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Permutable subgroups and the Maier-Schmid theorem for nilpotent-by-finite groups

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ABSTRACT. We prove that every core-free permutable subgroup of a nilpotent-by-finite group G is contained in the hypercentre of G. We also discuss some properties of permutable subgroups of arbitrary groups with respect to commutators and to the upper central series.

INTRODUCTION

By definition, a *permutable* (or *quasinormal*) subgroup of a group G is a subgroup H such that $HK = KH$, or equivalently $HK \leq G$, for all $K \leq G$. Permutability is a very strong, if somewhat elusive, embedding property, whose study dates back to papers by Øysten Ore in the thirties of the 20th century. General information on permutable subgroups and their properties can be found, among other sources, in [7, §7.1], [11] and [8].

The Maier-Schmid theorem referred to in the title is the following celebrated result, first proved in [9]: *if G is a finite group, then every core-free permutable subgroup of G is contained in the hypercentre of G.* Here, as usual, saying that a subgroup H of G is core-free means that the normal core H_G of H in G is trivial. Consequences of the Maier-Schmid theorem are that all permutable subgroups of finite groups are subnormal, and that those which are also core-free are nilpotent—as a matter of fact both properties were already known at the time when the Maier-Schmid theorem appeared.

Passing to infinite groups, only part of this information is preserved, and in weakened form. Indeed, while permutable subgroups of arbitrary groups are not necessarily subnormal, they are always ascendant (see [7, 7.1.7]). On the other hand, there are examples in the literature (see [6, 2] and [4, Theorem 10]) showing that if H is a core-free permutable subgroup of an infinite group G it may happen that H is not contained in the hypercentre of G, thus the Maier-Schmid theorem is not generally true for infinite groups. Moreover, such an H is bound to be residually (finite nilpotent) but can be rather far away from being nilpotent or at least hypercentral; for instance it can fail to be locally soluble ([6]).

On the positive side, the Maier-Schmid theorem is known to hold true in various classes of infinite groups, in most cases satisfying finiteness conditions (see [4] for results and background information), or also, for instance, for locally cyclic permutable subgroups of arbitrary groups ([1, 5]).

One of the results proved in [4] (Theorem 9) is the fact that the Maier-Schmid theorem holds for abelian-by-finite groups. The question of whether the same is, more generally, true of nilpotent-by-finite groups was left open in [4]. A partial answer was given in [3], where it was shown that the theorem holds for *periodic* core-free permutable subgroups of nilpotent-by-finite groups. Here we answer the question in full:

Theorem 2.6. *Let G be a nilpotent-by-finite group and let H be a core-free permutable subgroup of G. Then $H \leq Z_n(G)$ for some positive integer n. As a consequence, H is nilpotent and subnormal in G.*

Knowing that H is contained in the n-th centre $Z_n(G)$ of G for some finite n, rather than just in the hypercentre $\bar{Z}(G)$ of G, is of course a useful piece of extra information, which is not usually available in other cases when extensions of the Maier-Schmid theorem can be proved. For instance, the well-known example by Iwasawa discussed in the introduction of [4] provides a hypercentral group G with a permutable core-free subgroup not contained in $Z_\omega(G)$.

To this respect, therefore, core-free permutable subgroup of nilpotent-by-finite groups behave as those of finite groups. However, it is worth mentioning that they do not share a property of permutable subgroups of abelian-by-finite groups. Indeed, all core-free permutable subgroups of abelian-by-finite groups are finite (see Corollary 2.2 below), but nilpotent groups may have infinite permutable core-free sugroups, as witnessed by Example 1.7.

Some possible further extensions of Theorem 2.6 are ruled out by Theorem 10 of [4]: for every prime p there exists a metabelian p-group G with a core-free permutable subgroup which is not contained in the hypercentre of G.

A side scope of this paper is that of presenting some elementary properties of (not necessarily core-free) permutable subgroups of arbitrary groups, related to commutators and relative orders of elements. This is the content of Section 1. In particular, Theorem 1.6 extends to permutable subgroups some properties of the series obtained by intersecting a normal subgroup with terms of the upper central series of a group.

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Notation. Most of our terminology and notation is standard; part of it has been introduced above.

We write $H \mathrel{\mathsf{p}} G$ to mean that H is a permutable subgroup of G, while $H \mathrel{\widehat{\mathsf{p}}} G$ means that H is permutable and core-free.

