A note on torsion-free locally cyclic quasinormal subgroups

GIOVANNI CUTOLO AND ANTONELLA LEONE

Abstract. We discuss properties of subgroups as in the title. Our main result is a theorem of Maier-Schmid type: all locally cyclic, core-free quasinormal subgroups of a group G necessarily are contained in the hypercentre of G. This property was known to hold for periodic subgroups of this kind.

1. Introduction

A subgroup H of a group G is called *quasinormal* (or also *permutable*) if and only if $HK = KH$, that is, $HK \leq G$ for all $K \leq G$. As the name suggests, this property is a weak form of normality; indeed normal subgroups obviously are quasinormal, and quasinormal subgroups are ascendant (with defect at most $\omega+1$, by unpublished work by Napolitani and Stonehewer). General information on quasinormal/permutable subgroups may be found for instance in [4] or [6].

A measure of how much a quasinormal subgroup H of a group G is close to being normal is given by the restrictions that quasinormality of H forces on the factor group H^G/H_G. Here the fundamental result is a theorem by Maier and Schmid [5] showing that if G is finite then this factor group is hypercentrally embedded in G. This theorem is usually stated as follows: *if H is a core-free quasinormal subgroup of a finite group G, then $H \leq \bar{Z}(G)$.* Here, as usual, $\bar{Z}(G)$ denotes the hypercentre of G, and H being core-free means $H_G = 1$. The Maier-Schmid Theorem has been extended to several classes of infinite groups (see [3] for results and references), but it is known that it cannot be extended to arbitrary groups. Indeed, [2] provides an example of a core-free quasinormal subgroup not contained in the hypercentre in a finitely generated p-group, while an example in [3] shows that the Maier-Schmid Theorem may fail to hold even in metabelian p-groups. An indication of how partial is our understanding of quasinormal subgroups is the fact that it still appears to be unknown whether the Maier-Schmid Theorem holds for finite quasinormal subgroups of arbitrary groups.

Special attention has been devoted to cyclic, and slightly more generally to locally cyclic, quasinormal subgroups. For instance, Busetto ([1], Teorema 1.11; also see [6], Theorem 5.2.12) proved that the Maier-Schmid Theorem holds for any group G and periodic, locally cyclic core-free quasinormal subgroups. A more recent paper by Stonehewer and Zacher, [7], systematically studies the extension of previously known results about the embedding of a finite cyclic quasinormal subgroup H to the case when H is infinite cyclic, or even locally cyclic. A question not considered in [7] is whether Busetto's result still holds in this case. We answer the question in the positive, by proving the following:

Theorem. *Let H be a locally cyclic, core-free quasinormal subgroup of the group G. Then $H^G \leq Z_{\omega+1}(G)$. More precisely, if H is torsion-free then $H^G = [H,G] \rtimes H$, where $[H,G]$ is periodic and $[H,G] \leq Z_\omega(G)$.*

Standard examples, originally due to Iwasawa, show that in the situation of the theorem H is not necessarily contained in $Z_\omega(G)$, thus $\omega+1$ is the best possible bound. Indeed, let p be a prime and A an abelian p-group of infinite exponent. Let $G = A \rtimes H$, where $H = \langle h \rangle$ is infinite cyclic and h acts on A by means of the mapping $a \mapsto a^\pi$ for a given p-adic integer π such that $\pi \equiv_p 1$, and $\pi \equiv_4 1$ if $p = 2$. Then $H_G = 1$ and H is quasinormal in G. Moreover, $A = Z_\omega(G)$ intersects H trivially—it is clear that G is hypercentral of length $\omega+1$.

As said, the periodic case of our theorem is due to Busetto [1]; in this case $H \leq Z_\omega(G)$. In the same paper Busetto also obtained valuable information on the nonperiodic (that is, torsion-free) case, which we collect as Lemma 1 below. He showed that $H^G = TH$, where T, the torsion sugroup of H^G, is in $Z_\omega(G)$. Our theorem adds to that the information that H^G/T is a central factor of G. It is worth remarking that this is no longer true if the hypothesis $H_G = 1$ is dropped (see the final section of the paper, or [1, 7]), whereas Busetto's result is still valid in this case. Even though we will not make use of this in our proofs, we also record that with these more relaxed hypotheses Theorem 4.4 in [7] shows that if H^G/T is not central, then H^G is abelian.

