On Groups That Are Dominated by Countably Many Proper Subgroups

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In memoriam: Brian Hartley and David McDougall

Abstract

In this work we study groups for which there is a countable set of proper subgroups with the property that every proper subgroup is contained in some member of the set.

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1. Introduction

This article is the third in a series of studies of countability restrictions on the partially ordered set of subgroups of a group. In [16] and [6] the authors considered the property that a group have only countably many subgroups (CMS). This is a very strong property and its consequences for the group structure are considerable. For example, in [6] the authors were able to classify all soluble groups with CMS: they are precisely the soluble minimax groups without abelian factors of type $p^\infty \times p^\infty$ for any prime $p$.

In a subsequent paper [2] the present authors studied the much weaker property that a group have countably many maximal subgroups (CG). Modules and rings with countably many maximal submodules or right ideals respectively played an important part in

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the investigation. As a result numerous classes of soluble groups with CG were identified and several examples of finitely generated soluble groups with uncountably many maximal subgroups were described.

In the present work we study a property that is intermediate between the properties CMS and CG. Let $\mathcal{S}$ and $\mathcal{T}$ be non-empty sets of proper subgroups of a group $G$. Then $\mathcal{T}$ is said to dominate $\mathcal{S}$ if each member of $\mathcal{S}$ is contained in a member of $\mathcal{T}$. If there is a countable set of proper subgroups of $G$ that dominates $\mathcal{S}$, then $\mathcal{S}$ is said to be countably dominated or CD in $G$. Finally, should the set of all proper subgroups of $G$ be dominated by a countable subset $\mathcal{S}$, then we will say that $G$ is countably dominated or a CD-group: we will also say that $\mathcal{S}$ dominates $G$. It is clear how the property CD should be defined for modules.

Evidently every CMS-group is a CD-group, and by Lemma 2.1 below every CD-group is a CG-group. In fact the three properties are distinct. Indeed the direct product $G$ of groups of each prime order has CD since the set of subgroups of prime index dominates $G$. However, $G$ is not a CMS-group, for plainly it has uncountably many subgroups. Also the group $p^{\infty} \times p^{\infty}$ has no maximal subgroups, so it is a CG-group, but it is not a CD-group by a simple direct argument (or Lemma 3.3 below). Hence the general situation is:

$$\text{CMS} \subset \text{CD} \subset \text{CG}.$$ 

Lemma 2.2 also shows that for finitely generated groups the properties CD and CG are one and the same. On the other hand, the wreath product $\mathbb{Z}_p \wr \mathbb{Z}$ is a finitely generated metabelian group, so it satisfies max-$n$ and hence is a CG-group by [2: Theorem 6]. But the group does not have the property CMS, since it is not a minimax group. Indeed it appears that the property CD is much closer to CG than to CMS.

**Results**

In Section 2 we give a number of general results about the property CD; we also describe various sources of groups with this property. In Section 3 properties of $p$-adically irreducible modules are developed which are needed in the proofs of the main theorems in Sections 4 and 5. Virtually nilpotent CD-groups are characterized in Section 4. The approach adopted here involves modules over finite
groups whose underlying abelian groups are divisible \( p \)-groups. The theory of such modules was developed by Zaicev [17] and Hartley [8]. Our first result is:

**Theorem 4.1.** Let the group \( G \) have a nilpotent normal subgroup \( N \) of finite index and write \( F = G/N \) and \( \bar{N} = N^{ab} \). Then \( G \) is a CD-group if and only if \( \bar{N} \) has a finitely generated \( F \)-submodule \( X \) such that \( (\bar{N}/X)_p \) is a Černikov group whose finite residual is a near direct sum of finitely many, pairwise non-near isomorphic, \( p \)-adically irreducible \( F \)-submodules for each prime \( p \).

The terminology employed here is as follows. A near direct sum of modules is a sum of submodules in which the intersection of any one submodule with the sum of all the others is bounded as a \( \mathbb{Z} \)-module, i.e., it has finite exponent. Two modules are said to be near isomorphic if they have isomorphic quotients modulo bounded submodules. Finally, if \( p \) is a prime, a module whose underlying abelian group is a \( p \)-group is said to be \( p \)-adically irreducible if it is unbounded, but every proper submodule is bounded.

There is little prospect of classifying soluble CD-groups, as there are too many different types. Indeed even the case of metanilpotent groups with min-\( n \) presents a significant challenge. Nevertheless our conclusions about these groups are quite complete.

**Theorem 5.1.** Let \( G \) be a metanilpotent group satisfying min-\( n \) and let \( A = \gamma_\infty(G) \), the last term of the lower central series of \( G \). Then \( G \) has the property CD if and only if the following hold:

(i) \( G/A \) is a nilpotent Černikov group whose finite residual is locally cyclic;

(ii) \( A^{ab} \) has the property CD as a \( G \)-module.

Some light on nature of the second condition in Theorem 5.1 is shed by two equivalent descriptions.

**Theorem 5.7.** Let \( A \) be an artinian module over a nilpotent Černikov group \( Q \). Then the following are equivalent.

(i) \( A = A_1 + A_2 + \cdots + A_n + S \) where the \( A_i \) are pairwise non-near isomorphic, \( p \)-adically irreducible \( Q \)-modules for various primes \( p \) and \( S \) is a bounded \( Q \)-submodule.
(ii) A has countably many submodules.

(iii) A has the property CD as a $Q$-module.

Here the symbol $+$ denotes a near direct sum. The proof of Theorem 5.7 calls for a detailed analysis of the structure of artinian modules over nilpotent Černikov groups. This in turn rests on important work of Hartley and McDougall [10] on the structure of such modules in the non-modular case. Notice the similarity between the module conditions in Theorems 4.1 and 5.7.

The situation for soluble groups satisfying min-$n$ with derived length $> 2$ is certainly more complex: for there are uncountable groups of this type (see [9]), whereas all CD-groups are countable.

The property CD is not inherited by subgroups, even in the finitely generated case, as is shown by the group $\mathbb{Z}_p \wr \mathbb{Z}$ – further examples are given at the end of Section 5. Thus we should expect the property that every subgroup of a group have CD – which will be denoted by SCD – to be much stronger than CD. As evidence of this we present a result which describes the soluble SCD-groups.

**Theorem 6.1.** A soluble group $G$ is an SCD-group if and only if it has finite abelian ranks and there are no factors in $G$ of type $p^\infty \times p^\infty$ for any prime $p$.

Recall that a soluble group has finite abelian ranks if it has a series of finite length whose factors are abelian groups with all ranks finite. Notice that the groups described in Theorem 6.1 are similar to the soluble CMS-groups, the difference being that infinitely many primes may occur as orders of elements. Moreover, Theorem 6.1 enables us to characterize virtually soluble SCD-groups as well, with the help of Lemma 2.4 below.

Our final result Theorem 6.2 characterizes the periodic generalized radical groups which are SCD-groups.

**Notation**

(i) CD: countably dominated.

(ii) CMS: countably many subgroups.

(iii) CG: countably many maximal subgroups.

(iv) $H_K$: the $K$-core of $H$. 

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(v) $G^{ab}$: the abelianization of $G$.
(vi) $\bar{Z}(G)$: the hypercentre of $G$.
(viii) $\pi(G)$: the set of primes dividing orders of elements of $G$.
(ix) $\gamma_i(G)$: a term of the lower central series of $G$.
(x) $A_1 + A_2 + \cdots + A_n$: the near direct sum of submodules $A_i$.
(xi) $A \approx B$: modules $A$ and $B$ are near isomorphic.
(xii) Der$(G, A)$, Inn$(G, A)$: sets of derivations and inner derivations.

All modules are right modules.

2. General properties

Our first observation is that the property CD is closely related to that of having countably many maximal subgroups, denoted by CG. It will be convenient to establish this fact in a slightly more general form. First recall that two proper subgroups $H$ and $K$ of a group $G$ are said to be comaximal if $G = \langle H, K \rangle$.

**Lemma 2.1.** Let $G$ be a CD-group. Then $G$ is a countable CG-group and every set of pairwise comaximal subgroups of $G$ is countable.