If x is an element of a group G and $H \leq G$, the order of x relative to H is $|x|_H := |\langle x\rangle/\langle x\rangle \cap H|$, which equals $|H\langle x\rangle : H|$ if $H\langle x\rangle \leq G$. If $|x|_H$ is finite we say that x is periodic modulo H, or also π-periodic modulo H if $|x|_H$ is a π-number for a set π of primes. If $X \leq G$ the set of all elements of X which are periodic (resp. π-periodic) modulo H is the *isolator* $I_X(H)$ (resp. the π-isolator $I_{X,\pi}(H)$) of H in X; note that we don't require $H \leq X$ here. As is well-known, such isolators are subgroups if X is locally nilpotent. We shall also use expressions like "X is periodic modulo H" to mean $X = I_X(H)$, or "X has finite exponent modulo H" to mean that there exists a positive integer λ such that $X^\lambda \leq H$. Furthermore, $\pi(G : H)$ is the set of all primes p such that $I_{G,p}(H) \supset H$, and $\pi(G) = \pi(G : 1)$.

As usual, G_π denotes the π-component of G, that is set $I_{G,\pi}(1)$ of all π-elements of G, whenever this set is a subgroup. We write π' for the set of all primes not in π and follow the usual habit of writing p for π when $\pi = \{p\}$.

For the sake of definiteness, we use $\mathbb{N}^+$ to denote the set of positive integers and reserve $\mathbb{N}$ for the set of natural numbers.

1. Orders and commutators

We shall make frequent use of this key property of permutable subgroups:

Lemma 1.1 (see, for instance, [7, 7.1.7])**.** *Let G be a group and $H \mathrel{\mathsf{p}} G$. Then:*

(i) *all elements of $G \smallsetminus N_G(H)$ are periodic modulo H.*
(ii) *As a consequence: H is inert in G (that is, $|H : H \cap H^g|$ is finite for all $g \in G$).*

We shall also need the following elementary observations related to nilpotency. We could not locate a suitable reference, so we sketch a proof for the reader's convenience.

Lemma 1.2. *Let G be a group, π a set of primes, $X, Y \leq G$ and assume $X \leq Z_n(\langle X, Y\rangle)$ for some $n \in \mathbb{N}^+$. Then the following conditions are equivalent:*

(i) *there exists a π-number λ such that $[X, Y]^\lambda = 1$;*
(ii) *there exists a π-number λ such that $X^\lambda \leq C_G(Y)$.*

Furthermore, if X is periodic modulo $C_G(Y)$, then $[X, Y]$ is periodic and $[X, Y]_\pi = [I_{X,\pi}(C_G(Y)), Y]$.

Proof. Arguing by induction on n, it can be checked that $[X^{\lambda^{n-1}}, Y] = 1$ if $[X, Y]^\lambda = 1$. This settles the implication (i) $\Rightarrow$ (ii). To prove the converse, since $[X, Y]$ is nilpotent it can be assumed that $[X, Y]$ is abelian, and a second induction argument shows that $[X^\lambda, Y] = 1$ implies $[X, Y]^{\lambda^{n-1}} = 1$ in this case.

Finally, assume that X is periodic modulo $C_G(Y)$. Let P and Q be the π- and the π'-isolators of $C_G(Y)$ in X respectively; then $X = PQ$. By applying (ii) $\Rightarrow$ (i) to the cyclic subgroups of P, we see that $[P, Y]$ is generated by π-subgroups. But $[X, Y]$ is nilpotent, therefore $[P, Y]^Q \leq [X, Y]_\pi$. Similarly, $[Q, Y]^P \leq [X, Y]_{\pi'}$. Then $[X, Y] = [PQ, Y] = [P, Y]^Q \times [Q, Y] = [P, Y] \times [Q, Y]^P$, hence $[X, Y]$ is periodic and $[P, Y] = [X, Y]_\pi$, as required. □

With reference to the equivalence (i) $\iff$ (ii), the argument also provides bounds for the exponents involved, a fact that we shall not need.

In the next lemma we record a 'non-divisibility' property of commutators arising from non-normal permutable subgroups.