Some more results on the embedding of locally cyclic quasinormal subgroups, which we gain as a by-product of the proof of our theorem, are collected in the final section. For instance, we observe that our proof, together with some remarks in [7], implies that *if H is a locally cyclic non-normal quasinormal subgroup of a group G, then all subgroups of H are quasinormal in G.* This was already known in the case when H is cyclic or periodic (see [1]).

2. Proof of the Theorem

We shall make use of the following known results, due to Busetto and Stonehewer.

2010 *Mathematics Subject Classification.* 20E15, 20F26, 20E20.

Key words and phrases. quasinormal subgroups; permutable subgroups; locally cyclic; hypercentral embedding.

Lemma 1 ([1], Teorema 2.2, Corollario 2.9)**.** *Let H be a torsion-free, locally cyclic quasinormal subgroup of the group G. Then the periodic elements of H^G form a subgroup T of $Z_\omega(G)$, and $H^G = T \rtimes H$.*

Lemma 2 (see [4], Lemmas 7.1.7 and 7.1.9)**.** *Let H be a quasinormal subgroup of the group G and $g \in G$. Then either $H \lhd S := H\langle g\rangle$ or S/H_S is finite.*

The proof of our main theorem is based on the following lemma, which actually is more general in scope, as the hypothesis does not require core-freeness.

Lemma 3. *Let H be a torsion-free locally cyclic, quasinormal non-normal subgroup of the group G, and suppose $H \not\leq \bar{Z}(G)$. Then $H^2 < H$. Fix $h \in H \smallsetminus H^2$. Let $T = \mathrm{tor}(H^G)$, $C = C_G(H)$ and $K = C_G(H^G/T)$. Then $|N_G(H)/C| \leq 2 = |G/K|$ and, for all $x \in G \smallsetminus K$,*

(i) *x acts on H^G/T like the inversion map;*
(ii) *if $x \notin N_G(H)$, then x is periodic;*
(iii) *$d := hh^x$ is a 2-element, and $H_{\langle x\rangle} = H^{|d|}$;*
(iv) *$\langle d\rangle = H^{\langle x\rangle} \cap T = H^{\langle x\rangle} \cap \langle x\rangle \leq Z(C)$;*
(v) *H^G is abelian and $C \lhd G$;*
(vi) *x acts like the inversion map on $H^G/(\langle x\rangle \cap H^G)$;*
(vii) *K/C is abelian and x acts like the inversion map on it. Thus $G/C = (K/C) \rtimes \langle xC\rangle$, where xC has order 2.*

Moreover, if $D := \langle hh^x \mid x \in G \smallsetminus K\rangle$, then $D \leq T \leq Z_\omega(G)$, and

(viii) *$H_G \neq 1$ if and only if $\exp D$ is finite. In this case $\exp D = |H/H_G|$.*

Proof. G/K is isomorphic to a group of automorphisms of $H^G/T = HT/T \simeq H$ (see Lemma 1), hence it is abelian and has at most one nontrivial periodic element; such an element must induce the inverting map on H^G/T. If gK is an element of infinite order in G/K, then $H \cap \langle g\rangle = 1$ and Lemma 2 yields $g \in N_G(H)$. Thus $G = N_G(H) \cup L$, where L/K is the torsion subgroup of G/K. As $N_G(H) < G$ it follows that $G = L$, so $|G/K| = 2$. Also, $K \cap N_G(H) = C$, therefore $|N_G(C)/C| \leq 2$. Fix $x \in G \smallsetminus K$. Then (i) is now clear and implies $\langle x\rangle \cap H = 1$. If x is not periodic, it normalizes H by Lemma 2. Thus we have (ii).