**Proof.** By hypothesis there is a countable set of proper subgroups $\mathcal{X}$ that dominates the set of all proper subgroups of $G$. Let $\mathcal{S}$ be a set of pairwise comaximal subgroups of $G$. Each $H \in \mathcal{S}$ is contained in some $H^* \in \mathcal{X}$. Moreover, if $H$ and $K$ are distinct members of $\mathcal{S}$, then $H^* \neq K^*$ since $G = \langle H^*, K^* \rangle$. Since $\mathcal{X}$ is countable, it follows that $\mathcal{S}$ is countable. Also distinct maximal subgroups are pairwise comaximal, so $G$ has the property CG. It remains to prove that $G$ is countable. For each $X \in \mathcal{X}$ choose $g_X \in G \setminus X$ and observe that $L = \langle g_X \mid X \in \mathcal{X} \rangle$ is not contained in any member of $\mathcal{X}$; therefore $L = G$ and $G$ is countable. \[\square\]

There is a partial converse of the last result.

**Lemma 2.2.** A finitely generated CG-group is a CD-group.

**Proof.** If $G$ is finitely generated CG-group, then every proper subgroup of $G$ is contained in a maximal subgroup. Hence the set of maximal subgroups, which is a countable set, dominates $G$. \[\square\]
The next lemma will be used several times as a reduction tool.

**Lemma 2.3.** Let $N$ be a nilpotent normal subgroup of a group $G$. Then $G$ satisfies $\text{CD}$ if and only if $G/N'$ does.

**Proof.** Clearly a quotient of a $\text{CD}$-group is also a $\text{CD}$-group, so only the sufficiency requires proof. Now if $H < G$, then $HN' < G$; for otherwise $N = (H \cap N)N'$ and hence $N = H \cap N \leq H$, so that $H = G$. Since $G/N'$ is a $\text{CD}$-group, it follows that $G$ is too. \qed

Next we discuss closure properties of the class $\text{CD}$. It is a simple observation that $G = p^\infty \times p^\infty$ is not a $\text{CD}$-group. Indeed $G$ has uncountably many subgroups of type $p^\infty$ and these are pairwise comaximal, so Lemma 2.1 shows that $G$ is not a $\text{CD}$-group. Thus the property $\text{CD}$ is not closed under forming direct products or extensions.

As has already been observed, the property $\text{CD}$ is not closed under the formation of subgroups. In fact a construction of Ol’šanski˘ı [13: Theorem 1] demonstrates that every countable group can be embedded in a finitely generated $\text{CD}$-group. Also examples at the end of Section 5 tell us that the property $\text{CD}$ is not even inherited by normal subgroups of finite index.

While $\text{CD}$ is not closed under taking extensions, certain weak forms of extension closure are valid.

**Lemma 2.4.** Let $G$ be a group and let $N \triangleleft G$. If $N$ has $\text{CD}$ and $G/N$ satisfies the maximal condition on subgroups, then $G$ has $\text{CD}$.

**Proof.** Let $\mathcal{X}$ be a countable set of subgroups which dominates $N$. Denote by $\mathcal{Y}$ the set consisting of all proper subgroups of $G$ containing $N$ and all subgroups of the form $XS$, where $X \in \mathcal{X}$ and $S$ is a finitely generated subgroup of $G$. Then $\mathcal{Y}$ is countable since $G/N$ satisfies max and $N$ is countable by Lemma 2.1. We argue that $\mathcal{Y}$ dominates $G$. To this end let $H < G$. If $HN < G$ then $H \leq HN \in \mathcal{Y}$. Otherwise, $HN = G$ and $H \cap N < N$, so there exists $X \in \mathcal{X}$ such that $H \cap N \leq X$. Furthermore, since $H/H \cap N$ is finitely generated, there is a finitely generated subgroup $S$ such that $H = (H \cap N)S$. Since $S$ normalizes $H \cap N$, we have $H \cap N \leq XS$ and $H \leq XS \in \mathcal{Y}$. Thus $\mathcal{Y}$ dominates $G$. \qed

The following proposition, which is a variation on Lemma 2.4, is analogous to certain results in [2], especially Theorems 2 and 9. The
Proof uses the fact that an infinite, virtually soluble group of finite exponent necessarily has uncountably many maximal subgroups [2: Theorem 4].

**Proposition 2.5.** Let $G$ be a group with a polycyclic normal subgroup $N$.

(i) If $G/N$ has CG, then $G$ has CG.

(ii) If $G/N$ has CD, then $G$ has CD.

**Proof.** First note that polycyclic groups are CMS-groups and hence are CD-groups. In both parts of the proof induction on the derived length of $N$ is used, which allows us to assume that $N$ is abelian. Let $\mathcal{M}$ denote the set of maximal subgroups of $G$ that do not contain $N$.

Assume that $G/N$, but not $G$, has CG; then $\mathcal{M}$ is uncountable. Since $N$ has CMS, there exists $B < N$ such that $B = N \cap M$ for uncountably many $M \in \mathcal{M}$. Now $B \triangleleft NM = G$ for all such $M$, and it is easy to see that $N/B$ is minimal normal in $G/B$. Since we can pass to $G/B$, we may assume that $N$ is minimal normal in $G$, which means that it is a finite elementary abelian $p$-group for some prime $p$.

Uncountably many $M \in \mathcal{M}$ are a complements of $N$ in $G$, so by the well known correspondence between derivations and complements of normal subgroups $\text{Der}(M, N)$ is uncountable. Let $C = C_M(N)$. As $N$ and $M/C$ are finite, so is $\text{Der}(M/C, N)$. In view of the exact sequence $0 \to \text{Der}(M/C, N) \to \text{Der}(M, N) \to \text{Der}(C, N)$, it follows that $\text{Der}(C, N) = \text{Hom}(C, N)$ is uncountable. Consequently $C/C'C^p$ must be infinite. Now $M/C'C^p$ is abelian-by-finite and has finite exponent. In addition, $M \simeq G/N$, so that $M/C'C^p$ is a CG-group; therefore by [2: Theorem 4] it is finite, a contradiction which completes the proof of (i).

To prove (ii) assume that $G/N$ has CD. By (i) and Lemma 2.1 the group $G$ has CG. Set $S = \{H < G \mid G = HN\}$; we will show that $S$ is dominated by a countable set of proper subgroups. Let $H \in S$. Then $H \cap N = (H \cap N)^H < N$ and $H \cap N$ is contained in a maximal $H$-invariant proper subgroup of $N$, say $B(H)$, since $N$ is polycyclic. Thus $HB(H)$ is a maximal subgroup of $G$ and it belongs to $S$. Consequently $\{HB(H) \mid H \in S\}$ is a countable set that dominates $S$ and $G$ has CD. \qed
We close the section by providing some further sources of CD-groups. For example, finitely generated nilpotent-by-polycyclic-by-finite groups are CD-groups. To prove this reduce to the case of finitely generated abelian-by-polyclic-by-finite groups using Lemma 2.3. Such groups are well known to satisfy max-n, so they are CG-groups by [2: Theorem 6] and the claim is established.

Some classes of locally nilpotent CD-groups can be identified by using the next result.

**Proposition 2.6.** Let \( G \) be a locally nilpotent group. Assume that \( G \) has a nilpotent normal subgroup \( A \) such that \( A^{ab} \) is finitely generated as a \( G \)-module and \( G/A \) has CD. Then \( G \) has CD.

**Proof.** By Lemma 2.3 we may assume that \( A \) is abelian. By hypothesis the group \( A/[A,G] \) is finitely generated and hence polycyclic, so \( G/[A,G] \) has CD by Proposition 2.5(ii). Thus it is enough to show that there cannot exist \( H < G \) such that \( G = H[A,G] \). Assuming that \( H \) is such a subgroup, we have \( B = H \cap A \triangleleft HA = G \) and, since \( A \) is finitely generated as a \( G \)-module, \( B \leq C \) for some maximal submodule \( C \) of \( A \). Then \( HC \) is a maximal subgroup of \( G \). Since \( G \) is locally nilpotent, \( [A,G] \leq G' \leq HC \), which leads to \( G = H[A,G] \leq HC \), a contradiction. \( \Box \)

For example, if \( p \) is a prime and \( F \) is any finite \( p \)-group, the wreath product \( F \ wr \ p^\infty \) is a CD-group. Finally, we record a very different source of CD-groups. Recall that a barely transitive group is a transitive permutation group on an infinite set such that the orbits of each proper subgroup are finite.

**Proposition 2.7.** Let \( G \) be a barely transitive permutation group which is not finitely generated. Then \( G \) is a CD-group.

