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On a construction by Giudici and Parker on commuting graphs of groups

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ABSTRACT. Given a connected graph Δ, a group G can be constructed in such a way that Δ is often isomorphic to a subgraph of the commuting graph $\mathcal{KG}(\Delta)$ of G. We show that, with one exception, $\mathcal{KG}(\Delta)$ is connected, and in this latter case its diameter is at most that of Δ. If Δ is a path of length $n > 2$, then $\operatorname{diam}(\mathcal{KG}(\Delta)) = n$.

1. Introduction

In [2], M. Giudici and C. Parker show that for every positive integer n there exists a finite 2-group of nilpotency class 2 whose commuting graph is connected and has diameter greater than n. Their construction can be interpreted as a variant of a general procedure which makes possible to define a class-2 nilpotent group starting from a graph (see [3] for an early example).

In this case the construction is as follows. Let Δ be a (simple, undirected) connected graph. We define $\mathcal{G}(\Delta)$ as the group generated by the vertices of Δ subject to the following relations: for all $a, b, c, d \in V(\Delta)$, $a^2 = [a, b, c] = 1$ and $[a, b] = [c, d]$ if $d_\Delta(a, b) = d_\Delta(c, d) > 1$, while $[a, b] = 1$ if $d_\Delta(a, b) = 1$ (d_Δ denotes distance in Δ). Then $\mathcal{G}(\Delta)$ is a nilpotent 2-group of class at most 2 and exponent at most 4. Let n be the (possibly infinite) diameter $\operatorname{diam}\Delta$ of Δ. Write G for $\mathcal{G}(\Delta)$ and, for all $i \in \mathbb{N}$ such that $1 < i \leqslant n$, let $c_i = [u, v] \in G$ where u, v are (any) vertices of Δ such that $d_\Delta(u, v) = i$. Then G' is elementary abelian with $\{c_2, c_3, \ldots, c_n\}$ or $\{c_i \mid 1 < i \in \mathbb{N}\}$ as a basis, according to whether n is finite or infinite, so the rank of G' is $n - 1$ in the former case, infinitely countable in the latter. Also $G^{\mathrm{ab}} = G/G'$ is elementary abelian; it has rank $|V(\Delta)|$. It is of course possible that $G' < Z(G) < G$; for instance if Δ is a path of length 2 then G is isomorphic to the direct product of a dihedral group of order 8 and a group of order 2, as a more extreme case: G would be abelian if Δ were a complete graph.

In the definition of $\mathcal{G}(\Delta)$, the choice of letting adjacent vertices of Δ commute is justified by the fact that it often allows to identify Δ with a subgraph of the commuting graph of $\mathcal{G}(\Delta)$. Here, with reference to an arbitrary group G with center $Z = Z(G)$, we define the *commuting graph* $\mathcal{K}(G)$ of G as the graph with the set $\{aZ \mid a \in G \smallsetminus Z\}$ of all nontrivial cosets of Z in G as the vertex set, and in which any two distinct vertices aZ and bZ are adjacent if and only if $[a, b] = 1$ in G. Note that a slightly different definition of commuting graph is more common in the literature; it requires $G \smallsetminus Z$ as the vertex set and essentially the same adjacency condition: two distinct vertices are adjacent iff they commute in G. Calling $\mathcal{K}^*(G)$ this second graph, it is clear that $\mathcal{K}(G)$ is a retract of $\mathcal{K}^*(G)$: this latter is obtained from $\mathcal{K}(G)$ by replacing every vertex with a complete graph on $|Z|$ vertices and adding edges so that any two distinct vertices of $\mathcal{K}^*(G)$ will be adjacent iff the corresponding cosets coincide or are adjacent in $\mathcal{K}(G)$. Thus, $\mathcal{K}(G)$ is a minor of $\mathcal{K}^*(G)$, and embeds in it. What is relevant to our purposes is that if $a, b \in G$, then the distances of a and b in $\mathcal{K}^*(G)$ and aZ and bZ in $\mathcal{K}(G)$ are the same, and $\operatorname{diam}(\mathcal{K}^*(G)) = \operatorname{diam}(\mathcal{K}(G))$. Since we shall almost exclusively be concerned with connectedness and metric questions, this means that up to irrelevant details all our results will still remain valid if $\mathcal{K}(G)$ is replaced by $\mathcal{K}^*(G)$. An advantage of working with $\mathcal{K}(G)$ rather than $\mathcal{K}^*(G)$ is that if G and H are groups, then $\mathcal{K}(G)$ and $\mathcal{K}(H)$ are isomorphic not only if G and H are isomorphic, but also if G and H are merely isoclinic. For instance $\mathcal{K}(G) \simeq \mathcal{K}(H)$ for all subgroups H of G such that $G = HZ(G)$.

