Theorem A cannot be strenghtened up to stating that, in the given hypotheses, $w(G)$ is either finite or cyclic. For all $n \in \mathbb{Z}$ let as call p_n the nth power word x^n. In our example the nontrivial values of the word considered are all aperiodic, this is of some relevance in view of the proof strategy developed in [1].

Recall that an automorphism of (finite) order p of an abelian group A is said to be splitting if and only if $1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} = 0$. We are interested here in the case when p is prime. In this case $1 + \alpha + \alpha^2 + \cdots + \alpha^i \in \operatorname{Aut} A$ for all $i < p - 1$ (the inverse being $1 + \alpha^{i+1} + \alpha^{2(i+1)} + \cdots + \alpha^{j(i+1)}$, where j is chosen such that $(j + 1)(i + 1) \equiv_p 1)$. Also recall that for every integer $n > 1$ and every prime p, unless $p = n = 2$, there exists a finite abelian group A of exponent n having a splitting automorphism of order p. For instance, A can be a homocyclic group of rank $p - 1$ and exponent n, and α can be defined as the automorphism described by the companion matrix of the polynomial $1 + X + X^2 + \cdots + X^{p-1}$ with respect to a given $\mathbb{Z}_n$-basis of A. (The easiest case is when $p = 2$; in this case α is just the inverting automorphism.)

Example 1.5. For every integer $n > 1$ there exists a (metabelian, finite-by-cyclic) group G with a finite p_n-covering, consisting of infinite cyclic subgroups only, such that $G^n = p_n(G)$ is neither finite nor cyclic (nor abelian, actually).

In fact, let p be a prime not dividing n and let A be a finite abelian group of exponent n having a splitting automorphism α of order p. Let $G = A \rtimes \langle x\rangle$, where x has infinite order and $a^x = a^\alpha$ for all $a \in A$. Each element of G has the form $x^k a$ for some $k \in \mathbb{Z}$ and $a \in A$, and its powers can be computed as follows. Let $t \in \mathbb{Z}$. If p divides k then $[x^k, A] = 1$ and $(x^k a)^t = x^{kt} a^t$. If p does not divide k then x^k acts on A as the splitting automorphism α^k, of order p. It follows that $(x^k a)^p = x^{kp} a^{1+\alpha^k+\alpha^{2k}+\cdots+\alpha^{(p-1)k}} = x^{kp}$ and, after writing $t = pq + r$ (where r is the remainder of t modulo p), $(x^k a)^t = x^{kpq}(x^k a)^r = x^{kt} a^{\sigma_{k,t}}$, where $\sigma_{k,t} = 1 + \alpha^k + \alpha^{2k} + \cdots + \alpha^{(r-1)k}$. Note that $\sigma_{k,t} \in \operatorname{Aut} A$, unless p divides t, in which case $\sigma_{k,t} = 0$. Thus, in the case when p does not divide kt we have $(x^k a)^t = (x^t b)^k$, where $b = a^{\sigma_{k,t}\sigma_{t,k}^{-1}}$. Now it is not hard to describe $p_n\{G\} = \{g^n \mid g \in G\}$: the n-th powers of elements of $\langle x^p\rangle A$ form the subgroup $\langle x^{pn}\rangle$, while the remaining n-th powers in G have the form $(x^k a)^n = (x^n b)^k$ for some integer k not divisible by p and some $a, b \in A$. It follows that $\mathcal{F} := \{\langle x^n b\rangle \mid b \in A\}$ is a finite sharp p_n-covering of G; note that its elements are infinite. On the other hand it is now clear that $G^n = A \rtimes \langle x^n\rangle$ is infinite and noncyclic.

Comparison between Theorem A and Lemma 1.4 on one side, and Theorems B and C on the other could suggest that the full conclusion that $w(G)$ is either cyclic or finite can be obtained, at least, in the case of nilpotent groups. This is not the case, though. The next example also shows that the offending covering set can be of very small cardinality, even of size 2.