**Proof.** Let \( H \) be a point stabilizer in \( G \). It is well-known that \( G \) is countable and that its proper subgroups are residually finite – see [1]. Hence \( G \) is locally graded and [1: Lemma 2.1] shows that \( H \) is not contained in a maximal subgroup of \( G \). It follows that \( G \) can be written as the union of a strictly increasing sequence \((X_i)_{i \in \mathbb{N}}\) of proper subgroups of \( G \) containing \( H \) each of which is finitely generated modulo \( H \). Let \( U < G \). By bare transitivity \(|U : H \cap U| \) is finite, whence \( \langle H,U \rangle = \langle H,F \rangle \) for some finite subset \( F \) of \( G \). It follows that \( U \leq X_i \) for some \( i \in \mathbb{N} \). Therefore the set \( \{X_i \mid i \in \mathbb{N}\} \) dominates \( G \) and \( G \) has CD. \( \Box \)
3. Near direct sums and $p$-adically irreducible modules

In this section we present certain module theoretic results that are needed in Sections 4 and 5. Two modules $A$ and $B$ over the same ring are said to be near isomorphic, in symbols $A \approx B$, if there exist submodules $A_0 \leq A$ and $B_0 \leq B$ which are bounded as abelian groups and are such that $A/A_0 \cong B/B_0$ as modules. If $A$ and $B$ are $(\mathbb{Z}_r)$ divisible, it is easy to show that $A \approx B$ if and only if $A \cong B/B_0$ for some bounded submodule $B_0$ of $B$, by using the fact that $A \cong A/A[n]$ for $n > 0$.

A critical concept for us is that of a near direct sum. Let $A$ be a module and $\{A_i|i \in I\}$ a family of submodules of $A$. Then $A$ is said to be the near direct sum of the submodules $A_i$ if $A = \sum_{i \in I} A_i$ and $A_i \cap (\sum_{j \neq i \in I} A_j)$ is bounded for each $i \in I$. If all these intersections are annihilated by a fixed positive integer $t$, the sum is called a $t$-near direct sum. The notation

$$A = A_1 + A_2 + \cdots + A_n$$

will be used to indicate that $A$ is the near direct sum of submodules $A_1, A_2, \ldots, A_n$.

Let $p$ be a prime and let $A$ be a $p$-torsion module, i.e., its underlying abelian group is a $p$-group. Then $A$ is $p$-adically irreducible if all its proper submodules are bounded, but $A$ itself is unbounded; in this event $A$ is clearly divisible. A case of special interest is that of a divisible $p$-torsion module over $\mathbb{Z}F$ where $F$ is a finite group. Zaicev [17] (see also [8: p.218]) proved that if $A$ is a such module with $B$ a submodule, then there is a submodule $C$ such that $A$ is the $|F|$-near direct sum of $B$ and $C$. Zaicev drew the conclusion that $A$ is an $|F|$-near direct sum of $p$-adically irreducible submodules.

Near direct sums of pairwise non-near isomorphic modules that are $p$-adically irreducible feature prominently in this work. In certain respects they behave like semisimple modules, as the next result shows.

**Lemma 3.1.** Let $A$ be a divisible $p$-torsion $F$-module where $F$ is a finite group. Assume that $A$ is the near direct sum of a family $\{S_i|i \in I\}$ of pairwise non-near isomorphic, $p$-adically irreducible submodules. If $B$ is a divisible submodule of $A$, then $B = \sum_{i \in J} S_i$ for some $J \subseteq I$.

**Proof.** There is a positive integer $\ell$ such that $A/A[p^\ell] = \bigoplus_{i \in I} (S_i + p^\ell S_i)$ and $B/A[p^\ell] = \bigoplus_{i \in J} (S_i + p^\ell S_i)$. Since $A/A[p^\ell]$ is divisible, we have $B/A[p^\ell] = \bigoplus_{i \in J} (S_i + p^\ell S_i)$. Therefore, $B = \sum_{i \in J} S_i$. \qed
Let $S$ be a $p$-adically irreducible submodule of $A$. For each $i \in I$ composition of the natural homomorphism $S \to A/A[p^\ell]$ with the projection $A/A[p^\ell] \to (S_i + A[p^\ell])/A[p^\ell]$ yields a homomorphism $\theta_i: S \to (S_i + A[p^\ell])/A[p^\ell]$. Since $S \not\subseteq A[p^\ell]$, there is a $j \in I$ for which $\theta_j \neq 0$. Thus $(S)\theta_j$ is a non-zero divisible submodule of the $p$-adically irreducible module $(S_j + A[p^\ell])/A[p^\ell]$. Therefore $\theta_j$ is surjective. Since $S$ is $p$-adically irreducible, $\operatorname{Ker}(\theta_j)$ is bounded and $S \cong (S_j + A[p^\ell])/A[p^\ell] \approx S_j$. Since the modules $S_i$ are pairwise non-near isomorphic, $\theta_i = 0$ for all $i \in I \setminus \{j\}$, which means that $S \leq S_j + A[p^\ell]$. Hence $S = p^\ell S \leq S_j$, so that $S = S_j$.

Now consider a non-zero divisible submodule $B$ of $A$. By Zaitsev’s theorem quoted above, $B$ is a near direct sum of $p$-adically irreducible submodules. By the first part of the proof each of these $p$-adically irreducible submodules is one of the $S_i$.

The next result is surely known.

**Lemma 3.2.** Let $p$ be a prime and $F$ a finite group. Then the number of near-isomorphism classes of $p$-adically irreducible $\mathbb{Z}F$-modules is at most $|F|$.

**Proof.** Let $A$ be a divisible $p$-torsion $\mathbb{Z}$-$F$-module with finite $\mathbb{Z}$-rank $r$. Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of $p$-adic integers and the field of $p$-adic numbers respectively. The dual of $A$ is $A^* = \operatorname{Hom}(A, p^{\infty})$, which is a free $\mathbb{Z}_p$-module of rank $r$. Set $A^* = A^* \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$, which is a $\mathbb{Q}_p$-vector space of dimension $r$. Clearly $A$ is $p$-adically irreducible if and only if $A^*$ is $\mathbb{Q}_p$-simple.

Let $A$ and $B$ be $p$-adically irreducible $F$-modules; then $A$ and $B$ have finite rank since $F$ is finite. Clearly $A \approx B$ if and only if $A^* \mathbb{Q}_p^F \cong B^*$. A count of irreducible characters shows that there are at most $|F|$ isomorphism classes of simple $\mathbb{Q}_p F$-modules, so the lemma follows. □

We will have several opportunities to use the next observation.

**Lemma 3.3.** Let $F$ be a group, $p$ a prime and $A$ a $\mathbb{Z}F$-module. Assume that $A = A_1 + A_2 + C$ where $A_1$ and $A_2$ are near isomorphic, $p$-adically irreducible submodules and $C$ is a submodule of $A$. Then there is an uncountable set $\mathcal{B}$ of pairwise comaximal submodules of $A$ such that $A/B$ is $p$-adically irreducible for $B \in \mathcal{B}$. Hence $A$ does not have the property CD as a $\mathbb{Z}F$-module.
Proof. In the first place $A$ has a quotient $\bar{A} = U \oplus V$ where $U$ and $V$ are isomorphic $p$-adically irreducible $ZF$-modules. Note that $\text{End}_{ZF}(U)$ is uncountable, since it has a subring isomorphic with $\hat{\mathbb{Z}}$. Thus $U$ has uncountably many complements in $\bar{A}$: let $X \neq Y$ be two of them. Then $\bar{A}/Y \cong U$ is $p$-adically irreducible, while $X + Y/Y$ is unbounded since $X \neq Y$; hence $\bar{A} = X + Y$. It follows that the complements of $U$ in $\bar{A}$ form an uncountable set of pairwise comaximal submodules of $A$. The final statement follows by an argument in the proof of of Lemma 2.1.

We end the section with an example showing that near isomorphic, $p$-adically irreducible modules need not be isomorphic.

Example. Let $P = A \oplus A$ where $A \simeq 2^\infty$, and consider the automorphisms $\alpha, \beta$ of $P$ defined by

$$\alpha: (a, b) \mapsto (b, a) \quad \text{and} \quad \beta: (a, b) \mapsto (a, -b)$$

where $a, b \in A$. Thus $\alpha$ and $\beta$ have order 2 and $F = \langle \alpha, \beta \rangle$ is a dihedral group of order 8. Observe that $P$ is $2$-adically irreducible as a $ZF$-module. Now let $u$ denote the element of order 2 in $A$. Then $x = (u, u)$ is a fixed point of $F$ and $P \cong P/\langle x \rangle$. A simple calculation shows that $P$ and $P/\langle x \rangle$ are not $ZF$-isomorphic.

4. Virtually nilpotent groups with CD

In this section we give a complete characterization of the virtually nilpotent groups which have the property $CD$. What emerges is a criterion that largely hinges on the structure of certain associated modules.