Lemma 1.3. *Let G be a group and $H \mathrel{\mathsf{p}} G$. Assume that $x \in G$ and $h \in H$ are such that $c := [x, h] \notin H$. Then $c \in H\langle x\rangle$ and $|x|_H$ is finite; moreover, if $n = |x|_{H\langle c\rangle}$:*

(i) *$|x|_H = n \cdot |c|_H$;*
(ii) *$c \in Hx^{nt}$ for some integer t coprime with $|c|_H$;*
(iii) *there exists no $z \in C_G(\langle x, h\rangle)$ such that $c = z^n$.*

Proof. Clearly $c \in H\langle x\rangle$, since $H \mathrel{\mathsf{p}} G$. Also, x does not normalise H, as $c \notin H$, hence x is periodic modulo H by Lemma 1.1. Next, $|x|_H = |H\langle x\rangle : H| = |H\langle x\rangle : H\langle c\rangle|\,|H\langle c\rangle : H| = nm$, where $m = |c|_H$, yielding (i).

By looking at (relative) orders, we see that $\langle x\rangle \cap H\langle c\rangle = \langle x^n\rangle$ and $\langle x\rangle \cap H = \langle x^{nm}\rangle$. Then $c \in Hx^{nt}$ for some integer t. If there is a prime p dividing both t and m, then $c \in H\langle x^{nt}\rangle \leq H\langle x^{np}\rangle < H\langle c\rangle$, because $|x^{np}|_H = m/p < m = |c|_H$. This is a contradiction, hence t and m are coprime, which proves (ii).

Finally, assume $c = z^n$ for some $z \in C_G(\langle x, h\rangle)$. Choose $s \in \mathbb{Z}$ such that $st \equiv_m -1$; then $c^{st} \in c^{-1}H$. Let $x_1 = xz^s$. We have $c = [x_1, h]$, hence $H\langle c\rangle \leq H\langle x_1\rangle$ as above. Now $x_1^n = x^n c^s \in H\langle c\rangle$ and $x_1^{nt} = x^{nt}c^{st} \in Hcc^{st} = H$. Therefore $|x_1^n|_H$ divides t. But $|x_1^n|_H$ also divides m, as $x_1^n \in H\langle c\rangle$. Since t and m are coprime $x_1^n \in H$. Let $n_1 = |x_1|_{H\langle c\rangle}$. We have proved that $n_1 m = |x_1|_H$ divides n, hence $n_1 < n$. We can repeat the argument after substituting x_1, n_1 and $z_1 := z^{n/n_1}$ for x, n and z; indeed $z_1 \in C_G(\langle x_1, h\rangle)$ and $z_1^{n_1} = c = [x_1, h]$. Further iteration provides a strictly decreasing sequence $(n_i)_{i\in\mathbb{N}^+}$ of positive integers; this contradiction completes the proof. □

Lemma 1.4. *Let G be a group and $H \mathrel{\mathsf{p}} G$. Fix a prime p, and let A be an abelian p-subgroup of G. Let $x \in I_{G,p}(H)$ and $h \in H$ be such that $c := [x,h] \in A \smallsetminus H$. If $A \cap H$ has finite exponent and $[A,\langle x\rangle], [A,h] \le H$, then either A has finite exponent, or $[A,\langle x,h\rangle] \neq 1$ and A it is the direct product of a group of finite exponent by a Prüfer p^∞-group.*