Example 1.6. Let $G = (\langle a\rangle \times \langle c\rangle) \rtimes \langle b\rangle$, where a has infinite order, b and $c = [a, b]$ have order 2 and $c \in Z(G)$. If $g = a^i c^j b^k$ is a generic element of G, where $i \in \mathbb{Z}$ and $j, k \in \{0, 1\}$ then $g^2 = a^{2i}$ if $k = 0$ and $g^2 = a^{2i}c^i$ if $k = 1$. Therefore the set of all squares in G is $\langle a^2\rangle \cup \langle a^2 c\rangle$. The two cyclic subgroups appearing here form a sharp p_2-covering of G and $G^2 = \langle a^2, c\rangle$ is neither finite nor cyclic.

We close this section with a very simple lemma that we shall use repeatedly in the next section.

Lemma 1.7. *Let G be a group with a finite w-covering for some word w. Let u be a nonperiodic element in $w\{G\}$; let g be a periodic element of $C_G(u)$ and q be a positive multiple of the order of g. If $k_\lambda := gu^{q^\lambda} \in w\{G\}$ for infinitely many positive integers λ then $g = 1$.*

Proof. Let $\mathcal{F}$ be a finite w-covering of G. There exist $K \in \mathcal{F}$ and $\lambda, \mu \in \mathbb{N}$ such that $0 < \lambda < \mu$ and $k_\lambda, k_\mu \in K$. Then $u^{q^{\lambda+1}} = k_\lambda^q \in K$. As $\lambda < \mu$ we deduce that $u^{q^\mu} \in K$, hence $g \in K$ because $k_\mu \in K$. But K is infinite cyclic, because u is aperiodic, and g is periodic. Hence $g = 1$. □

2. Lower commutator words

Now we consider a specific family of commutator words: for all integer $n > 1$ let $k_n = [x_1, x_2, \ldots, x_n]$ be the weight-n lower commutator word. In this section we specialise the problem discussed in the previous section to the case when $w = k_n$ for some n, aiming at proving Theorems B and C (see the introduction). So, we fix $n > 1$ and suppose that G is a group with a finite k_n-covering set. Then $\gamma_n(G)$ is finitely generated and $C := C_G(\gamma_n(G))$ has finite index τ in G, by Lemma 1.1, and the torsion elements in $\gamma_n(G)$ form a finite subgroup T. Also, C stabilises the series $1 \leq \gamma_n(G) \leq \gamma_{n-1}(G) \leq \cdots \leq G' \leq G$; it follows that $[\gamma_t(G), \gamma_{n-t+1}(C)] = 1$ for all $t \in \{1, 2, \ldots, n\}$. When $t = 1$ this just means $\gamma_n(C) \leq Z(G)$. Further, let $\gamma_{\{n\}}(G) = \{[x_1, x_2, \ldots, x_n] \mid x_1, x_2, \ldots, x_n \in G\}$, the set of all values of k_n in G. We consider the notation just established as fixed throughout this section.

Lemma 2.1. *Suppose $n > 2$. Let s, t be integers such that $0 < s \leq t < n$ and $t > 1$; let $g_1, g_2, \ldots, g_{s-1}, g_{s+1}, \ldots, g_t \in G$ and assume that at least one of these elements is in C. Also let $c \in \gamma_{n-t}(C)$. Then the mapping*

$$x \in G \longmapsto [g_1, g_2, \ldots, g_{s-1}, x, g_{s+1}, \ldots, g_t, c] \in \gamma_n(G)$$

is a homomorphism. When $t = n-1$ its image $\{[g_1, g_2, \dots, g_{s-1}, x, g_{s+1}, \dots, g_{n-1}, c] | x \in G\}$ is either cyclic or finite.

Moreover, for all $g \in \gamma_{n-1}(G)$ the mapping

$$c \in C \longmapsto [g, c] \in [g, C]$$

is an epimorphism, and if $g \in \gamma_{\{n-1\}}(G)$ then $[g, C]$ is either cyclic or finite.