It is also worth noting that, by construction, $\mathcal{K}(G)$ cannot have just one vertex nor a vertex of eccentricity 1 (i.e., adjacent to all other vertices), so either it is empty (which happens iff G is abelian) or its diameter is greater than 1.

The main result in [2] can be rephrased as follows: if Δ is a (finite) path of length $n > 2$, then $\mathcal{KG}(\Delta) := \mathcal{K}(\mathcal{G}(\Delta))$ is a connected graph whose diameter can be arbitrarily large, in fact bounded below in terms of n. The authors also note that some computational evidence suggests that this diameter could actually be precisely n. We shall prove that this is the case. We also show that, up to small exceptions, if Δ is an arbitrary connected graph, $\mathcal{KG}(\Delta)$ is connected of diameter at most $\operatorname{diam}\Delta$. Our main result is the following.

Theorem. *Let Δ be a connected graph of (possibly infinite) diameter n, let $G = \mathcal{G}(\Delta)$ and $\Gamma = \mathcal{KG}(\Delta)$. Then:*

(i) *if $n > 2$, then Γ is connected and* $\operatorname{diam}(\Gamma) \leqslant n$. *Moreover, if Δ is a path, then* $\operatorname{diam}(\Gamma) = n$;

(ii) *if $n = 2$, then either Γ is connected and* $\operatorname{diam}(\Gamma) = 2$, *or $|G/Z(G)| = 4$ and Γ is the graph on three vertices and no edges.*

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Which of the two possible cases does actually occur for a given graph of diameter 2 is discussed in Proposition 2.4. Finally, Proposition 5.9 shows for a few small values of $n = \operatorname{diam}\Delta > 2$ that the maximum diameter n for $\operatorname{diam}(\mathcal{KG}(\Delta))$ is only attained when $\mathcal{KG}(\Delta)$ is isomorphic to the graph constructed from a path of the same length.

2. Notation and preliminaries

We use standard terminology for groups and graphs (a reference for the latter is [1]); note that we admit the empty graph (the one with empty vertex set) and that $v \sim_\Delta w$ (or simply $v \sim w$, if Δ is suggested by the context) means that v and w are adjacent vertices of the graph Δ.

In accordance with standard terminology, an element g of a p-group G (where p is a prime) has *breadth* a positive integer $\operatorname{br}_G(g)$ iff g has (exactly) $p^{\operatorname{br}_G(g)}$ conjugates in G. In this paper G will almost always be a nilpotent 2-group of class 2 and G' will have exponent 2; in these cases $\operatorname{br}_G(g)$ is the rank (that is, the dimension as a vector space over the integers modulo 2) of $G/C_G(g)$ or, equivalently, of $[G, g]$.

With reference to a connected graph Δ and the group $G = \mathcal{G}(\Delta)$, for each $g \in G$ we shall write $\bar{g}$ for the coset $gZ(G) \in G/Z(G)$. Furthermore, for every finite set X of vertices of Δ, we denote by $\text{Þ}(X)$ the product in G of the elements of X in a prefixed order (which one is irrelevant to our purposes, since we will only be interested in commutators involving $\text{Þ}(X)$; of course we let $\text{Þ}(X) = 1$ if $X = \varnothing$) and let $\overline{\text{Þ}}(X) = \text{Þ}(X)Z(G) = \prod_{g\in X} \bar{g}$; this latter is of course independent of the ordering of the factors and is either $Z(G)$ or a vertex of $\mathcal{KG}(\Delta)$.

As usual, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the floor: $\max\{n \in \mathbb{Z} \mid n \leqslant x\}$ and the ceiling: $\min\{n \in \mathbb{Z} \mid n \geqslant x\}$ of the number x.

For any sentence φ we write θ_φ for the truth value of φ: so θ_φ is 1 or 0 according to whether φ is true or false. This symbol will be mostly used in Section 3.

Just for the sake of further reference we state an obvious lemma.