Statements (iii) and (iv) trivially hold if $x \in N_G(H)$, hence there is no loss in assuming $x \notin N_G(H)$ while proving them. We know, from Lemma 2 again, that $H\langle x\rangle/H_{\langle x\rangle}$ is finite and, by Maier-Schmid Theorem, x acts nilpotently on $H^{\langle x\rangle}/H_{\langle x\rangle}$, and hence on $H^G/TH_{\langle x\rangle}$. But x acts like the inverting automorphism on this group, hence $|H/H_{\langle x\rangle}| = |H^G/TH_{\langle x\rangle}|$ is a power of 2. It follows that $H^2 < H$. As a further consequence, $H^{\langle x\rangle}/H_{\langle x\rangle}$ is itself a 2-group, as the product of finitely many (two, as a matter of fact: see [6], Lemma 6.3.4) 2-groups.

Now, $H^{\langle x\rangle} \cap \langle x\rangle \leq \mathrm{tor}(H^{\langle x\rangle}) = H^{\langle x\rangle} \cap T$. On the other hand $H^{\langle x\rangle} = (H^{\langle x\rangle} \cap \langle x\rangle)H = (H^{\langle x\rangle} \cap T)H$ and $H \cap \langle x\rangle = H \cap T = 1$, therefore $|H^{\langle x\rangle} \cap \langle x\rangle| = |H^{\langle x\rangle} : H| = |H^{\langle x\rangle} \cap T|$; it follows that $H^{\langle x\rangle} \cap T = H^{\langle x\rangle} \cap \langle x\rangle$. By the previous paragraph, this group is a (finite) 2-group. Recall that $h \in H \smallsetminus H^2$ and let $d = hh^x$. Then $H = \langle h\rangle H_{\langle x\rangle}$ and $d \in H^{\langle x\rangle} \cap T \leq \langle x\rangle$, so d is a 2-element of $\langle x\rangle$. Also, $(H\langle d\rangle)^x = \langle h^x\rangle H_{\langle x\rangle}\langle d\rangle = H\langle d\rangle$, hence $H^{\langle x\rangle} = H\langle d\rangle$, thus $\langle d\rangle = H^{\langle x\rangle} \cap T$. For all $c \in C$ we have $cx \notin K$ and $d = hh^{cx}$, so we might substitute cx for x in the argument so far. Therefore $d \in \langle cx\rangle \cap \langle x\rangle$, hence $[d, c] = 1$. This shows that $d \in Z(C)$. Now (iv) is proved. An immediate consequence is $[H, d] = 1$; hence $H^{\langle x\rangle} = \langle d\rangle \times H$, from which it easily follows that d has the same order as $H/H_{\langle x\rangle}$, yielding (iii). Since $d \in Z(C)$ we also have $[H^x, C] = 1$, or equivalently $C^{x^{-1}} \leq C$. This remains true if x is replaced by any element of $G \smallsetminus K$ (even if this element normalizes H), hence all such elements normalize C. Therefore $C \lhd G$, which also shows that H^G is abelian, as all conjugates of H lie in $Z(C)$. Thus also (v) is proved (that H^G is abelian would also follow from Theorem 4.4 of [7]).

Now let θ_x be the endomorphism $a \mapsto aa^x$ of H^G. It follows from (i) that x induces the inversion map on $H_{\langle x\rangle}$; then (iv) yields $H^{\theta_x} \leq \langle x\rangle$. By applying this remark to H^g in place of H, for all $g \in G$ we have $(H^G)^{\theta_x} \leq \langle x\rangle$, that is, (vi).

Next we prove (vii). To this end, by a standard argument, it will be enough to show that all $x \in G \smallsetminus K$ have order 2 modulo C. Recall that $H = H_{\langle x\rangle}\langle h\rangle$ and $[H_{\langle x\rangle}, x^2] = 1$ because x induces the inversion map on $H_{\langle x\rangle}$. Moreover, if $d = hh^x$, we have $h^{x^2} = (h^{-1}d)^x = (h^{-1}d)^{-1}d = h$. Thus $x^2 \in C$ and we obtain the required conclusion.