Proof. Let s, t and $g_1, g_2, \dots, g_{s-1}, g_{s+1}, \dots, g_t, c$ be as in the hypothesis and consider the mapping $\theta \colon x \in G \longmapsto [g_1, g_2, \dots, g_{s-1}, x, g_{s+1}, \dots, g_t] \in C \cap \gamma_t(G)$. The first mapping in the statement is then $\alpha \colon x \in G \mapsto [x^\theta, c] \in \gamma_n(G)$. We know that θ is a homomorphism modulo $\gamma_{t+1}(G)$. Thus, for all $x, y \in G$ there exists $h \in \gamma_{t+1}(G)$ such that $(xy)^\theta = hx^\theta y^\theta$. Now $[h, c] = 1$ by a remark preceding this lemma, and $(xy)^\alpha = [(xy)^\theta, c] = [hx^\theta y^\theta, c] = [x^\theta y^\theta, c] = [x^\theta, c][x^\theta, c, y^\theta][y^\theta, c]$. But $[x^\theta, c, y^\theta] = 1$, because $[x^\theta, c] \in \gamma_n(G)$ and $y^\theta \in C$. Therefore α is a homomorphism. If $t = n-1$ then $\operatorname{im}\alpha$ is a subgroup of G contained in $\gamma_{\{n\}}(G)$, hence it is either cyclic or finite. That the second mapping in the statement is an epimorphism is straightforward, and the proof follows as in the previous case. □

Lemma 2.2. *$\gamma_n(C)$ is either finite or cyclic.*

Proof. In view of [1], we may assume $n > 2$. For every $N \lhd C$ we define a binary relation $\doteqdot_N$ in $S := \gamma_{\{n\}}(C) \smallsetminus N$ as follows. For all $x, y \in S$, let $x \doteqdot_N y$ if and only if $\langle x, y\rangle$ is either finite or cyclic. If $g = [a_1, \dots, a_n]$ and $g' = [a'_1, \dots, a'_n]$ are elements of S (for some $a_1, \dots, a_n, a'_1, \dots, a'_n \in C$) and $a_i \neq a'_i$ for at most one of the subscripts i then g and g' belong to the image X of one of the homomorphisms defined in Lemma 2.1 in one of the cases in which the same lemma shows that X is either finite or cyclic. Then $g \doteqdot_N g'$. Now let $\approx_N$ denote the transitive closure of $\doteqdot_N$. Of course $\approx_N$ is an equivalence relation in S. We claim that all elements of S are pairwise $\approx_N$-equivalent. In order to prove that, arguing by contradiction, suppose that $\bar a := (a_1, \dots, a_n)$ and $\bar a' := (a'_1, \dots, a'_n)$ are two n-tuples of elements of C such that $g := [a_1, \dots, a_n]$ and $g' := [a'_1, \dots, a'_n]$ are two elements in S such that $g \not\approx_N g'$. Also suppose that $\bar a$ and $\bar a'$ have been chosen to have minimal distance d subject to this requirement; d is defined as the number of subscripts i such that $a_i \neq a'_i$; of course $d > 0$. Let i be the least integer such that $a_i \neq a'_i$. There exists $b \in C$ not belonging to the kernel of either of the homomorphisms $C \to \gamma_n(C)N/N$ defined by $x \mapsto [a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n]N$ and $x \mapsto [a_1, \dots, a_{i-1}, x, a'_{i+1}, \dots, a'_n]N$ (see Lemma 2.1). So we have that $g_1 := [a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n]$ and $g'_1 := [a_1, \dots, a_{i-1}, b, a'_{i+1}, \dots, a'_n]$ are two elements of S which are defined by n-tuples of elements of C whose distance is less than d. Therefore $g_1 \approx_N g'_1$. But $g_1 \doteqdot_N g$ and $g'_1 \doteqdot_N g'$ by a remark in the early part of the proof. It follows that $g \approx_N g'$, a contradiction. Thus our claim is established: all elements of S are $\approx_N$-equivalent. In the case when $N = 1$ this shows that the nontrivial elements of $\gamma_{\{n\}}(C)$ are either all periodic or all aperiodic. In the former case $\gamma_n(C)$ is finite, so we may assume the latter. Let p be a prime and $N = (\gamma_n(C))^p$. If $x, y \in \gamma_{\{n\}}(C) \smallsetminus N$ and $x \doteqdot_N y$ then $\langle x, y\rangle$ is cyclic (as x and y have infinite order); it follows that $N\langle x\rangle = N\langle y\rangle$. Then the same happens if we merely assume that $x \approx_N y$ rather than $x \doteqdot_N y$. Now, given any $g \in \gamma_{\{n\}}(C) \smallsetminus N$, all elements of $\gamma_{\{n\}}(C) \smallsetminus N$ are $\approx_N$ equivalent to g. So $\gamma_{\{n\}}(C) \smallsetminus N \subseteq N\langle g\rangle$, hence $\gamma_n(C) = N\langle g\rangle$. It follows that $\gamma_n(C)$ is cyclic. □