Lemma 2.1. *Let Δ be a connected graph and $G = \mathcal{G}(\Delta)$. Let X, Y be finite subsets of $V(\Delta)$. Then $[\text{Þ}(X), \text{Þ}(Y)] = 1$ (in G) if and only if, for all integers $i > 1$, the number of all pairs $(x, y) \in X \times Y$ such that $d_\Delta(x, y) = i$ is even.*

Proof. For all integers i such that $1 < i \leqslant \operatorname{diam}(\Delta)$, let $\lambda_i = \big|\{(x, y) \in X \times Y \mid d_\Delta(x, y) = i\}\big|$, moreover, let $c_i = [u, v]$ for any $u, v \in V(\Delta)$ such that $d_\Delta(u, v) = i$. Then $[\text{Þ}(X), \text{Þ}(Y)] = \prod_{x\in X, y\in Y}[x, y] = \prod_{i=2}^{n} c_i^{\lambda_i}$, where $n = \max\{d_\Delta(x, y) \mid x \in X \text{ and } y \in Y\}$. Now the result follows from the fact that, as remarked earlier, the commutators c_i form a basis of G'. □

Of course, in the previous lemma, the condition $[\text{Þ}(X), \text{Þ}(Y)] = 1$ exactly means that one of the following holds: either one of $\overline{\text{Þ}}(X)$ and $\overline{\text{Þ}}(Y)$ is $Z(G)$, or $\overline{\text{Þ}}(X) = \overline{\text{Þ}}(Y)$, or $\overline{\text{Þ}}(X)$ and $\overline{\text{Þ}}(Y)$ are adjacent vertices of $\mathcal{KG}(\Delta)$.

It is worth noting that the assignment $\Delta \mapsto \mathcal{G}(\Delta)$ does not appear to be part of a description of a meaningful functor from the category of connected graphs to that of groups, in that a graph morphisms $\varphi\colon \Delta_1 \to \Delta$ (i.e., an adjacency preserving mapping $V(\Delta_1) \to V(\Delta)$) of graphs does not usually induce, at least in the obvious sense, a group homomorphism $\mathcal{G}(\Delta_1) \to \mathcal{G}(\Delta)$. It does when φ preserves distances, so we have a functor from the category of connected graphs and distance-preserving mappings to that of groups.[1] We shall make use of the fact that this functor preserves embeddings, and state this property in the form that we need, as part (i) of the next lemma. If Δ_1 is a connected subgraph of the graph Δ, we say that it is *isometrically embedded* in Δ iff any two vertices of Δ_1 have the same distances in Δ and in Δ_1. This is certainly the case when Δ is a tree, but also when Δ_1 is a subpath of a path of minimal length among those joining two given vertices in Δ.

Lemma 2.2. *Let Δ be a connected graph and G a group.*

(i) *If Δ_1 is an isometrically embedded, connected subgraph of Δ, then the inclusion map $V(\Delta_1) \hookrightarrow V(\Delta)$ extends to a group monomorphism $\mathcal{G}(\Delta_1) \rightarrowtail \mathcal{G}(\Delta)$.*
(ii) *If H is a subgroup of G, then the inclusion map $H \hookrightarrow G$ induces a graph embedding $\mathcal{K}(H) \to \mathcal{K}(G)$. This embedding is an isomorphism if and only if $G = HZ(G)$.*

Proof. If $\iota\colon V(\Delta_1) \hookrightarrow V(\Delta)$ preserves distances, that is, $d_\Delta(a, b) = d_{\Delta_1}(a, b)$ for all $a, b \in V(\Delta_1)$, then ι preserves the relators in the presentations defining $\mathcal{G}(\Delta_1)$ and $\mathcal{G}(\Delta)$; hence it induces a homomorphism $\mathcal{G}(\Delta_1) \to \mathcal{G}(\Delta)$, which is easily seen to be monic.

Now we prove (ii). Let T be a right transversal of $Z(H)$ in H. Then the assignments $gZ(H) \mapsto gZ(G)$, where g ranges over the elements of $T \smallsetminus Z(H)$, define an injective mapping from $V(\mathcal{K}(H))$ to $V(\mathcal{K}(G))$ which clearly is a graph embedding $\mathcal{K}(H) \to \mathcal{K}(G)$, as required. This mapping is surjective (that is, an isomorphism) if and only if $G = TZ(G)$, which in turn is equivalent to $G = HZ(G)$, as this latter equality implies $Z(H) = H \cap Z(G)$. □

We have already seen, in the introduction, that there are (finite, connected) graphs Δ_1 such that $\mathcal{G}(\Delta_1)$ is not abelian; but any such Δ_1 is certainly embedded (not isometrically) in a complete graph K, and of course $\mathcal{G}(K)$ is abelian. This shows how things can go wrong in (i) if the hypothesis of isometric embedding is removed. As

[1] As a matter of fact, the construction discussed in this paper could be generalised by starting from an arbitrary (pseudo)metric space rather than from a connected graph. This construction provides a functor from (pseudo)metric spaces to groups.

regards (ii), note that the embedding $\mathcal{K}(H) \to \mathcal{K}(G)$ is usually not isometric and is not always uniquely determined by H: it depends on the choice of T unless $Z(H) = H \cap Z(G)$.