Finally, let $D := \langle hh^x \mid x \in G \smallsetminus K\rangle$. We already know that $D \leq T$, and $T \leq Z_\omega(G)$ by Lemma 1. Each $x \in G \smallsetminus K$ acts like the inversion on $H_{\langle x\rangle}$, hence a subgroup of H is normalized by x if and only if it is contained in $H_{\langle x\rangle}$. Therefore $H_G = \bigcap\{H_{\langle x\rangle} \mid x \in G \smallsetminus K\} = \bigcap\{H^{|hh^x|} \mid x \in G \smallsetminus K\}$ by (iii), and this gives (viii). □

More can be said on the groups considered in Lemma 3. For instance, it can be proved that the subgroup D is normal in G and hence coincides with $T = \mathrm{tor}(H^G)$. We postpone this discussion to the final section of the paper and focus on our main goal.

Proof of the Theorem. Let H be a locally cyclic, core-free quasinormal subgroup of the group G. In view of Lemma 1 and Teorema 1.11 of [1] we may assume that H is torsion-free and only have to show that $H \leq \bar{Z}(G)$.

Suppose that G and H provide a counterexample. Then they satisfy the hypothesis of Lemma 3. We employ the same notation introduced there, and note that, by (viii) of that lemma, D has infinite exponent.

Since K and $N_G(H)$ are proper subgroups of G, we can fix $x \in G \setminus (K \cup N_G(H))$. Recall that x is periodic, by Lemma 3 (ii). Let $E = (\langle x\rangle \cap H^G)^G$. Then $E \leq Z_n(G)$ for some positive integer n, because of Lemma 1, and DE/E still has infinite exponent, so that HE/E is core-free in G/E, by Lemma 3 (viii). Thus we may factor out E and assume $\langle x\rangle \cap H^G = 1$. Lemma 3 (vi) shows that now x acts like the inverting automorphism on H^G. As a further consequence, the action of x on H^G commutes with that of any element of G, in other words: $xC \in Z(G/C)$. Therefore G/C is elementary abelian, by Lemma 3 (vii).

This makes it possible to refine the argument for Lemma 3 (vi) to obtain a stronger conclusion. For all $y \in G\setminus K$ and $g \in G$, since G/C is abelian the automorphism induced by g on H^G commutes with the endomorphism $\theta_y\colon a \mapsto aa^y$ of H^G. Therefore, for all $a \in H^G$, the elements $(a^g)^{\theta_y} = (a^{\theta_y})^g$ and a^{θ_y} have the same order. By Lemma 3 (vi), they both are in $\langle y\rangle$; therefore $\langle (a^g)^{\theta_y}\rangle = \langle a^{\theta_y}\rangle$. But $H^{\theta_y} = \langle hh^y\rangle$, hence $\operatorname{im}\theta_y \leq \langle hh^y\rangle$—recall that h is a fixed element in $H^2 \setminus H$. This shows that y acts like the inversion map on $H^G/\langle hh^y\rangle$.

For all $u \in K$ let $d_u = hh^{xu}$, so that $d_u = hh^{-u} = [u, h]$. Also, since $u^2 \in C$ we have $d_u^u = d_u^{-1}$. The preceding paragraph shows that xu acts like the inversion map on $H^G/\langle d_u\rangle$, and we know that the same is true of x. Therefore u centralizes $H^G/\langle d_u\rangle$, hence $[H^G, u] = \langle d_u\rangle$. Then, for all $u, v \in K$ there exists $\lambda_{uv} \in \mathbb{N}$ such that $[d_u, v] = d_v^{\lambda_{uv}}$. As K/C is abelian, $d_v^{-2\lambda_{uv}} = [d_u, u, v] = [d_u, v, u] = d_u^{\lambda_{uv}\lambda_{vu}}$, hence $(d_v^2 d_u^{\lambda_{vu}})^{\lambda_{uv}} = 1$. If the order $|d_v|$ of d_v is greater than $2|d_u|$, this implies $d_v^{2\lambda_{uv}} = 1$, that is, $[d_u^2, v] = 1$. Now, $d_{vu} = d_v^u d_u$ has the same order as d_v, hence $[d_u^2, u] = [d_u^2, vu] = 1$. But $d_u^u = d_u^{-1}$, hence $d_u^4 = 1$. We have proved that, for all $u \in K$, either $d_u^4 = 1$ or there exists no $v \in K$ such that $|d_v| > 2|d_u|$. Since $D = \langle d_u \mid u \in K\rangle$ we deduce that D has finite exponent. This is a contradiction; the proof is complete. □