We shall make use of commutator formulae holding in some nilpotent groups. We collect them here for the convenience of the reader.

Lemma 2.3. *Let $k \in \mathbb{N}$, let p be a prime and let X be a nilpotent group of class $k+1$ and such that $\gamma_{k+1}(X)$ has exponent p. Let λ be a positive integer and assume that $\lambda > 1$ if $p = 2$. Let i be a positive integer such that $i \leq k$ and let $g_{i+1}, g_{i+2}, \dots, g_k \in X$.*

(i) *For all $a, b \in \gamma_i(X)$, $[a^{p^\lambda} b, g_{i+1}, g_{i+2}, \dots, g_k] = [a, g_{i+1}, g_{i+2}, \dots, g_k]^{p^\lambda} [b, g_{i+1}, g_{i+2}, \dots, g_k]$.*
(ii) *If $i > 1$, for all $g \in \gamma_{i-1}(X)$ and $a, b \in X$,*
$[g, a^{p^\lambda} b, g_{i+1}, g_{i+2}, \dots, g_k] = [g, a, g_{i+1}, g_{i+2}, \dots, g_k]^{p^\lambda} [g, b, g_{i+1}, g_{i+2}, \dots, g_k]$.

Proof. We will use the following standard result: if u, v, t are positive integers, $x \in \gamma_u(X)$ and $y \in \gamma_v(X)$ then $[y, x^t] \equiv [y, x]^t [y, x, x]^{\binom{t}{2}}$ and $[x^t, y] \equiv [x, y]^t [x, y, x]^{\binom{t}{2}}$ modulo $\gamma_{3u+v}(X)$ (see, for instance, [3], Corollary 1.1.7). Actually, we will apply these formulae for $t = p^\lambda$, in which case p divides $\binom{t}{2}$—this is where the hypothesis on λ becomes relevant. Also note that $\gamma_{i+2}(X)\big(\gamma_{i+1}(X)\big)^p \leq Z_{k-i}(X)$, hence $[ga, g_{i+1}, g_{i+2}, \dots, g_k] = [a, g_{i+1}, g_{i+2}, \dots, g_k]$ for all $a \in X$ and $g \in \gamma_{i+2}(X)\big(\gamma_{i+1}(X)\big)^p$.

The proof of (i) is by induction on $k - i$, the result being trivial if $i = k$. Let $c = [a^{p^\lambda} b, g_{i+1}, g_{i+2}, \dots, g_k]$. We have $[a^{p^\lambda} b, g_{i+1}] = [a^{p^\lambda}, g_{i+1}][a^{p^\lambda}, g_{i+1}, b][b, g_{i+1}]$; in order to compute c this product can be reduced modulo $N := \gamma_{i+3}(X)\big(\gamma_{i+2}(X)\big)^p$. By the formulae in the first paragraph, $[a^{p^\lambda}, g_{i+1}] \equiv [a, g_{i+1}]^{p^\lambda}$ modulo $\gamma_{3i+1}(X)\big(\gamma_{2i+1}(X)\big)^p \leq N$. Also, $[a^{p^\lambda}, g_{i+1}, b] \in N$. Therefore $c = \big[[a, g_{i+1}]^{p^\lambda}[b, g_{i+1}], g_{i+2}, \dots, g_k\big]$. By induction hypothesis now we obtain $c = [a, g_{i+1}, g_{i+2}, \dots, g_k]^{p^\lambda}[b, g_{i+1}, g_{i+2}, \dots, g_k]$. Thus (i) is proved. Now, if g, a and b are as in (ii), then $[g, a^{p^\lambda} b] \equiv [g, a]^{p^\lambda}[g, b]$ modulo $\gamma_{i+2}(X)\big(\gamma_{i+1}(X)\big)^p$. Hence $[g, a^{p^\lambda} b, g_{i+1}, g_{i+2}, \dots, g_k] = \big[[g, a]^{p^\lambda}[g, b], g_{i+1}, g_{i+2}, \dots, g_k\big]$, and the result in (ii) follows from (i). □