Proof. Let $q = \exp(A\cap H)\,|x|_H$ and $X = \langle A, x\rangle$, so that $X' = [A,\langle x\rangle] \le H$ and $Y := X \cap H \lhd X$. For all $g \in X$, write $\bar g$ for gY; note that $|g|_H$ is the order $|\bar g|$ of $\bar g$ in the abelian p-group X/Y. Let $a \in A \smallsetminus A[q]$, where $A[q] = \{b \in A \mid b^q = 1\}$. Then $|\bar a| > |\bar x|$, hence $|\overline{xa}| = |\bar a|$; also, $|\bar c| < |\bar a|$ since $|\bar c| \le |\bar x|$ by Lemma 1.3. We have $[xa,h] = c[a,h] \in cH$ because A is abelian and $[A,h] \le H$. Then Lemma 1.3 shows $|\overline{xa}| = |\bar c|\,n$, where $n = |xa|_{H\langle c\rangle}$, and $\langle \bar c\rangle = \langle \overline{xa}\rangle^n$. As $|x|_{H\langle c\rangle}\,|\bar c| = |\bar x| < |\overline{xa}|$, the former equation gives $|x|_{H\langle c\rangle} < n$, hence $x^n \in H\langle c^p\rangle$, that is, $\bar x^n \in \langle \bar c^p\rangle$. Thus we obtain $\langle \bar c\rangle = \langle \overline{xa}^n\rangle = \langle \bar a^n\rangle < \langle \bar a\rangle$, since $|\bar c| < |\bar a|$. We have shown that, for all $a \in A \smallsetminus A[q]$, it holds $A \cap H < (A\cap H)\langle c\rangle < (A\cap H)\langle a\rangle$. It follows that $A^q \simeq A/A[q]$ has rank 1. If $\exp A$ is infinite, then A^q is a Prüfer group and hence a direct factor in A. Finally, we show that in this latter case $[A,\langle x,h\rangle] \neq 1$. Indeed assume that $\exp A$ is infinite and $[A,\langle x,h\rangle] = 1$. Let $G_0 = \langle A,x,h\rangle$ and $H_0 = H \cap G_0$, so that $H_0 \mathrel{\mathsf{p}} G_0$; then $A \cap H \le Z(G_0)$. Denote by $*$ the natural epimorphism $G_0 \twoheadrightarrow G_0/(A\cap H)$. Then $c^* \notin H_0^*$ and $\exp A^*$ is infinite, moreover $\langle c^*\rangle < \langle a^*\rangle$ for all elements $a^* \in A^*$ of sufficiently large order, so that $c^* \in (A^*)^{p^\lambda}$ for all $\lambda \in \mathbb{N}$. Since $[A^*,\langle x^*,h^*\rangle] = 1$ this contradicts Lemma 1.3. Now the proof is complete. □

The following lemma is a special case of Lemma 1.4 and will play a key role in the upcoming proofs.

Lemma 1.5. *Let G be a group, $X \le G$ and $Y \le H \mathrel{\mathsf{p}} G$. Assume $[X,Y] \le Z(\langle X,Y\rangle)$. Then $Q := [X,Y]/([X,Y]\cap H)$ is periodic and, for all primes p, the p-component of Q has finite exponent.*

Proof. At the expense of replacing G with $\langle X,Y\rangle$ and H with $H \cap \langle X,Y\rangle$, if needed, we may assume $[X,Y] \le Z(G)$. We may also pass to the quotient $G/([X,Y]\cap H)$, so we assume $[X,Y]\cap H = 1$, hence $Q \simeq [X,Y]$. If $[X,Y]$ is not periodic, then it is easily seen that G is generated by elements of infinite order modulo H. In this case $H \lhd G$ by Lemma 1.1 (i), hence $[X,Y] \le [X,Y]\cap H = 1$. Therefore $[X,Y]$ is periodic. Fix a prime p. Since $[X,Y] \le Z(G)$ the set $\{[x,y] \mid x \in X,\ y \in Y\}$ is closed under taking powers. From this, and from the fact that $[X,Y]$ is periodic and abelian it is not hard to deduce that every element of $A := [X,Y]_p$ can be written as $c = \prod_{i\in I}[x_i,h_i]$ for a finite set I, where, for each $i \in I$, $x_i \in X$, $h_i \in Y$ and $c_i := [x_i,h_i]$ has p-power order. If $c \neq 1$, then $c_i \neq 1$ for some fixed $i \in I$. Now $|x_i|_H$ is finite, otherwise x_i normalises H and so $c_i \in H \cap [X,Y] = 1$. Then, at the expense of replacing x_i with x_i^λ for a suitable p'-number λ, we may assume $x_i \in I_{G,p}(H)$. Lemma 1.4 now shows that A has finite exponent. The lemma is proved. □

We are in a position to prove one of the results anticipated in the introduction.

Theorem 1.6. *Let G be a group and $H \mathrel{\mathsf{p}} G$; for all $i \in \mathbb{N}$ also let $S_i = H \cap Z_i(G)$.*

(i) *If p is a prime and $\exp(S_1)_p$ is finite, then $\exp(S_{n+1}/S_n)_p$ is finite for all $n \in \mathbb{N}$;*
(ii) *if $\exp S_1$ is finite, then $\exp S_n^G$ is finite for all $n \in \mathbb{N}$ such that $\pi([S_n,G])$ is finite;*
(iii) *if both $\exp S_1$ and $\pi([G,H]:H)$ are finite, then $\exp S_n^G$ is finite for all $n \in \mathbb{N}$.*

Proof. To prove (i), fix a prime p and assume that $(S_1)_p$ has finite exponent. Let $\bar P = P/S_1 = (S_2/S_1)_p$. Then $[P,G] \le (Z(G))_p$ and $[P,G]\cap H \le (S_1)_p$. It follows from Lemma 1.5 that $e := \exp[P,G]$ is finite. Then $P^e \le Z(G)\cap H = S_1$, so that $\bar P^e = 1$. This proves the result in the case $n = 1$; an easy induction argument now completes the proof.