3. Further remarks

In this final section we collect some more information on the groups G with a torsion-free, locally cyclic, quasinormal subgroup H which is neither normal nor contained in the hypercentre. Firstly, the subgroups of H still are quasinormal:

Proposition 4. *Let H be a locally cyclic, quasinormal non-normal subgroup of the group G. Then all subgroups of H are quasinormal in G.*

Proof. If H is periodic, this statement is Corollario 1.9 of [1]. If H is not periodic, Theorem 4.3 of [7], together with Lemma 2, shows that all subgroups of H are quasinormal in G unless some $g \in N_G(H)$ induces an automorphisms of infinite order on H. But in this latter case $H \not\leq \bar{Z}(H\langle g\rangle)$, hence Lemma 3 applies to show $|N_G(H)/C_G(H)| \leq 2$, a contradiction. □

We follow the notation employed in Lemma 3, thus $C = C_G(H) = C_G(H^G)$, $K = C_G(H^G/T)$, where $T = \operatorname{tor}(H^G)$, h is a fixed element of $H \setminus H^2$, and $D = \langle hh^x \mid x \in G \setminus K\rangle$. It is an easy consequence of the same lemma that $H^G = H \times D^G$ and so $T = D^G$.

For all $x \in G \setminus K$ we proved that $d := hh^x$ is a power of x, hence of x^2, since $[H, d] = 1$. As a matter of fact, $d \in \langle x^4\rangle$, otherwise $H\langle x\rangle/H^2\langle x^4\rangle$ would be isomorphic to the dihedral group of order 8, and $H\langle x^4\rangle/H^2\langle x^4\rangle$ would not be quasinormal in it.

Proposition 5. *Let H be a a torsion-free, locally cyclic, quasinormal non-normal subgroup of the group G, and suppose $H \not\leq \bar{Z}(G)$. Then, in the notation of Lemma 3, we have:*

(i) *G/C is a 2-group;*
(ii) *$D = \operatorname{tor}(H^G)$;*
(iii) *for all $g \in K$, $H^G/C_{H^G}(g)$ is a finite 2-group of rank at most 2.*

Proof. (i) Let $g \in G$. If g has infinite order modulo C then g normalizes H, by Lemma 2. But then $g^2 \in C$ by Lemma 3, a contradiction. Therefore G/C is periodic. Now, by Lemma 3 again, $T = D^G$ is a 2-group, and $H^G \leq \bar{Z}(K)$ by Lemma 1, thus $K/C_K(T)$ is a 2-group. It follows that K/C is a 2-group. Therefore, G/C is a 2-group, as required.

(ii) We already observed that $D^G = \operatorname{tor}(H^G)$, so we only need to prove that $D \lhd G$. For all $x \in G \setminus K$ let $d_x = hh^x$. If $D \ntrianglelefteq G$ then there exist $x, y \in G \setminus K$ such that $(d_x)^y \notin D$. Let $S := \langle h, x, y\rangle$. By Proposition 4, $\langle h\rangle$ is quasinormal, hence Lemma 4.2 of [7] yields $\langle h\rangle^S = \langle h\rangle\langle [h, x]\rangle\langle [h, y]\rangle = \langle h\rangle\langle h^{-2}d_x\rangle\langle h^{-2}d_y\rangle = \langle h\rangle\langle d_x, d_y\rangle$. Thus $\langle d_x, d_y\rangle = \operatorname{tor}(\langle h\rangle^S) \lhd S$; it follows that $(d_x)^y \in \langle d_x, d_y\rangle \leq D$. This contradiction proves (ii).