Next lemma is a special case of Theorem C, and actually is a key step in proving it and Theorem B.

Lemma 2.4. *If G is nilpotent and $\gamma_n(G)/T$ is cyclic then $\gamma_n(G)$ is either finite or cyclic.*

Proof. Assume false. Then $\gamma_n(G)$ is infinite and $T \neq 1$. We may further assume that T is minimal normal in G, hence T has prime order, say p, and $T \leq Z(G)$. Since G is nilpotent and $\gamma_n(G)/T$ is infinite cyclic this latter factor is central in G and $\gamma_{n+1}(G) \leq T$. Lemma 2.2 shows that $\gamma_n(G) \nleq Z(G)$, hence $\gamma_{n+1}(G) = T$ and G has class $n+1$. Among the nontrivial commutators of weight n in G some have infinite order, because $\gamma_n(G) \neq T$, and some are periodic: those of weight $n+1$. We shall see that this leads to a contradiction, by following the pattern of the proof of Lemma 2.2.

Let $a_1, \dots, a_n, a'_1, \dots, a'_n \in G$; suppose that $g := [a_1, \dots, a_n] \notin T$ and $g' := [a'_1, \dots, a'_n] \in T$. We aim at showing that $g' = 1$. Let $q = 4$ if $p = 2$ and $q = p$ otherwise. Assume first that the n-tuples $\bar{a} = (a_1, \dots, a_n)$ and $\bar{a}' = (a'_1, \dots, a'_n)$ have distance 1, that is: for some subscript i we have $a_i \neq a'_i$ but $a_j = a'_j$ for all $j \neq i$. Then $g^{q^\lambda} g' = [a_1, \dots, a_{i-1}, a_i^{q^\lambda} a'_i, a_{i+1}, \dots, a_n] \in \gamma_{\{n\}}(G)$ for all $\lambda \in \mathbb{N}$, by Lemma 2.3, and so $g' = 1$ by Lemma 1.7. In the general case, suppose that $\bar{a}$ and $\bar{a}'$ are chosen to have the least possible distance d subject to $g' \neq 1$. Let i be the least integer such that $a_i \neq a'_i$. Then $(a_1, \dots, a_{i-1}, a_i, a'_{i+1}, \dots, a'_n)$ has distance 1 from $\bar{a}'$, hence $[a_1, \dots, a_{i-1}, a_i, a'_{i+1}, \dots, a'_n] \in T$, and distance $d-1$ from $\bar{a}$, hence the minimality requirement on d yields $[a_1, \dots, a_{i-1}, a_i, a'_{i+1}, \dots, a'_n] = 1$. Similarly $[a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n] = 1$. Then $[a_1, \dots, a_{i-1}, a_i^q a'_i, a'_{i+1}, \dots, a'_n] = g'$ and $[a_1, \dots, a_{i-1}, a_i^q a'_i, a_{i+1}, \dots, a_n] = g^q$ (see Lemma 2.3). As the n-tuples used to obtain these two commutators have distance $d-1$ this is a contradiction. Therefore $g' = 1$. Now the proof is complete. □

The proof of Theorem B immediately reduces to the case when $\gamma_n(G)$ is torsion-free, and we shall assume this hypothesis in the next few lemmas. In this case Lemma 1.1 (ii) shows that $\gamma_n(G)$ is abelian, and we can draw a useful consequence from Lemma 2.1.