Once again, assume that Δ is a connected graph. We show that, in order to discuss $\mathcal{KG}(\Delta)$, we can replace Δ with a certain subgraph (a retract, in fact) of its. Indeed, for all $a \in V(\Delta)$ let $N^+(a) = \{x \in V(\Delta) \mid d_\Delta(a,x) \leqslant 1\}$. It is easily seen that, for all $a, b \in V(\Delta)$, the following three conditions are equivalent:

- $N^+(a) = N^+(b)$;
- $d_\Delta(a,b) \leqslant 1$, and $d_\Delta(a,x) = d_\Delta(b,x)$ for all $x \in V(\Delta) \smallsetminus \{a,b\}$;
- $\bar{a} = \bar{b}$.

Define an equivalence relation ρ in $V(\Delta)$ by letting $a \;\rho\; b$ if and only if these conditions hold. Fix a complete set R of representatives of the ρ-equivalence classes. The subgraph of Δ induced by R is clearly isomorphic to the graph obtained by contracting each ρ-equivalence class in one single vertex; therefore, up to isomorphism, this subgraph of Δ is independent of the choice of R; we call it Δ_ρ. The following is immediately checked.

Lemma 2.3. *In the notation just established, Δ_ρ is connected and isometrically embedded in Δ; moreover,* $\mathrm{diam}(\Delta_\rho) = \mathrm{diam}(\Delta)$ *unless Δ is complete. Let G_ρ be the image of the embedding $\mathcal{G}(\Delta_\rho) \rightarrowtail G := \mathcal{G}(\Delta)$ described in Lemma 2.2 (i). Then $G = G_\rho Z(G)$ and so $\mathcal{KG}(\Delta_\rho) \simeq \mathcal{KG}(\Delta)$.*

Proof. For all $v \in V(\Delta)$ let v^* be the vertex of Δ_ρ which is ρ-equivalent to v (so $v^* = v$ if $v \in V(\Delta_\rho)$). For all $a, b \in V(\Delta)$, if P is a path in Δ joining a and b, then the vertices in $\{v^* \mid v \in V(P)\}$ form a path in Δ_ρ joining a^* and b^* of length at most that of P, therefore $d_\Delta(a,b) = d_{\Delta_\rho}(a^*, b^*)$ unless $a^* = b^*$. This shows that Δ_ρ is connected and isometrically embedded in Δ, and $\mathrm{diam}(\Delta_\rho) = \mathrm{diam}(\Delta)$ if Δ_ρ is not complete. Now the lemma follows from Lemma 2.2 and the fact that $v \in v^* Z(G)$ for all $v \in V(\Delta)$. □

The reduction provided by Lemma 2.3 is useful in settling the case of graphs of diameter 2. If Σ is any finite, connected graph and $k = |V(\Sigma)|$, we define the *remoteness* matrix of Σ as the $k \times k$ matrix (a_{vw}) over the field $\mathbb{F}_2$ of order 2, with rows and columns labelled by the vertices of Σ (with respect to a fixed linear order), where each entry a_{vw} is 0 or 1 according to whether $d_\Delta(v,w) \leqslant 1$ or $d_\Delta(v,w) > 1$.

Proposition 2.4. *Let Δ be a connected graph of diameter 2, and let Δ_ρ the subgraph of Δ defined as in the previous paragraphs. Let $G = \mathcal{G}(\Delta)$ and $\Gamma = \mathcal{KG}(\Delta)$. Then either Γ is connected of diameter 2, or $|G/Z(G)| = 4$ and Γ is the graph with three vertices and no edges. The latter case occurs if and only if Δ_ρ is finite and its remoteness matrix has rank 2.*

Proof. Let $Z = Z(G)$. We have $|G'| = 2$, because $\mathrm{diam}(\Delta) = 2$. Then $|G/C_G(g)| = 2$ for all $g \in G \smallsetminus Z$. If $|G/Z| > 4$, this shows that for all $g, h \in G$ we have $Z < C_G(g) \cap C_G(h)$; then, if $g, h \notin Z(G)$ then $\bar{g} \sim \bar{c} \sim \bar{h}$ for some $c \in (C_G(g) \cap C_G(h)) \smallsetminus Z(G)$. It follows that $\mathcal{KG}(\Delta)$ is connected of diameter 2 in this case: note that, as remarked in the introduction, $\mathcal{KG}(\Delta)$ certainly has at least a pair of non-adjacent vertices. In the remaining case G/Z is non-cyclic of order 4, and it is clear that the three nontrivial cosets of Z in G are isolated vertices in $\mathcal{KG}(\Delta)$. Therefore $\mathcal{KG}(\Delta)$ has three vertices and no edges.