Theorem 4.1. Let the group $G$ have a nilpotent normal subgroup $N$ of finite index and write $F = G/N$ and $\bar{N} = N^{ab}$. Then $G$ is a $CD$-group if and only if $\bar{N}$ has a finitely generated $F$-submodule $X$ such that $(\bar{N}/X)_p$ is either finite or a Černikov group whose finite residual is a near direct sum of finitely many, pairwise non-near isomorphic, $p$-adically irreducible $F$-submodules for each prime $p$.

Proof. In the first place Lemma 2.3 shows that we can take $N$ to be abelian. Starting with the necessity, we assume that $G$ is a $CD$-group. Initially we will suppose that $G$ is periodic and we can then assume that $N$ is a $p$-group. Let $A$ denote the maximum divisible
subgroup of $N$. Since $G/N^p$ is a $\text{CD}$-group of finite exponent, it is finite by [2: Theorem 4]. It follows that $N/A$, and hence $G/A$, is finite because it is reduced.

We view $A$ as a $\text{ZF}$-module. By Zaicev’s theorem $A$ is the near direct sum of a family $\{A_i \mid i \in I\}$ of $p$-adically irreducible $F$-submodules. Suppose that $A_i \approx A_j$ for distinct $i, j \in I$. Then by Lemma 3.3 there is an uncountable set $\mathcal{Y}$ of $F$-submodules of $A$ that are pairwise comaximal in $A$ and are such that $A/Y$ is $p$-adically irreducible for all $Y \in \mathcal{Y}$. Choose a transversal $T$ to $A$ in $G$ and let $\mathcal{Y}_1 = \{(TY \mid Y \in \mathcal{Y})\}$. If $Y \in \mathcal{Y}$, the group $A/Y$ is not finitely generated, whence neither is $G/Y$. Therefore $G \notin \mathcal{Y}_1$ and $\mathcal{Y}_1$ is a set of pairwise comaximal subgroups of $G$. Now the mapping $Y \in \mathcal{Y} \mapsto (TY)Y \in \mathcal{Y}_1$ is injective; for, if $(TY)Y = (TY)^*$ with distinct $Y, Y^*$ in $\mathcal{Y}$, then $YY^* = A$ and hence $G = (TY)Y \in \mathcal{Y}_1$, a contradiction. It follows that $\mathcal{Y}_1$ is uncountable, which is in contradiction to Lemma 2.1. Hence the modules $A_i$ are pairwise non-near isomorphic and by Lemma 3.2 the index set $I$ must be finite. As a consequence $N$ is a Černikov group. Since $A$ is the finite residual of $N$, we see that $N$ has the required structure.

At this point we drop the hypothesis of periodicity. There is a free abelian subgroup $X$ such that $N/X$ is periodic. Since $G/N$ is finite, we can replace $X$ by its core in $G$ and assume that $X \triangleleft G$. Let $p$ be any prime; then $G/X^p$ is a periodic $\text{CD}$-group, so the first part of the proof shows that $X/X^p$ is finite. Thus $X$ is finitely generated and also $(N/X)_p$ is a Černikov group. Moreover, if $D/X^p$ is the finite residual of $(N/X^p)_p$, then $DX/X$ is the finite residual of $(N/X)_p$, which completes the proof of necessity.

Next we establish sufficiency; assume that $G$ satisfies the conditions. First of all assume that $X = 1$, so that $G$ is periodic. Note that as a consequence $G$ is countable. We will construct a countable set that dominates $G$ as follows. Let $R$ denote the finite residual of $N$. For each $p \in \pi(N)$ let $C_p$ be the (finite) set of all proper subgroups of $G$ containing $N_p^pR_p$. Furthermore, if $R_p \neq 1$, then $R_p$ is the near direct product of a finite family $\{S_i \mid i \in I\}$ of pairwise non-near isomorphic, $p$-adically irreducible $F$-modules. Denote by $\mathcal{M}_p$ the set of all $F$-submodules that have the form $\langle S_j \mid j \neq i, j \in I\rangle$ for some $i \in I$. Now let $X_p$ be the set of all subgroups $ELN_p^p$, where $E$ is a finite subgroup of $G$ and $L \in \mathcal{M}_p$; note that subgroups in $X_p$ are proper. Define $\mathcal{X}$ be the union of all the sets $C_p$ and $X_p$ just
defined. Clearly $\mathcal{X}$ is countable; we will prove that it dominates $G$. To this end let $U < G$; we will argue that $U \leq X$ for some $X \in \mathcal{X}$.

In the first place we may assume that $N \not\subseteq U$, since otherwise $U \in \mathcal{X}$. Choose $p \in \pi(N/N \cap U)$; then $N \cap (UN_p) = (N \cap U)N_p < N$, so that $UN_p < G$ and, on replacing $U$ by $UN_p$, we may assume that $N_p \leq U$. If $UR_p < G$, take $X$ to be $UR_p \in \mathcal{C}_p$. Therefore we suppose that $UR_p = G$, so that $R_p \neq 1$ and $U \cap R_p \not\subseteq G$. Let $S$ be the finite residual of $U \cap R_p$. Then $S$ is a proper divisible $F$-submodule of $R_p$ and thus Lemma 3.1 shows that $S \leq L$ for some $L \in \mathcal{M}_p$. Moreover, $U/SN_p'$ is finite since $G/N_p'$ is Černikov. Therefore $U = ESN_p'$ for some finite $E \leq G$. If $X = ELN_p'$, then $U \leq X \in \mathcal{X}_p \subseteq \mathcal{X}$ and therefore $\mathcal{X}$ dominates $G$ and $G$ has CD.

Finally, consider the general case, so $X$ is the finitely generated submodule of $N$ specified in the hypothesis. Then $G/X$ is a periodic CD-group by the previous part of the proof. It now follows from Proposition 2.5 that $G$ is a CD-group. \hfill \Box

The full structural consequences of the condition CD are shown in the next result.

**Corollary 4.2.** Let $G$, $N$ and $F$ be as in the theorem and assume that $G$ has CD. Then:

(i) the group $N$ has finite abelian ranks;

(ii) there is a finitely generated subgroup $X$ of $N$ such that $G/X^G$ is periodic and $(N/X^G)_p$ is a Černikov group whose finite residual is the near direct sum of finitely many, pairwise non-near isomorphic, $p$-adically irreducible $F$-submodules for all primes $p$.

**Proof.** By the theorem $N^{ab}$ has finite ranks, so by the well known tensor product property of the lower central series the lower central factors of $N$ also have finite ranks; therefore $N$ has finite abelian ranks. Consequently, $N$ has a finitely generated subgroup $X$ such that each factor $X_{\gamma_i}(N)/X_{\gamma_i+1}(N)$ is periodic. Therefore $G/X^G$ is periodic, so we may assume that $G$ is periodic; we can also take $N$ to be a $p$-group. Since $N$ is a nilpotent Černikov group, $N'$ is finite and central in $N$. If $R$ denotes the finite residual of $N$, then $RN'/N'$ is the finite residual of $N^{ab}$. Furthermore $R \approx RN'/N'$. Since $G/N'$ is a CD-group, it follows from the theorem that $R$ inherits the relevant properties of the module $RN'/N'$. \hfill \Box
We single out two special cases of Theorem 4.1 in which the characterization takes a simpler form.

**Corollary 4.3.**

(i) A periodic nilpotent group is a CD-group if and only if each of its infinite primary components is a finite extension of a Prüfer group.

(ii) A nilpotent Černikov group is a CD-group if and only if its finite residual is locally cyclic (in which case the group is a CMS-group).

**Proof.** Let $G$ be a periodic nilpotent CD-group. Applying Corollary 4.2 with $N = G$, we conclude that each primary component of $G$ is Černikov, so that the finite residual $R$ of $G$ is central. Hence the $p$-adically irreducible subgroups of $R_p$ are $p^\infty$-groups and $R$ is locally cyclic. The converse follows at once from Theorem 4.1. It is immediate that (ii) is valid.

Observe that the class of nilpotent CD-groups is not subgroup closed. For example, let $G = \langle x \rangle \times A$ where $A \cong 2^\infty \oplus 2^\infty$ and $(a_1, a_2)x = (a_1, a_1 + a_2)$; then $G$ is a nilpotent CD-group of class 2, but $A$ is not a CD-group. On the other hand it follows from Corollary 4.3 and Proposition 2.5 that the class of polycyclic-by-periodic nilpotent CD-groups is subgroup closed.