(iii) Let $x \in G \setminus K$. From the proof of Lemma 3 (vi), recall the endomorphism $\theta_x\colon a \mapsto aa^x$ of H^G. We have $\operatorname{im}\theta_x \leq \langle x\rangle \cap H^G \leq D$, hence $H^G/\ker\theta_x$ is a cyclic 2-group, and x acts on $\ker\theta_x$ like the inversion map. Now let $g \in K$. By substituting xg for x in the argument, we see that $H^G/\ker\theta_{xg}$ is a cyclic 2-group as well. Now, if $E = \ker\theta_x \cap \ker\theta_{xg}$, both x and xg act on E like the inversion map, hence $E \leq C_{H^G}(g)$, which proves the result. □

Examples of groups G and subgroups H satisfying the properties required in Proposition 5 are already known, some are in [1] and [7]. Here we produce a few more, with the purpose of shedding some light on the possible structure of H^G. For the convenience of the reader we prove a condition for quasinormality which is no doubt known.

Lemma 6. *Let $G = A \rtimes H$ be a group, where A is periodic and H acts hypercentrally on it. Then H is quasinormal in G if and only if H induces on A power automorphisms fixing all elements of order 4 in A.*

Proof. If H is quasinormal in G and $B \leq A$, then $B^H \leq A \cap HB = (A \cap H)B = B$, thus H normalizes B. Therefore H acts on A by means of power automorphisms.

Conversely, suppose that H induces by conjugation power automorphisms on A. To prove that H is quasinormal in G it is enough to show that $H\langle ah\rangle = \langle ah\rangle H$ for all $a \in A$ and $h \in H$. Thus, there is no loss in assuming that $A = \langle a\rangle$ is cyclic. We can also factor out $C_H(A)$; this makes G a finite nilpotent group, and we may assume that it is a p-group for some prime p. Now H is isomorphic to a subgroup of $\operatorname{Aut} A$, and the hypothesis on the elements of order 4 in A implies that H is cyclic. So G is metacyclic, and the known characterization of finite meta-hamiltonian p-groups (see, for instance, [6], Theorems 2.3.1 and 5.1.1) shows that all subgroups of G are quasinormal. □

Our first example shows that D may have arbitrary rank.

Example 7. Let A be an abelian 2-group of finite exponent $2^n > 4$, and let H be a torsion-free, locally cyclic group such that $H^2 < H$. Let $h \in H \setminus H^2$. Form the split extension $B = A \rtimes H$, where $[A, H^2] = 1$ and $a^h = a^{1+2^{n-1}}$ for all $a \in A$. Let $G = B \rtimes \langle x\rangle$, where x has order 2^m, for some positive integer $m < n$, $[A, x] = 1$ and $k^x = k^{-1}$ for all $k \in H$. Then $H^2 \lhd G$ and, modulo H^2, H acts by means of the power automorphism $g \mapsto g^{1+2^{n-1}}$ on the abelian group $A\langle x\rangle$. It follows from Lemma 6 that H is quasinormal in G. Thus H satisfies the hypothesis of Proposition 5. Now $\operatorname{tor}(H^G) = [A, h] = A^{2^{n-1}}$, an elementary abelian group that we can make of arbitrary rank.

This example also shows that every torsion-free, locally cyclic group which is not 2-divisible can occur as H in Proposition 5. It is also easy to see that the exponent of $\operatorname{tor}(H^G)$ can be an arbitrary power of 2: for all positive integers λ, in the group $G = \langle h, x, d \mid x^4 = d = hh^x,\ [h, d] = 1 = d^{2^\lambda}\rangle$ the subgroup $\langle h\rangle$ is quasinormal and $\langle d\rangle = \operatorname{tor}(\langle h\rangle^G)$ actually has order 2^λ. This group occurs as a subgroup of the group discussed in the next example.

Example 8. Let H be a torsion-free, locally cyclic group such that $H^2 < H$, and fix $h \in H \setminus H^2$. Let λ_1, λ_2 be positive integers. Then there exists a group $G = \langle x_2\rangle\langle x_1\rangle H$, in which H is quasinormal but not normal nor contained in $\bar{Z}(G)$, and $\operatorname{tor}(H^G) = \langle d_1\rangle \times \langle d_2\rangle$, where $d_1 = hh^{x_1}$ and $d_2 = hh^{x_2}$ have orders 2^{λ_1} and 2^{λ_2} respectively.