Lemma 2.5. *If $\gamma_n(G)$ is torsion-free and s, t, k are non-negative integers such that $s+t+k+1 = n > 2$ and $k > 0$, then $[{}_sG, C, {}_tG, \gamma_k(C)]$ is cyclic and central in G, and it is trivial if $\gamma_n(C) = 1$.*

Proof. Let $X := [{}_sG, C, {}_tG, \gamma_k(C)]$. By repeatedly using Lemma 2.1 we obtain $X^{\tau^{s+t}} = [{}_sG^\tau, C, {}_tG^\tau, \gamma_k(C)] \leq \gamma_n(C)$. Since $\gamma_n(G)$ is finitely generated torsion-free abelian and $\gamma_n(C)$ is cyclic, by Lemma 2.2, X is cyclic and $X = 1$ if $\gamma_n(C) = 1$. Next, since $\gamma_n(G)$ is abelian, $[X, G]^{\tau^{s+t}} = [X^{\tau^{s+t}}, G] \leq [\gamma_n(C), G] = 1$, hence $[X, G] = 1$. The result follows. □

We can also argue by induction on n, the case $n = 2$ having been settled in [1]. The next lemma isolates a special case in which this argument yields the result very easily.

Lemma 2.6. *Suppose that $\gamma_n(G)$ is torsion-free but not cyclic and n is minimal for such an example to exist. If G/C is cyclic then $\gamma_n(C) \neq 1$.*

Proof. Assume, by contradiction, $\gamma_n(C) = 1$, and let $\mathit{\Gamma} = \gamma_{n-1}(G)$. Since G/C is cyclic we have $G' = [G, C]$, hence $[\mathit{\Gamma}, C] = [C, {}_{n-2}G, C]$. Then $[\mathit{\Gamma}, C] = 1$ by Lemma 2.5. As a consequence $\mathit{\Gamma}/Z(\mathit{\Gamma})$ is cyclic and $\mathit{\Gamma}$ is abelian. Now fix $x \in G$ such that $G = C\langle x \rangle$. Then $K := C_{\mathit{\Gamma}}(x) \leq Z(G)$ and the mapping $\theta\colon g \in \mathit{\Gamma} \mapsto [g, x] \in \gamma_n(G)$ is an epimorphism whose kernel is K. Moreover $(\gamma_{\{n-1\}}(G))^\theta \subseteq \gamma_{\{n\}}(G)$. It follows that $\gamma_{n-1}(G/K) = \mathit{\Gamma}/K$ is torsion-free and $\gamma_{\{n-1\}}(G/K)$ is covered by finitely many cyclic subgroups. Therefore, the minimality hypothesis on n implies that $\gamma_{n-1}(G/K)$ is cyclic, but $\gamma_{n-1}(G/K) \simeq \gamma_n(G)$. Thus we have a contradiction and the proof is complete. □

Now we are ready for the main result of this section, from which Theorems B and C will follow.

Proposition 2.7. *Let G be a group such that $\gamma_{\{n\}}(G)$ is covered by a finite set of cyclic subgroups and $\gamma_n(G)$ is torsion-free. Then $\gamma_n(G)$ is cyclic.*

Proof. We continue to argue by contradiction and employ the notation in use thus far in this section. We assume that n is the minimal integer for which there exists a counterexample G. By Lemma 1.1 (ii) and Proposition 1.3, $\gamma_n(G)$ is abelian and G has exactly one sharp k_n-covering of minimal size, call it $\mathcal{F}$. As a further minimality requirement on G we may assume that, among all counterexamples relative to the (now fixed) integer n, our group G is such that $|G/C| + |\mathcal{F}|$ has the least possible value.