It is worthwhile recording the situation for abelian groups in regard to the property CD.

**Corollary 4.4.** An abelian group $A$ is a CD-group if and only if there is a finitely generated subgroup $B$ such that $A/B$ is periodic and each of its infinite primary components is a finite extension of a Prüfer group.

5. Metanilpotent groups with min-$n$ and CD

Our aim in this section is to give necessary and sufficient conditions for a metanilpotent group satisfying min-$n$ to have the property CD. In this endeavour it is essential to keep in mind that a metanilpotent group with min-$n$ is locally finite and countable by work of Baer [4] and McDougall [12] respectively. It was shown in [2] that all metanilpotent groups with min-$n$ are CG-groups. However, the question of which of these groups have the property CD is more
subtle and progress requires detailed knowledge of the structure of artinian modules over nilpotent Černikov groups. Our principal aim is to establish the following result.

**Theorem 5.1.** Let $G$ be a metanilpotent group satisfying $\min n$ and let $A = \gamma_\infty(G)$, the last term of the lower central series of $G$. Then $G$ has the property $\text{CD}$ if and only if the following hold:

(i) $G/A$ is a nilpotent Černikov group whose finite residual is locally cyclic.

(ii) $A^{ab}$ has the property $\text{CD}$ as a $G$-module.

**Proof.** First note that $A$ and $G/A$ are nilpotent and by Lemma 2.3 we can assume that $A$ is abelian. To prove necessity assume that $G$ has $\text{CD}$ and note that (i) holds by Corollary 4.3. Let $S$ denote the set of all proper $G$-submodules of $A$ and suppose that $A$ does not have $\text{CD}$ as a $G$-module. Write $A_0 = A \cap \bar{Z}(G)$ and note that $\bar{Z}(G)$ is a Černikov group by a theorem of Baer [3] – see also [14: Theorem 5.22]. For the moment let $A_0 = 1$. Since $H^0(G/A, A) = 0$, we have $H^2(G/A, A) = 0$ by [11: 10.3.2], so that $G$ splits over $A$, say $G = H \rtimes A$. Notice that $HS < G$ for $S \subseteq S$. As $G$ is a $\text{CD}$-group, there is a countable set of proper subgroups $T$ such that for each $S \subseteq S$ we have $HS \leq T(S)$ for some $T(S) \subseteq T$. Then $S \leq A \cap T(S)$ and $A \cap T(S)$ is a proper $G$-submodule of $A$. Hence $S$ is dominated by $\{A \cap T(S)|S \subseteq S\}$, which is a countable set of proper $G$-submodules of $A$. Therefore $A$ has the property $\text{CD}$ as a $G$-module.

Returning to the general case, we note that by the previous paragraph the set of $S \subseteq S$ such that $SA_0 < A$ is $\text{CD}$ in $G$. What remains to be proved is that the set $S_1 = \{S \subseteq S|SA_0 = A\}$ is $\text{CD}$ in $G$. Let $S \subseteq S_1$. Then $A/S$ is a Černikov group since $A/S \simeq A_0/S \cap A_0$. Evidently we may assume that $A/S$ is either finite or else a divisible abelian $p$-group of finite rank which is $p$-adically irreducible as a $G$-module. Since $A$ is an artinian $G$-module, it has finitely many submodules of finite index. Thus it is enough to show that the set $S_2$ of all $S \subseteq S_1$ for which $A/S$ is a $p$-adically irreducible $G$-module is $\text{CD}$ in the module $A$. Assume that this is false.

Write $C = C_G(A_0)$; then $F = G/C$ is finite by a result of Baer [3] – see also [14: Theorem 3.29.2]. Since $A/S \simeq A_0/S \cap A_0$, the subgroup $C$ centralizes $A/S$, so the latter is an $F$-module. By
Lemma 3.2 there are finitely many near isomorphism types of \( p \)-adically irreducible \( F \)-modules, from which it follows that there is an uncountable subset \( S_3 \subseteq S_2 \) such that if \( S, T \in S_3 \), then \( A/S \) and \( A/T \) are near isomorphic as \( F \)-modules.

Suppose that \( S + T = A \) for some \( S, T \in S_3 \). Then \( A/S \cong T/S \cap T \) and \( A/T \cong S/S \cap T \). But then \( G/(S \cap T) \) is a Černikov \( \text{CD} \)-group whose finite residual contains two near isomorphic, \( p \)-adically irreducible submodules \( S/(S \cap T) \) and \( T/(S \cap T) \). Since this contradicts Theorem 4.1, we have \( S + T \neq A \) for all \( S, T \in S_3 \).

Now fix \( S \in S_3 \) and let \( T \in S_3 \) be arbitrary. Since \( S + T < A \) and \( A/T \) is \( p \)-adically irreducible, \( (S + T)/T \cong S/S \cap T \) is finite. Since \( S \) is \( G \)-artinian, it has finitely many \( G \)-submodules of finite index. Hence there exist uncountably many \( T \in S_3 \) such that \( S \cap T = U \) is fixed. But \( G/U \) is a Černikov group, so it has countably many finite subgroups. This gives the contradiction that there are countably many \( T \)'s; thus necessity is established.

Turning to sufficiency, we suppose that properties (i) and (ii) hold, but the set \( \mathcal{H} \) of all proper subgroups of \( G \) is not \( \text{CD} \) in \( G \). Now \( G/A \) is a \( \text{CD} \)-group by Corollary 4.3, so the set \( \{ H \in \mathcal{H} | HA < G \} \) is \( \text{CD} \) in \( G \). Therefore the set \( \mathcal{H}_1 = \{ H \in \mathcal{H} | HA = G \} \) is not \( \text{CD} \) in \( G \).

For any \( H \in \mathcal{H}_1 \) we have \( H \cap A \lhd HA = G \); thus \( H \cap A \) is a proper submodule of \( A \). By condition (ii) the set of proper submodules of \( A \) is dominated by some countable set \( S \) of proper submodules. For each \( S \in S \) define \( \mathcal{H}_1(S) = \{ H \in \mathcal{H}_1 | H \cap A \leq S \} \). Since \( S \) is countable, there must exist \( S \in S \) such that \( \mathcal{H}_1(S) \) is not \( \text{CD} \) in \( G \). Factoring out by \( S \), we reach the situation where \( G = H \times A \) for \( H \in \mathcal{H}_2 \), a subset of \( \mathcal{H}_1 \) which is not \( \text{CD} \) in \( G \) and is therefore uncountable. Notice that \( H \) is a Černikov group.

Let \( A_0 = A \cap \bar{Z}(G) \) and recall that \( \bar{Z}(G) \) is a Černikov group. For the moment suppose that \( A_0 = 1 \). Then \( H^1(G/A, A) = 0 \) by [11: 10.3.2]. Hence \( \text{Der}(G/A, A) = \text{Inn}(G/A, A) \), which is countable. It follows that there are only countably many complements \( H \) in \( G = H \times A \), which gives the contradiction that \( \mathcal{H}_2 \) is countable and completes the proof in this case.

We now proceed to the general case. Since \( (A/A_0) \cap \bar{Z}(G/A_0) = 1 \), the group \( G/A_0 \) is a \( \text{CD} \)-group by the previous argument, so the set of \( H \in \mathcal{H}_2 \) such that \( HA_0 < G \) is \( \text{CD} \) in \( G \). It follows
that the set $\mathcal{H}_3 = \{ H \in \mathcal{H}_2 | HA_0 = G \}$ cannot be CD in $G$ and thus is uncountable. For $H \in \mathcal{H}_3$ we have $A = A \cap (HA_0) = A_0$, from which it follows that $A$, and hence $G$, is Černikov. In addition $A = [A, G]$, which implies that $H^1(G/A, A)$ has finite exponent by [11: 10.3.6]. From this it is routine to deduce that $H^1(G/A, A)$ is finite. Therefore Der$(G/A, A)$ is countable and again there are countably many complements $H$. However, this means that $\mathcal{H}_3$ is countable, a final contradiction. □

Artinian modules over centre-by-finite groups

Let $G$ be a metanilpotent group with min-$n$ and write $A = \gamma_\infty(G)$ and $Q = G/A$. Then $A^{ab}$ is an artinian module over the nilpotent Černikov group $Q$; note also that $Q$ is centre-by-finite. Theorem 5.1 makes it clear that the key to determining whether $G$ is a CD-group is the $Q$-module structure of $A^{ab}$. Thus such modules merit our attention. In particular, we need to understand the effect of the module property CD on $A^{ab}$.