To construct such a group, start with $A := \langle d_1\rangle \times \langle d_2\rangle \times H$, where d_1 and d_2 have the required orders. Set $\lambda = \max\{\lambda_1, \lambda_2\}$. Then A has two automorphisms, α_1 and α_2, defined by

$$a^{\alpha_i} = a^{-1}d_i^t,\ \text{for all } t \in \mathbb{N} \text{ and } a \in h^tH^{2^\lambda}; \qquad d_i^{\alpha_i} = d_i; \qquad d_j^{\alpha_i} = d_j^{-1},$$

if $\{i, j\} = \{1, 2\}$. It is easy to check that α_1 and α_2 commute, and both have order 2. For an arbitrary integer $\mu > 1$ we may extend A to a group $\langle x_1\rangle A$, where x_1 acts like α_1 on A and $x_1^{2^\mu} = d_1$. Let $y = x_1^{-1-2^{\mu-1}}h$. Then also y acts like α_1 on A, and $y^2 = x_1^{-2-2^\mu}h^{\alpha_1}h = x_1^{-2}d_1^{-1}h^{-1}d_1h = x_1^{-2}$. Since $[\alpha_1, \alpha_2] = 1$ this implies that we can extend α_2 to an automorphism β of $\langle x_1\rangle A$ mapping x_1 to y. Now, since $y^2 = x_1^{-2}$ and $[x_1^2, h] = 1$,

$$x_1^{\beta^2} = y^\beta = (x_1^{-1-2^{\mu-1}}h)^\beta = y(y^2)^{-1-2^{\mu-2}}h^{-1}d_2 = x_1^{-1-2^{\mu-1}}h(x_1^{-2})^{-1-2^{\mu-2}}h^{-1}d_2 = x_1d_2, \qquad \text{so} \qquad x_1^{\beta^4} = x_1d_2^2.$$

Thus β^4 acts on $A\langle x_1\rangle$ as the inner automorphism determined by d_2. So there exists a group $G = \langle x_2\rangle\langle x_1\rangle A$, where x_2 induces β on $\langle x_1\rangle A$ and $x_2^4 = d_2$. We claim that G and H have the required properties. It is clear that $G = \langle x_2\rangle\langle x_1\rangle H$, $H \nleq \bar{Z}(G)$ and $H^G = \langle d_1\rangle \times \langle d_2\rangle \times H$. So, we only have to check that H is quasinormal in G. First, we show that x_1 inverts x_2^2. Indeed $(x_2^2)^{x_1} = x_2^2[x_2^2, x_1] = x_2^2[x_1, \beta^2]^{-1} = x_2^2d_2^{-1} = x_2^{-2}$, because $d_2 = x_2^4$. To prove that H is quasinormal it will be enough to show that $S := \langle H, g\rangle = \langle g\rangle H$ for all $g \in G$. We may assume that $g \notin C := C_G(H)$. Now $C = \langle x_2^2\rangle \times \langle x_1^2\rangle \times H \lhd G$, and $|G/C| = 4$, hence $g = xc$, where $c = x_2^{2i}x_1^{2j}a \in C$ for suitable integers i, j and $a \in H$, and $x \in \{x_1, x_2, x_2x_1\}$. We assume $a \in h^tH^{2^\lambda}$, where $t \in \mathbb{N}$, hence $a^{x_1}a = d_1^t$, $a^{x_2}a = d_2^t$ and $a^{x_2x_1}a = a^2(d_1d_2)^{-t}$. Consider the case $x = x_1$. In this case $g^2 = x_1^2c^{x_1}c = x_1^2x_1^{4j}d_1^t$, because x_1 inverts x_2^2. It follows that $\langle g\rangle$ contains $\langle x_1^{2^\mu}\rangle = \langle d_1\rangle$. Now, $\langle d_1\rangle H \lhd S$ hence $S = \langle g\rangle\langle d_1\rangle H = \langle g\rangle H$. The case when $x = x_2$ can be settled similarly. In the remaining case $x = x_2x_1$. We shall check that $|S : H|$ equals the order $|\langle g\rangle \colon \langle g\rangle \cap H|$ of g modulo H. We have $(x_2x_1)^2 = x_2^2x_1^{-1-2^{\mu-1}}hx_1 = x_2^2x_1^{-2^{\mu-1}}h^{-1}d_1 = x_2^2x_1^{2^{\mu-1}}h^{-1}$, so $g^2 = x_2^2x_1^{2^{\mu-1}}h^{-1}c^{x_2x_1}c = x_2^2x_1^{2^{\mu-1}}h^{-1}a^2(d_1d_2)^{-t}$. The factors appearing in this product commute pairwise, hence $g^4 = d_2d_1h^{-2}a^4(d_1d_2)^{-2t} \in H^S = \langle d_1d_2\rangle \times H$. Thus $|S/H^S| = 4$, and the order of g^4 modulo H is $|d_1d_2| = 2^\lambda = |H^S : H|$. It follows that $|\langle g\rangle \colon \langle g\rangle \cap H| = |S : H|$, as claimed, and this implies $S = H\langle g\rangle$, also in this case. Now we have proved that H is quasinormal in G. □