Similarly, to prove (ii) and (iii) it is enough to consider the case $n = 2$. Assume that $\exp S_1$ is finite. Since $[S_2,G]\cap H \le S_1$ we may apply Lemma 1.5 to show that each primary component of $[S_2,G]$ has finite exponent. But $\pi([S_2,G])$ is finite, hence $e := \exp[S_2,G]$ is finite. Then $S_2^e \le S_1$, hence S_2 has finite exponent and the same is true of $S_2^G \le Z_2(G)$. Thus (ii) is proved. Finally, (iii) is an easy consequence of (ii). Indeed, if $\exp S_1$ and $\pi([G,H]:H)$ are finite, then $\pi([S_2,G])$ is finite because $[S_2,G]\cap H$ has finite exponent as a subgroup of S_1. □

Example 1.7. Let π_1 and π_2 be disjoint sets of odd primes, and let $\pi = \pi_1 \cup \pi_2$. Assume that π_2 is either empty or infinite, and that π_1 is infinite in the former case.

For all $p \in \pi$ let A_p be a cyclic group of order p^2, and let α_p be the automorphism $a \in A_p \mapsto a^{1+p} \in A_p$. Now let $A := \mathrm{Dr}_{p\in\pi} A_p$, and let $H = \langle h\rangle \times \mathrm{Dr}_{p\in\pi_1}\langle h_p\rangle$ and $G = A \rtimes H$ be defined as follows. For all $p \in \pi_1$, we let h_p have order p and act like α_p on A_p and trivially on $A_{p'}$. If $\pi_2 = \varnothing$, we let $h = 1$, otherwise h has infinite order and acts trivially on A_{π_1} and like α_p on A_p for all $p \in \pi_2$.

Then G is a nilpotent group of class 2. It is not hard to see that H is permutable in G (as a consequence, for instance, of [5, Lemma 6]; here we use the fact that $2 \notin \pi$), and clearly $H_G = 1$. In the notation of Theorem 1.6, $S_1 = H \cap Z(G) = 1$ and $S_2 = H \cap Z_2(G) = H$. Of course H may be made periodic of infinite exponent, or non-periodic; also note that $[S_2,G] = \langle a_p^p \mid p \in \pi\rangle$ involves infinitely many primes.

With reference to Theorem 1.6, this example shows that, even if G is nilpotent, assuming that $\exp S_1$ is finite is not enough, by itself, to conclude that $\exp S_2$ is finite, nor, even, that S_2 is periodic or $\exp(\operatorname{tor} S_2)$ is finite.

The same example also suggests that the other standard property of the factors of the series obtained by intersecting the upper central series of a group with a normal subgroup cannot be extended to permutable subgroups. Indeed, in the group just constructed, S_1 is torsion-free but S_2/S_1 is not unless $\pi_1 = \varnothing$. This contrasts with the well-known fact that, still in the setting of Theorem 1.6, if $H \lhd G$ and S_1 is π-torsion-free for a set π of primes, then each factor S_{n+1}/S_n must also be π-torsion-free.

A consequence of Theorem 1.6 is the following useful variant of Lemma 1.5.

Lemma 1.8. *Let G be a group, $Y \leq H \mathrel{\mathsf{p}} G$ and $X \leq I_{Z_n(G),p}(C_G(Y))$ for some $n \in \mathbb{N}$ and a prime p. If $(H \cap Z(G))_p$ and $[X, Y] \cap H$ have finite exponents, then also $[X, Y]$ has finite exponent.*