Firstly, we note that if $Q$ is a nilpotent Černikov group and $A$ an artinian $Q$-module, then $G = Q \rtimes A$ is a metanilpotent group satisfying min-$n$. By the results of Baer and McDougall $G$ is locally finite and countable, from which it follows that $A$ is countable and periodic.

The structure of artinian modules over centre-by-finite groups has been analyzed in the non-modular case by B. Hartley and D. McDougall in the important paper [10]. We will describe their results in some detail since they are critical for this investigation.

For any prime $p$ let $Q$ be a countable, locally finite $p'$-group. Let $\{ M_\lambda | \lambda \in \Lambda \}$ denote a complete set of non-isomorphic, simple $\mathbb{Z}_p Q$-modules and let the rank of $M_\lambda$ be $n_\lambda$. If $V$ is a divisible abelian $p$-group of rank $n_\lambda$, we can endow $V[p]$ with a $Q$-module structure by identifying it with the abelian group $M_\lambda$. Since $Q$ is a locally finite $p'$-group, the $Q$-module structure of $V[p]$ can be extended to $V[p^n]$ – see for example [10: Lemma 3.2]. In this way we obtain $Q$-modules $V_\lambda(n) = V[p^n]$, $n = 1, 2, \ldots$. Let $V_\lambda(\infty) = \bigcup_{n=1,2,\ldots} V_\lambda(n)$; this is the $Q$-injective hull of $V[p]$. Then $V_\lambda(n + 1)/V_\lambda(n) \cong M_\lambda$. It is easy to show that $V_\lambda(\infty)$ is a uniserial, $p$-adically irreducible $Q$-module. Moreover, the $V_\lambda(n)$ are the only proper submodules of $V_\lambda(\infty)$. Finally, note that the $Q$-modules $V_\lambda(n)$ are noetherian and artinian, while $V_\lambda(\infty)$ is artinian.
Example (i). Let $Q$ be a group of type $q^\infty$ and let $M$ be the simple Carin $Q$-module over $\mathbb{Z}_p$ where $p \neq q$ – see [5] or else [11: 1.5.1]. Recall that $M$ is the field obtained by adjoining $q^n$th roots of unity for $n = 1, 2, \ldots$ to the field of $p$ elements, and giving $M$ the natural $Q$-module structure. Then there is a corresponding $p$-adically irreducible $Q$-module $V$ such that $M \cong V[p]$.

The main result of [10] is a Krull-Schmidt Theorem for artinian $Q$-modules (Theorems A and C). In the following theorem $V_\lambda(n)$ and $V_\lambda(\infty)$ have the above meanings.

**Theorem.** Let $p$ be a prime, $Q$ a countable centre-by-finite $p'$-group and $A$ an artinian $Q$-module which is a $p$-group. Then $A$ is the direct sum of finitely many indecomposable submodules each of which is isomorphic to some $V_\lambda(n)$ or $V_\lambda(\infty)$. Moreover the decomposition is unique up to an automorphism of $A$.

Another result that will be important here is:

**Lemma 5.2.** Let $A$ be an artinian module over a nilpotent Černikov group $Q$. Then $Q_p/C_{Q_p}(A_p)$ is finite for all primes $p$.

This may be deduced from [12: Theorem 3.2] and it is not hard to prove directly. We remark that $Q/C_p(A_p)$ might not be a $p'$-group, so we could be faced with a modular situation – see the example (v) at the end of this section. Another useful fact is:

**Lemma 5.3.** Let $Q$ be a nilpotent Černikov group and $A$ an artinian $Q$-module. If $A$ is bounded, then it is noetherian.

**Proof.** We may assume that $Q$ acts faithfully on $A$ and that $A$ is a $p$-group. Lemma 5.2 shows that $Q = P \times R$ where $P$ is a finite $p$-group and $R$ is a $p'$-group. Since $Q/R$ is finite, $A$ is $R$-artinian. Now $A$ being bounded, the Hartley-McDougall decomposition shows that $A$ is the direct sum of finitely many $R$-submodules of types $V_\lambda(i)$, $i = 1, 2, \ldots$. Since each $V_\lambda(i)$ is noetherian, the result follows.

**Corollary 5.4.** Let $G$ be a metanilpotent group satisfying min-$n$ and set $A = \gamma_\infty(G)$. If $A$ has finite exponent, then $G$ has the property CD.

**Proof.** First note that $G/A$ is a Černikov group. By Lemma 5.3 the $G/A$-module $A^{ab}$ is noetherian. Since $A$ is countable, it has countably many submodules and the result follows by Theorem 5.1.
Next we establish a basic result about artinian modules over nilpotent Černikov groups that will allow us to analyse the condition (ii) in Theorem 5.1 in the modular case.

**Proposition 5.5.** Let $Q$ be a nilpotent Černikov group and $A$ an artinian $Q$-module which is a $p$-group. Then there is a near direct decomposition

$$A = (A_1 + A_2 + \cdots + A_n) + A[p^f],$$

where $f$ is a natural number and the $A_i$ are $p$-adically irreducible submodules.

**Proof.** We may assume without loss that $Q$ acts faithfully on $A$. By Lemma 5.2 we have $Q = P \times R$, where $P$ is a finite $p$-group, of order $p^k$ say, and $R$ is a Černikov $p'$-group. Of course, $A$ may be assumed to be unbounded.

Since $A$ is artinian, there is a $Q$-submodule $A_1$ which is minimal subject to being unbounded. Then $A_1$ is a $p$-adically irreducible $Q$-module. Since $A_1$ is divisible, it is $R$-injective by [10: Lemma 2.3] and we can write $A = A_1 \oplus B$ for some $R$-submodule $B$. Let $\pi : A \to A_1$ be the canonical $R$-projection and define a map $\bar{\pi} : A \to A_1$ by $(a)\bar{\pi} = \sum_{x \in P} (ax)\pi x^{-1}$, $a \in A$. By a standard calculation $\bar{\pi}$ is a $Q$-homomorphism and $(A_1)\bar{\pi} = p^k A_1 = A_1$. Since $\bar{\pi}^2 = p^k \bar{\pi}$, we have for any $a \in A$ that $(p^k a - (a)\bar{\pi})\bar{\pi} = 0$, so that $p^k a - (a)\bar{\pi} \in K_1 = \ker(\bar{\pi})$ and thus $A/(A_1 + K_1)$ is bounded. Notice that $A_1 \cap K_1 = A_1[p^f]$, so $A_1 + K_1 = A_1 + K_1$. If $K_1$ is unbounded, we repeat the argument for $K_1$ to get a $p$-adically irreducible submodule $A_2$ and a submodule $K_2$ such that $K_1/(A_2 + K_2)$ is bounded and $A_1 + K_2 = A_2 + K_2$.

Continuing in this manner, we obtain a sequence of $p$-adically irreducible submodules $A_1, A_2, \ldots$ and submodules $K_1, K_2, \ldots$ such that $A/(A_1 + A_2 + \cdots + A_n + K_n)$ is bounded and $A_1 + A_2 + \cdots + A_n + K_n$ is near direct. Since $A$ is artinian, this procedure must terminate finitely, which means that some $K_n$ is bounded. Hence $A/(A_1 + A_2 + \cdots + A_n)$ is bounded. Write $B = A_1 + A_2 + \cdots + A_n$. Since $B$ is a divisible subgroup, we have $A = B \oplus S$ where $S$ is a bounded subgroup. If $p^f S = 0$, then $S \leq A[p^f]$ and hence $A = B + A[p^f]$. □
From the last result we obtain some useful structural information about artinian modules over nilpotent Černikov groups.

**Proposition 5.6.** Let $Q$ be a nilpotent Černikov group and $A$ an artinian $Q$-module. Then there is a bounded $Q$-submodule $K$ such that $A/K \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where the $A_i$ are $p$-adically irreducible $Q$-modules for various primes $p$. Hence $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n$.

**Proof.** Evidently we may assume that $A$ is a $p$-group. By Proposition 5.5 we have $A = (A_1 + A_2 + \cdots + A_n) + A[p^\ell]$ for some $\ell > 0$ where the $A_i$ are $p$-adically irreducible submodules. Choose $m \geq \ell$ so large that $A[p^m]$ contains all the intersections $A_i \cap (\sum_{j \neq i} A_j)$, $i = 1, 2, \ldots, n$. Then

$$A/A[p^m] \cong A_1/A_1[p^m] \oplus A_2/A_2[p^m] \oplus \cdots \oplus A_n/A_n[p^m].$$

Finally, $A_i/A_i[p^m] \cong A_i$ via multiplication by $p^m$. $\square$

We are now in a position to characterize those artinian modules over nilpotent Černikov groups which have the module property CD.