The first step in the proof consists in showing that at least one element of $\mathcal{F}$ is normal in G. For all $K \in \mathcal{F}$ and $g \in G$ we have $K^g \in \mathcal{F}$, by Proposition 1.3, and hence $K \cap K^g = 1$ if $K \neq K^g$, by Lemma 1.2. Therefore either $K \lhd G$ or $K_G = 1$. Let M be a maximal subgroup of G containing C; then $M \lhd G$, because $\gamma_n(G) \leq C$. Also, $|M/C_M(\gamma_n(M))| < |G/C|$, so $\gamma_n(M)$ is cyclic by our minimality requirement on G. If $\gamma_n(M) \neq 1$ there must exist $K \in \mathcal{F}$ such that $K \cap \gamma_n(M) \neq 1$. But $K \cap \gamma_n(M) \lhd G$, hence $K_G \neq 1$ and so $K \lhd G$. Therefore, we may assume $\gamma_n(M) = 1$. In this case $[\gamma_n(G), {}_{n-1}M] = 1$, because $\gamma_n(G) \leq M$. Let i be the least positive integer such that $[\gamma_n(G), {}_iM] = 1$. If $i > 1$ then $[\gamma_n(G), {}_{i-1}M]^\tau = [\gamma_n(G), {}_{i-2}M, M^\tau] = 1$, as $M^\tau \leq C$, and $[\gamma_n(G), {}_{i-1}M] = 1$ because $\gamma_n(G)$ is torsion-free. This contradicts the definition of i, hence $i = 1$, that is, $[\gamma_n(G), M] = 1$. Therefore $M = C$. Then G/C is cyclic. But we have also shown that $\gamma_n(M) = 1$; this is a contradiction, in view of Lemma 2.6. We conclude that there is a $K \in \mathcal{F}$ such that $K \lhd G$. Given such a K, let $N/K = \mathrm{tor}(\gamma_n(G)/K)$. Then $\gamma_n(G/N)$ is torsion-free and centralised by C/N, and $\gamma_{\{n\}}(G/N)$ is covered by a set of cyclic subgroups which has size $|\mathcal{F}| - 1$. By minimality, it follows that $\gamma_n(G)/N$ is cyclic. Of course also N is cyclic. Thus $\gamma_n(G)$ has rank 2.

Now consider the conjugation action of G on $\gamma_n(G)$. For any $x \in G$ let $\theta\colon a \in \gamma_n(G) \mapsto [a,x] \in \gamma_n(G)$. Clearly $\theta \in \mathrm{End}(\gamma_n(G))$ and $\mathrm{im}\,\theta^{n-1} \subseteq \gamma_{\{n\}}(G)$, hence $\mathrm{im}\,\theta^{n-1}$ is cyclic. If $\mathrm{im}\,\theta$ is not cyclic then $\gamma_n(G)/\mathrm{im}\,\theta$ is finite, and it follows that $\mathrm{rk}(\mathrm{im}\,\theta^{n-1}) = \mathrm{rk}(\mathrm{im}\,\theta) = 2$, a contradiction. Thus $[\gamma_n(G), x]$ is cyclic for all $x \in G$. Let X and Y be the centralizers in G of N and $\gamma_n(G)/N$ respectively; of course they both contain C and have index at most 2 in G. Moreover $X \cap Y/C$ embeds in the group of all automorphisms of $\gamma_n(G)$ stabilizing the series $1 < N < \gamma_n(G)$. This latter group is torsion-free (it is infinite cyclic), while G/C is finite, hence $C = X \cap Y$. Suppose $X \neq G \neq Y$. Then there exists $x \in G$ inducing the inversion map on both N and $\gamma_n(G)/N$. Then $[\gamma_n(G), x]$ has rank 2, but we showed that $[\gamma_n(G), x]$ must be cyclic. This contradiction proves that one of X and Y must be G; therefore $|G/C| = 2$ (that $G \neq C$ follows from Lemma 2.2). Now let $x \in G \setminus C$, so that $G = C\langle x\rangle$. Then $\gamma_{n+1}(G) = [\gamma_n(G), x]$ is (infinite) cyclic, according to what we proved earlier. Hence $\bar{G} := G/(\gamma_{n+1}(G))^2$ is nilpotent but $\gamma_n(\bar{G})$ is neither cyclic nor finite. This is impossible by Lemma 2.4, and we have a contradiction, thus completing the proof. □