Proof. We argue by induction on n and assume $n > 1$, the result being obvious if $n \leq 1$. By Lemma 1.2, we know that $[X, Y]$ is a p-group. Let $*$ denote the natural epimorphism: $G \twoheadrightarrow G/Z(G)$. It follows from Theorem 1.6 (i) that $\big((Z_n(G))^* \cap H^*\big)_p$ has finite exponent, hence $[X, Y]^* \cap H^*$ has finite exponent, as well as $(H^* \cap Z(G^*))_p$. Applying the induction hypothesis to G^* we obtain that $\exp[X, Y]^*$ is finite. In view of Lemma 1.2 this means that $[X^\lambda, Y] \leq Z(G)$ for some $\lambda \in \mathbb{N}^+$. Therefore $[X^\lambda, Y] \cap H$ has finite exponent, and the same is true of $[X^\lambda, Y]$, by Lemma 1.5. A second appeal to Lemma 1.2 shows that $\exp[X, Y]$ is finite. □

2. The Maier-Schmid theorem for nilpotent-by-finite groups

In this section we shall prove our main result. We begin with a few lemmas. In the first one we collect some easy (and essentially known) remarks on core-free permutable subgroups. Note that the hypotheses on G in both (iii) and (iv) are satisfied if G is nilpotent-by-finite.

Lemma 2.1. *Let H be a permutable subgroup of a group G. Then:*
- (i) *If $H_G = 1$ and N is a normal subgroup of G such that $H \cap N$ has finitely many conjugates in G, then $H \cap N$ is finite.*
- (ii) *If $H_G = 1$ and N is a normal subgroup of finite index in G, then $H \cap Z(N)$ is finite.*
- (iii) *If G is locally residually finite, $H_G = 1$ and H has finitely many conjugates in G, then H^G is finite and contained in $Z_n(G)$ for some $n \in \mathbb{N}$.*
- (iv) *If all periodic images of G are locally finite and G is both finitely generated and periodic modulo H, then G/H_G is finite and $H^G/H_G \leq Z_n(G/H_G)$ for some $n \in \mathbb{N}$.*

Proof. (i): Since H is inert in G (Lemma 1.1 (ii)), also $H \cap N$ is inert. But $(H \cap N)_G = 1$, therefore $H \cap N$ is finite.

(ii) follows from (i) applied to $Z(N)$; (iii) is Proposition 1 of [4].

(iv): $|H^G : H|$ is finite by [7, Lemma 7.1.9]. But G/H^G is periodic and finitely generated, hence finite by the hypothesis on G. Then $|G : H|$ is finite, hence G/H_G is finite and the statement follows from the Maier-Schmid theorem. □

A direct consequence of part (ii) of the previous lemma is:

Corollary 2.2. *Every core-free permutable subgroup of an abelian-by-finite group is finite.*

Corollary 2.2 does not extend to nilpotent-by-finite groups. Indeed, the groups constructed in Example 1.7 are nilpotent of class 2 and have infinite, core-free permutable subgroups.

Also note that the normal closure of a subgroup as in Corollary 2.2 may well be infinite. An example, not much different from those in Example 1.7, is the following. If p is an odd prime and $G = A \rtimes \langle h \rangle$, where $|h| = p$, A is infinite homocyclic of exponent p^2 and $a^h = a^{1+p}$ for all $a \in A$, then G is abelian-by-finite, $\langle h \rangle \mathrel{\widehat{\mathsf{p}}} G$ and $\langle h \rangle^G$ is infinite.

A further consequence of Lemma 2.1 is that, in order to prove the main result of this section, we only need to consider the case of a permutable subgroup whose isolator is the whole group.

Lemma 2.3 ([3]). *Let G be a nilpotent-by-finite group and let $H \mathrel{\widehat{\mathsf{p}}} G$. If G is not periodic modulo H, then $H \leq Z_n(G)$ for some $n \in \mathbb{N}$.*

Proof. Let N be a nilpotent normal subgroup of finite index in G. If G is not periodic modulo H, then neither is N and Lemma 1.1 (i) yields $H \lhd HN$. Then H has finitely many conjugates in G, and the conclusion follows from Lemma 2.1(iii). □

Lemma 2.4. *Let $H \mathrel{\widehat{\mathsf{p}}} G$, where G is a nilpotent-by-finite group. Let T be the maximal periodic normal subgroup of H^G. Then $T \leq \bar{Z}(G)$.*