**Theorem 5.7.** Let $A$ be an artinian module over a nilpotent Černikov group $Q$. Then the following statements are equivalent.

(i) $A = A_1 + A_2 + \cdots + A_n + S$ where the $A_i$ are non near isomorphic, $p$-adically irreducible $Q$-modules for various primes $p$ and $S$ is a bounded $Q$-submodule.

(ii) $A$ has countably many submodules.

(iii) $A$ has the property CD as a $Q$-module.

**Proof.** Evidently we can assume that $A$ is a $p$-group for some prime $p$.

(i) $\Rightarrow$ (ii). Let $A$ have the decomposition in (i). Note that $S$ is noetherian by Lemma 5.3, so it has countably many submodules; thus we can assume that $n > 0$. Let $B = A_1 + A_2 + \cdots + A_{n-1} + S \neq A$. By induction on $n$ the submodule $B$ has countably many submodules and the same is true of $A/B$ since its proper submodules are bounded and hence noetherian.

Assume that nevertheless $A$ has uncountably many submodules $\{S_\lambda | \lambda \in \Lambda\}$. Then there are submodules $C$ and $D$ such that $S_\lambda \cap$
$B = C$ and $S_\lambda + B = D$ with $C, D$ fixed for uncountably many $\lambda$. Then $D/C = (S_\lambda/C) \oplus (B/C)$. If $D/B$ is bounded, it is noetherian and hence $\text{Hom}_Q(D/B, B/C)$ is countable. Otherwise $D = A$ and $\text{Hom}_Q(A/B, B/C) = 0$ since $A/B$ cannot be near $Q$-isomorphic with a submodule of $B/C$. This gives the contradiction that there are countably many complements $S_\lambda/C$ and hence countably many $S_\lambda$’s.

(ii) $\Rightarrow$ (iii). This implication is obvious.

(iii) $\Rightarrow$ (i). Assume that $A$ has CD as a $Q$-module. By Proposition 5.5 there is a decomposition $A = (A_1 + A_2 + \cdots + A_n) + A[p^\ell]$ where the $A_i$ are $p$-adically irreducible and $\ell > 0$. Now apply Lemma 3.3 to show that no two of the $A_i$ can be near isomorphic $Q$-modules.

Combining Theorems 5.1 and 5.7, we obtain a satisfying description of the metanilpotent groups with min-$n$ that have CD.

**Corollary 5.8.** Let $G$ be a metanilpotent group satisfying min-$n$ and let $A = \gamma_\infty(G)$. Then $G$ is a CD-group if and only if $G/A$ is a nilpotent Černikov group whose finite residual is locally cyclic and $A^{ab}$ has countably many $G$-submodules.

It is easy to find examples of metanilpotent groups with min-$n$ that are not CD-groups – and even some that are Černikov groups.

**Example (ii).** Consider the group $G = Q \times A$ where $Q = \langle x \rangle$ has order 2 and $A = A_1 \times A_2$ with $A_i \simeq 2^\infty$. Let $Q$ act on $A$ via $a^x = a^{-1}$, $a \in A$. Then $\gamma_\infty(G) = A$ has uncountably many $Q$-submodules, so $G$ is not a CD-group by Corollary 5.8. On the other hand, if we make $x$ act trivially on $A_1$ and invert in $A_2$, then $G$ is a CD-group since $A_1$ and $A_2$ are not near isomorphic as $Q$-modules. Notice that $G$ is a hypercentral group and $A$ is not a CD-group.

**Example (iii).** In this example $A = A_1 \oplus A_2$ where $A_i \simeq 5^\infty$. Then $A_i$ has an automorphism $\alpha$ of order 4. Let $Q = \langle x \rangle$ have order 4 and make $A$ into a $Q$-module by defining $a_1 x = (a_1)\alpha$ and $(a_2)x = a_2\alpha^{-1}$, where $a_i \in A_i$. Set $G = Q \times A$. Evidently $\gamma_\infty(G) = A$ and $G$ is a CD-group since $A_1$ and $A_2$ are not near isomorphic as $Q$-modules. In this case $G$ has trivial centre.

**Example (iv).** To obtain non-Černikov examples let $Q$ be a group of type $q^\infty$ and let $A$ be the $p$-adically irreducible $Q$-module (where
\( p \neq q \) derived from a simple Čarín \( \mathbb{Z}_p Q \)-module – see Example (i) above. Set \( G = Q \ltimes A \); this is a metabelian group with \( \text{min-}n \) that is not a Černikov group. Evidently \( \gamma_\infty(G) = A \) has only countably many submodules, so \( G \) is a CD-group. On the other hand, the group \( Q \ltimes (A \oplus A) \) is not a CD-group because the \( Q \)-module \( A \oplus A \) has uncountably many submodules.

From examples in (ii) and (iii) we can see that if a metanilpotent group with \( \text{min-}n \) is a CD-group, its finite residual need not have CD. It is more challenging to show that the Černikov residual need not inherit the property CD.

Example (v). There is a metabelian group with \( \text{min-}n \) which has CD, but whose Černikov residual does not have CD.

Let \( Q = R \times \langle z \rangle \) where \( R \) is a \( 2^\infty \)-group and \( |z| = 3 \). Let \( A = A_1 \oplus A_2 \) where \( A_i \) is the injective hull of the Čarín 3-module over \( R \); thus \( R \) acts on \( A_i \) via the field multiplication. Make \( A \) into a \( Q \)-module via the actions

\[(a_1, a_2)r = (a_1r, a_2r), \quad (a_1, a_2)z = (a_2, -a_1 - a_2),\]

where \( r \in R \) and \( a_i \in A_i \). Clearly \( A \) is \( R \)-artinian.

We prove that \( A \) is a 3-adically irreducible \( Q \)-module. If this is false, then \( A \) has a proper unbounded submodule \( B \). First note that \( B \nmid A_1 \); for otherwise we will also have \( B \supseteq A_2 \) and \( B = A \). Factoring out by a suitable \( A[3^m] \), we can assume that \( B \cap A_1 = 0 \).

Since \( B \) is unbounded, \( A = B \oplus A_1 \) and \( B \cong R \rightarrow A/A_1 \cong A_2 \). Let \( 1 \neq b \in B[3] \); then \( \langle b \rangle \langle z \rangle \) is a finite elementary abelian 3-group and hence \( z \) fixes some non-trivial element of it. Since \( B[3] \) is \( R \)-isomorphic with a simple Čarín module and \( b \mapsto b(z - 1) \) is an \( R \)-module endomorphism, we conclude that \( B[3](z - 1) = 0 \).

Suppose that \( B(z - 1) \neq 0 \) and write \( B_i = B[3^i] \). There is a least \( n \geq 2 \) such that \( B_n(z - 1) \neq 0 \). Since \( B_{i+1}(z - 1) \leq B_i \), we have \( B_{n+1}(z - 1)^3 \leq B_{n-1}(z - 1) = 0 \). Since \( z^3 = 1 \), it follows that

\[ 0 = B_{n+1}(z - 1)^3 = B_{n+1}(3z(1 - z)) = B_n(z - 1), \]

a contradiction which shows that \( B(z - 1) = 0 \). Let \( (a_1, a_2) \in B \); then \( (a_1, a_2) = (a_2, -a_1 - a_2) \), so that \( 3a_1 = 0 = 3a_2 \). Hence \( 3B = 0 \), a contradiction which proves \( A \) to be 3-adically irreducible.

In conclusion define \( G = Q \ltimes A \), which is a metabelian group satisfying \( \text{min-}n \). Then \( A = \gamma_\infty(G) \) has CD as a \( G \)-module by
Theorem 5.7 and Theorem 5.1 shows that $G$ is a CD-group. The finite residual of $G$ is $RA$ and the Černikov residual is $A$, neither of which is a CD-group. \hfill $\Box$

6. Groups whose subgroups are CD-groups.

As was mentioned in the introduction, soluble groups with CD are much harder to deal with than nilpotent groups. It is noteworthy that a periodic soluble CD-group can have abelian $p$-subgroups of infinite rank: for example the wreath product $\mathbb{Z}_p \wr p^\infty$ is a CD-group, yet the base group is infinite elementary abelian. This observation suggests that one of the difficulties in dealing with soluble CD-groups is the failure of subgroup closure for this property. With this in mind, we strengthen our hypothesis in this section and consider groups all of whose subgroups have CD: this property will be denoted by SCD.