Theorem B is an immediate consequence of Proposition 2.7: in the hypothesis of the theorem, if $\gamma_n(G)$ is infinite then $T = \mathrm{tor}(\gamma_n(G))$ is finite and $\gamma_n(G/T)$ is cyclic by Proposition 2.7. The proof of Theorem C requires some more argument:

Proof of Theorem C. Suppose that G is as stated in the hypothesis, and assume that $\gamma_n(G)$ is infinite. Let $T = \mathrm{tor}(\gamma_n(G))$. We know from Theorem B that T is finite and $\gamma_n(G)/T$ is (infinite) cyclic. Since G is locally finite-by-nilpotent this factor must be central in G. So, all we have to prove is that $T = 1$. Arguing by contradiction, we may assume that T is minimal normal in G. By Lemma 2.4 we deduce that $T \cap Z(G) = 1$. Fix $x \in Z_n(G) \setminus Z_{n-1}(G)$ and $t \in T \setminus 1$. The set $\{g \in G \mid [x,g] \in Z_{n-2}(G)\}$ is a proper subgroup of G, and also $C_G(t) < G$, because $T \cap Z(G) = 1$. Thus there exists $a_2 \in G$ such that $[x,a_2] \notin Z_{n-2}(G)$ and $[t,a_2] \neq 1$. Similarly, there exists $a_3 \in G$ such that $[x,a_2,a_3] \notin Z_{n-3}(G)$ and $[t,a_2,a_3] \neq 1$, and, by iterating the argument, we prove the existence of elements $a_2, a_3, \dots, a_n \in G$ such that $u := [x, a_2, a_3, \dots, a_n] \neq 1 \neq [t, a_2, a_3, \dots, a_n] =: g$. Note that $u \in Z(G)$, hence $u \notin T$. For any positive integer λ divisible by the order of g we have $u^\lambda g = [x^\lambda t, a_2, a_3, \dots, a_n]$, hence Lemma 1.7 may be invoked to obtain a contradiction. This contradiction completes the proof. □

We still have to prove that the conclusion of Theorem C does not necessarily hold in the more general hypotheses of Theorem B. The following example settles this point.

Example 2.8. For every integer $n > 2$ there exists a metabelian, residually nilpotent group G such that $\gamma_{\{n\}}(G)$ is covered by finitely many cyclic subgroups, although $\gamma_n(G)$ is neither cyclic nor finite.

Indeed, let $G = \big((\langle a\rangle \times \langle b\rangle) \rtimes \langle y\rangle\big) \rtimes \langle x\rangle$, where a and y have infinite order, b has order 2^{n-2} and x has order 2, and conjugation is defined as follows: $[a,y] = b$ and $[b,y] = 1$, x acts like the inversion map on $\langle a,b\rangle$, and $[y,x] = a$. Then $G' = \langle a,b\rangle$ and $\gamma_i(G) = \langle a^{2^{i-2}}, b^{2^{i-3}}\rangle$ if $2 < i \le n$, hence $\gamma_n(G)$ is infinite but not cyclic. On the other hand $\gamma_{\{n\}}(G)$ is contained in the set of all commutators of the form $c = [a^\lambda b^\mu, t_1, t_2, \dots, t_{n-2}]$, where $\lambda, \mu \in \mathbb{Z}$ and $t_1, t_2, \dots, t_{n-2}$ are in a (fixed) transversal T of $C_G(G')$ in G. Since G is metabelian and $b \in Z_{n-2}(G)$ such a c can be written as $c = [a, t_1, t_2, \dots, t_{n-2}]^\lambda$. Now, $G/C_G(G')$ is finite; it follows that the set $\{\,\langle [a, t_1, t_2, \dots, t_{n-2}]\rangle \mid t_1, t_2, \dots, t_{n-2} \in T\}$ is a (sharp, and certainly redundant) k_n-covering of G.

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