Proof. In view of Lemma 2.3 we may assume that G is periodic modulo H. Let N be a nilpotent normal subgroup of finite index in G. First, we shall prove $T \cap N \leq \bar{Z}(G)$. To this end, note that $G = XN$ for some finitely generated subgroup X of G, and let $a \in T \cap N$. Since $V := \langle a \rangle^X$ is finite, $V \leq H^G$ and $H_G = 1$, there exists a finitely generated $Y \leq G$ such that $X \leq Y$, $a \in H^Y$ and $V \cap H_Y = 1$. By Lemma 2.1 (iv), the group HY/H_Y is finite and $W := H^Y/H_Y \leq \bar{Z}(HY/H_Y)$. But V is X-isomorphic to a subgroup of W, hence $[V, {}_m X] = 1$ for some $m \in \mathbb{N}$. This argument shows that every factor of the series obtained by intersecting T with the upper central series of N has an ascending X-central (and hence G-central) series of length at most ω. Therefore $T \cap N \leq Z_{\omega c}(G)$, where c is the nilpotency class of N.

Now let $C = C_G(T/T\cap N)$. Since $T/T\cap N$ and hence G/C are finite, $T = S(T\cap N)$ and $G = QC$ for suitable finitely generated subgroups S and Q. Now S^Q is periodic (as a subgroup of T) and finitely generated (as a subgroup of $\langle S, Q\rangle$), hence it is finite. As in the previous paragraph, we can find a finitely generated $U \leq G$ such that $Q \leq U$, $S \leq H^U$ and $S^Q \cap H_U = 1$, and then deduce that HU/H_U is finite and $H^U/H_U \leq \bar{Z}(HU/H_U)$. Since, as a Q-group, S^Q is isomorphic to a subgroup of H^U/H_U we have $[S, {}_nQ] = 1$ for some $n \in \mathbb{N}$. Therefore $[T, {}_nG] = [S(T\cap N), {}_n(QC)] \leq T\cap N \leq Z_{\omega c}(G)$, so that $T \leq Z_{\omega c+n}(G)$. □

The next lemma is the key step in the proof of our main result.

Lemma 2.5. *Let G be a nilpotent-by-finite group and $H \mathrel{\widehat{\mathfrak{p}}} G$. Let K be a periodic normal subgroup of G contained in H^G and assume that all primary components of K have finite exponent. Then $L := (HK)_G \leq Z_n(G)$ for some $n \in \mathbb{N}$.*

Proof. Clearly $L = (L\cap H)K \leq H^G$. Let F be the Fitting subgroup of G. Then F is nilpotent. Let $L_1 = L \cap F$. A known result due to Baumslag (see for instance [10, Lemma 6.34]) shows that each primary component of K is nilpotent since it is nilpotent-by-finite and of finite exponent, hence $K \leq L_1$.

First consider the case when $\exp K$ is finite. Since $L \leq HK$ and G is nilpotent-by-finite, in this case there exists $\lambda \in \mathbb{N}^+$ such that $L^\lambda \leq H$. But then $L^\lambda \leq H_G = 1$, hence $L \leq \bar{Z}(G)$ by Lemma 2.4. Let p be a prime dividing $\exp L$, and let A be a factor of the finite series $1 \leq L_p \cap Z(F) \leq L_p \cap Z_2(F) \leq \cdots \leq (L_1)_p \leq L_p$. Then $\Gamma = G/C_G(A)$ is finite, because $L_p/(L_1)_p$ is finite and the remaining factors of the series are centralised by F. Since $L \leq \bar{Z}(G)$ it is then clear that Γ is a p-group and we may apply Baumslag's result again (to the external semidirect product $A \rtimes \Gamma$, for instance) to conclude that A has a finite Γ-central series. It follows that there is some $n \in \mathbb{N}$ such that $L \leq Z_n(G)$, as required. Moreover, if $\exp L_1$ and $|G/F|$ are coprime all the factors A of the form $(L_p \cap Z_{i+1}(F))/(L_p \cap Z_i(F))$ appearing in the previous argument are central in G. Then, in this case, the argument yields $L_1 \leq Z_c(G)$, where c is nilpotency class of F, and hence $L \leq Z_{\ell+c}(G)$, where $\ell = |L : L_1|$ is finite.