In general one cannot expect to say much about SCD-groups – after all Tarski $p$-groups have this property. However, we are able to characterize virtually soluble groups with SCD in terms of their abelian sections. By Lemma 2.4 it is sufficient to do this for soluble groups.

**Theorem 6.1.** A soluble group $G$ is an SCD-group if and only if it has finite abelian ranks and there are no factors in $G$ of type $p^\infty \times p^\infty$ for any prime $p$.

**Proof.** The necessity of the conditions follows from Corollary 4.4. Assume that the conditions hold in $G$. Since these are inherited by subgroups, it is enough to prove that $G$ is a CD-group.

We argue by induction on the derived length $d$ that $G$ is a CD-group. Let $d > 1$ and put $A = G^{(d-1)}$. Then $G/A$ is a CD-group, so it suffices to show that the set $S = \{ H < G \mid G = HA \}$ has a countable cofinal subset. By Corollary 4.4 the subgroup $A$ has CD; let $\mathcal{X}$ be a countable dominating set of proper subgroups in $A$. For any $X \in \mathcal{X}$, the group $A/X_G$ has a non-trivial $G$-invariant quotient which is either divisible or finite elementary abelian. Since $G$ has no $p^\infty \times p^\infty$ factors, there is a $G$-invariant subgroup $B(X) < A$ such that $X_G \leq B(X)$ and $A/B(X)$ is isomorphic to $\mathbb{Q}$ or $p^\infty$ or a finite elementary abelian $p$-group for some prime $p$. Let $\mathcal{C} = \{ B(X) \mid X \in \mathcal{X} \}$, which is a countable set.

Let $H \in S$. Then $D = H \cap A < HA = G$ and $D < A$; hence $D \leq X$ for some $X \in \mathcal{X}$, whence $D \leq X_G \leq B = B(X)$. Now
\((HB) \cap A = (H \cap A)B = B\), so that \(HB < G\) and \(G/B = (HB/B) \lt (A/B)\).

It follows that the set of all subgroups \(K\) of \(G\) such that \(K/B\) is a complement of \(A/B\) in \(G/B\) for some \(B \in \mathcal{C}\) is cofinal in \(\mathcal{S}\). To complete the proof it is enough to show that for each \(B \in \mathcal{C}\) there are only countably many such complements. We can factor \(G\) by such a \(B\), so let \(B = 1\). Hence \(A\) is finite or \(A \simeq \mathbb{Q}\) or \(A \simeq p^\infty\). We will show that \(A\) has countably many complements in \(G\).

Let \(H\) be a complement of \(A\) in \(G\) and set \(C = C_H(A) \lt G\). We wish to show that \(\text{Der}(H, A)\) is countable. There is an exact sequence

\[0 \to \text{Der}(H/C, A) \to \text{Der}(H, A) \to \text{Hom}(C, A),\]

so it is enough to prove that \(\text{Der}(H/C, A)\) and \(\text{Hom}(C, A)\) are countable. This is clear if \(A\) is finite, so assume it is infinite.

First suppose that \(A \simeq \mathbb{Q}\). Then \(\text{Aut}(\mathbb{Q})\) is a direct product of cyclic groups and it has finite torsion subgroup, while \(H/C\) has finite abelian ranks. It follows that \(H/C\) is finitely generated. Therefore \(\text{Der}(H/C, A)\) is countable. Since \(\text{Hom}(C, A)\) is a finite dimensional \(\mathbb{Q}\)-space, it too is countable.

Now let \(A\) be a \(p^\infty\)-group; then \(\text{Aut}(A)\) and hence \(H/C\), is abelian and residually finite. There is a finitely generated subgroup \(X/C\) such that \(H/X\) is periodic. If \(X \neq C\), then \(C_A(X)\) is finite and thus \(\text{Der}(H/X, C_A(X))\) is finite. Also \(\text{Der}(X/C, A)\) is clearly countable, which implies that \(\text{Der}(H/C, A)\) is countable by the cohomology sequence. If \(X = C\), then \(H/C\) is periodic and hence finite, since it is residually finite. Thus \(\text{Der}(H/C, A)\) is countable. Finally, consider \(\text{Hom}(C, A)\). There cannot be a surjective homomorphism \(C \to A\); for if there were, there would be a \(p^\infty\)-quotient \(C/N\) and then the factor \(C_A/N\) would be of type \(p^\infty \times p^\infty\). Consequently a homomorphism from \(C\) to \(A\) has finite image, from which it follow that \(\text{Hom}(C, A)\) is countable. \(\square\)

It is worth noting that a soluble group whose abelian subgroups have \(\text{CD}\) need not have this property. Indeed, the nilpotent group constructed in [6: Theorem 3.1] has a quotient of type \(p^\infty \times p^\infty\), so it does not satisfy \(\text{CD}\), yet all its abelian subgroups have \(\text{CMS}\) and hence \(\text{CD}\). On the other hand, a \textit{periodic} soluble group whose abelian subgroups are \(\text{CD}\)-groups does in fact have \(\text{CD}\). We will put this result in a more general setting. First recall that a \textit{generalized}
radical group is a group with an ascending series whose infinite factors are locally nilpotent.

**Theorem 6.2.** Let $G$ be a periodic generalized radical group. Then the following conditions on $G$ are equivalent:

(i) $G$ is an SCD-group;

(ii) all abelian subgroups of $G$ are CD-groups;

(iii) for each prime $p$ the Sylow $p$-subgroups of $G$ are either finite or finite extensions of a $p^\infty$-group.

**Proof.** Certainly (i) implies (ii). Assume that (ii) holds and let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ is a locally finite $p$-group whose abelian subgroups have finite rank. Thus $P$ is a Černikov group (see [7: Theorem 1.6.9]). In addition the finite residual of $P$ can have rank at most 1, so (iii) follows.

Next assume that (iii) holds. Since the hypothesis is inherited by subgroups, it is enough to prove that $G$ is a CD-group. The group $G$ is locally finite and satisfies min-$p$ for all primes $p$, so a theorem of Belyaev – see [7: Theorem 3.5.15] – implies it has a locally soluble subgroup of finite index. By Lemma 2.4 we may assume that $G$ is locally soluble and hence is radical. By [7: Proposition 5.4.8] the group $G$ is countable and it cannot contain a proper subgroup isomorphic to itself. Furthermore, for every prime $p$ the quotient $G/O_p'(G)$ is a Černikov group [7: Corollary 2.5.13].

We claim that $G$ has no proper subgroup $H$ such that $G = HO_{p'}(G)$ for every prime $p$. For, let $H$ be such a subgroup. By [7: Corollary 3.1.6], if $P$ is a Sylow $p$-subgroup of $H$, then $H = PQ$ for some $p'$-subgroup $Q$ of $G$, which implies that $G = P(QO_{p'}(G))$, whence it follows that $P$ is a Sylow $p$-subgroup of $G$. In the terminology of [7] this means that $H$ is a basic subgroup of $G$ relative to the set of all primes. But then $H \simeq G$ by [7: Theorem 5.3.10], so that $H = G$, thus justifying the claim.

It follows that every proper subgroup fails to contain $O_p'(G)$ for some prime $p$. Finally, $G/O_{p'}(G)$ is a CD-group by Theorem 4.1 and it follows that $G$ is a CD-group.

There is some similarity between the structures of soluble CMS-groups and soluble SCD-groups. However, this does not extend to locally soluble groups. Indeed, while locally soluble CMS-groups
are soluble by [6: Theorem 2.12], there are periodic locally nilpotent SCD-groups which are insoluble. For example, let $F_p$ be a finite $p$-group of derived length $d_p$ where the $d_p$ are unbounded. Then $G = {\text{Dr}}_p F_p$ is a periodic, residually finite, locally nilpotent group which is an SCD-group by Theorem 6.2. Of course $G$ is insoluble.

Concluding remarks.

The proof of Theorem 6.2 shows that a locally finite group with CD satisfies the equivalent conditions (ii) and (iii) in the theorem. It is tempting to conjecture that the converse is true, i.e., (iii) implies the property CD for locally finite groups. However, this is disproved by the existence of uncountable locally finite groups with finite Sylow subgroups [7: 5.4.11]: recall that all CD-groups are countable. Of course one can still ask whether a countable locally finite group $G$ satisfying min-$p$ and having no subgroups of type $p^{\infty} \times p^{\infty}$ for all $p$ must have CD and hence SCD. This question remains open. It is not hard to see that it would be sufficient to prove it for countable locally finite groups with finite Sylow subgroups.

Bibliography