What is of interest in this last example is that an analogous construction starting with three (rather than two) arbitrary integers $\lambda_1, \lambda_2, \lambda_3$ would not be possible. This follows from the argument in the proof of the main theorem. For, assume $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and let the group G have a torsion-free, locally cyclic quasinormal subgroup $H = \langle h\rangle H^{2^{\lambda_3}}$. In the usual notation of Lemma 3, let three elements $x_1, x_2, x_3 \in G \setminus K$ define elements $d_i = hh^{x_i}$. Then, by the proof of the theorem, the images of d_2 and d_3 modulo $\Omega_{\lambda_1}(H^G)$ satisfy a certain equality from which it follows that either the order $2^{\lambda_2-\lambda_1}$ of the image of d_2 is at most 4, or $\lambda_3 - \lambda_2 \leq 1$.

These considerations do not exclude the possibility that the structure of D be fairly arbitrary. We leave open the question: is it the case that every abelian 2-group of finite exponent may occur as the torsion subgroup of H^G, where H and G satisfy the hypothesis of Proposition 5? Another question is suggested by a common feature of our examples. Still in the situation of Proposition 5, is $G/C_G(H^G)$ necessarily abelian? If so, $G/C_G(H^G)$ would even be elementary abelian, by Lemma 3.

References

[1] G. Busetto, *Proprietà di immersione dei sottogruppi modulari localmente ciclici nei gruppi*, Rend. Sem. Mat. Univ. Padova **63** (1980), 269–284.

[2] G. Busetto and F. Napolitani, *On some questions concerning permutable subgroups of finitely generated groups and projectivities of perfect groups*, Rend. Sem. Mat. Univ. Padova **87** (1992), 303–305.

[3] M.R. Celentani, A. Leone and D.J.S. Robinson, *The Maier-Schmid problem for infinite groups*, J. Algebra **301** (2006), no. 1, 294–307.

[4] J.C. Lennox and S.E. Stonehewer, *Subnormal subgroups of groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1987, Oxford Science Publications.

[5] R. Maier and P. Schmid, *The embedding of quasinormal subgroups in finite groups*, Math. Z. **131** (1973), 269–272.

[6] R. Schmidt, *Subgroup lattices of groups*, de Gruyter Expositions in Mathematics, vol. 14, Walter de Gruyter & Co., Berlin, 1994.

[7] S.E. Stonehewer and G. Zacher, *Cyclic quasinormal subgroups of arbitrary groups*, Rend. Sem. Mat. Univ. Padova **115** (2006), 165–187.

E-mail address: cutolo@unina.it
URL: http://www.dma.unina.it/~cutolo

E-mail address: anleone@unina.it

Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Via Cintia — Monte S. Angelo, I-80126 Napoli, Italy