Now consider the general case. For every prime p, we have $J_{[p]} := I_{L_1,p'}(H) = (L_1 \cap H)K_{p'}$. Then $L_1^q \leq J_{[p]}$ for some power q of p, because $\exp K_p$ is finite. Let $B_{[p]} = (HL_1^q)_G$. Then $B_{[p]} \leq (HK)_G = L$, and $L_1^q \leq B_{[p]} \leq L \cap HL_1^q$ because $L_1^q \lhd G$. Moreover, $B_{[p]}^\ell \leq L^\ell \cap HL_1^q \leq L_1 \cap HL_1^q = (L_1 \cap H)L_1^q \leq J_{[p]}$. Now, $\exp(KB_{[p]}/B_{[p]})$ is finite and so, by the previous case, $L/B_{[p]} \leq Z_{n_p}(G/B_{[p]})$ for some $n_p \in \mathbb{N}$. Furthermore, if $p \notin \pi := \pi(G/F)$ then $\exp(L_1/B_{[p]})$, which is a power of p, is coprime with $|G/F|$ and hence with $|G/B_{[p]} : \mathrm{Fit}(G/B_{[p]})|$. Then, thanks to the closing remark in the previous paragraph, we may choose $n_p = \ell + c$, that is $C := [L, {}_{\ell+c}G] \leq B_{[p]}$. Thus we have $C^\ell \leq \bigcap\{B_{[p]}^\ell \mid p \in \pi'\} \leq \bigcap\{J_{[p]} \mid p \in \pi'\} = I_{L_1,\pi}(H) = (L_1 \cap H)K_\pi$. Since π is finite $\exp K_\pi$ is finite. It follows that C has finite exponent modulo H and then, as $H_G = 1$, we see that $\exp C$ is finite. By the previous case, $C \leq Z_m(G)$ for some $m \in \mathbb{N}$, hence $[L, {}_{\ell+c+m}G] = [C, {}_mG] = 1$, that is, $L \leq Z_{\ell+c+m}(G)$. □

Theorem 2.6. *Let G be a nilpotent-by-finite group and $H \mathrel{\widehat{\mathfrak{p}}} G$. Then $H \leq Z_n(G)$ for some positive integer n. As a consequence, H is nilpotent and subnormal in G.*

Proof. Let $F = \mathrm{Fit}\, G$, so that F is nilpotent and G/F is finite. Arguing by contradiction, we may assume that G and H provide a counterexample in which F has the minimal index in G and, among such counterexamples, also one in which $H_0 := H \cap F$ has minimal nilpotency class. By Lemma 2.3, G is periodic modulo H.

Let $A = Z(H_0)$ and $C = C_F(A)$. Then $H_0 \leq C$, hence F is periodic modulo C; it follows from Lemma 1.2 that $[A, F]$ is periodic and, for every prime p, $[A, F]_p = [A, P]$ where $P = I_{F,p}(C)$. Now, $H \cap Z(F)$ is finite by Lemma 2.1 (ii), hence we may apply Theorem 1.6 (i) to F and H_0 to deduce that $(H_0)_p$ has finite exponent. Then $\exp([A, P] \cap H_0)$ is finite and hence, by Lemma 1.8, $\exp[A, P]$ is finite.

Let $K = [A, F]^G$. Then $K \leq \mathrm{tor}(F \cap H^G)$ and all primary components of K have finite exponent, because $[A, F]$ has the same property. Let $L = (HK)_G$ and use $*$ to denote the natural epimorphism $G \twoheadrightarrow G/L$. Then $H^* \mathrel{\widehat{\mathfrak{p}}} G^*$, and $H^* \nleq Z_n(G^*)$ for all $n \in \mathbb{N}$ by Lemma 2.5.

Now $A^* \leq H^* \cap Z(F^*)$, as $[A, F] \leq L$. Then A^* is finite by Lemma 2.1 (ii); since it only has finitely many conjugates also $K_1^* := (A^*)^{G^*}$ is finite. Thus we can make use of Lemma 2.5 again to factor out $(H^*K_1^*)_{G^*}$. More explicitly, let $L_1 \lhd G$ be such that $L \leq L_1$ and $L_1^* = (H^*K_1^*)_{G^*}$. Using double asterisks to denote images modulo L_1, we have $H^{**} \mathrel{\widehat{\mathfrak{p}}} G^{**}$ and, for all $n \in \mathbb{N}$, $H^{**} \nleq Z_n(G^{**})$. The minimality requirement on $|G/F|$ implies $|G^{**}/F^{**}| = |G/F|$, that is, $L_1 \leq F$. Then $H^{**} \cap F^{**} = (H \cap F)^{**} = H_0^{**}$. But $(Z(H_0))^{**} = A^{**} = 1$, hence the nilpotency class of H_0^{**} is less than that of H_0, which contradicts our second minimality assumption. This contradiction completes the proof. □

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