We are only left to prove that $|G/Z| = 4$ is equivalent to the stated condition. The cosets $\bar{a} = aZ$, where a ranges over the vertices of Δ_ρ, are pairwise distinct and generate G/Z, by Lemma 2.3. It follows that G/Z is finite if and only if Δ_ρ is finite. We may therefore assume that Δ_ρ is finite. With reference to the notation of Lemma 2.3, let $H = G_\rho$. Then H is finite and $G = HZ$, so that $H' = G'$ and $H/Z(H) \simeq G/Z$. The mapping $(gH', hH') \in H^{\mathrm{ab}} \times H^{\mathrm{ab}} \mapsto [g,h] \in H'$ is bilinear, and since $|H'| = 2$, it can be regarded as a symmetric bilinear form over $\mathbb{F}_2$. With respect to the basis $\{aH' \mid a \in V(\Delta_\rho)\}$, this form is represented by the remoteness matrix A of Δ_ρ, and its radical is $Z(H)/H'$. Therefore $\mathrm{rk}(G/Z) = \mathrm{rk}(H/Z(H)) = \mathrm{rk}\, A$. Now we see that $|G/Z| = 4$ if and only if $\mathrm{rk}\, A = 2$, as required. □

3. Paths

In this section we discuss the case in which the graph Δ is a path. This is the case of the original construction in [2], where it is proved that, unless $\mathrm{diam}(\Delta) < 2$, the resulting graph is connected. Our argument does not use this information.

The following notation will be in use throughout this section. P is a fixed path of length an integer $n > 2$ and endvertices a, b. We let $G = \mathcal{G}(P)$, $Z = Z(G)$, and $\Psi = \mathcal{K}(G) = \mathcal{KG}(P)$. Also, for all $i \in I_n := \{0, 1, \dots, n\}$, we let a_i be the vertex of P at distance i from a, thus $a_0 = a$ and $a_n = b$. As a first easy observation, we note that the vertices of P, considered as elements of G, are linearly independent modulo Z.

Lemma 3.1. *Let $X \subseteq V(P)$. Then $\text{Þ}(X) \in Z$ if and only if $X = \varnothing$.*

Proof. Assume $\text{Þ}(X) \in Z$. Let $i \in I_n$ be such that $a_i \in X$. Then a_i is the only vertex of P at distance i from a. Applying Lemma 2.1 to X and $\{a\}$ yields $i \in \{0, 1\}$. On the other hand, a_i is the only vertex of P at distance $n - i$ from b, hence, by the same lemma applied to X and $\{b\}$, we have $i \in \{n, n-1\}$. As $n > 2$ by hypothesis, we obtain a contradiction. □

Lemma 3.1 shows that the mapping $X \mapsto \overline{\mathrm{P}}(X)$ is a bijection from the set of all nonempty subsets of $V(P)$ to $V(\Psi)$. A very special case of Lemma 3.1 is the fact that $v \in V(P) \mapsto \bar{v} \in V(\Psi)$ is a graph embedding of P into Ψ (this would be false without the hypothesis $n > 2$: if P had length 2 then its 'medium' vertex would lie in Z and so it would not correspond to a vertex in Ψ).

Making use of the former remark, we fix some more notation. Let $x \in V(\Psi)$. We let $\operatorname{supp}(x)$ be the subset J of I_n such that $x = \overline{\mathrm{P}}(\{a_i \mid i \in J\})$, further let $m(x) = \min(\operatorname{supp}(x))$, $M(x) = \max(\operatorname{supp}(x))$, $\lambda(x) = M(x) - m(x)$ and $\mu(x) = (M(x) + m(x))/2$. If $\lambda(x)$ is even, we let $\zeta(x) = \bar{a}_{\mu(x)}$, otherwise $\zeta(x) = \bar{a}_{\mu(x)-1/2}\,\bar{a}_{\mu(x)+1/2}$.

Lemma 3.2. *Let x, y be two connected vertices of Ψ.*

(i) *If $x \sim y$ then one of the following holds:*
  (a) *$\mu(x) = \mu(y)$;*
  (b) *$|\mu(x) - \mu(y)| = 1/2$ and $\{\lambda(x), \lambda(y)\} = \{0, 1\}$;*
  (c) *$|\mu(x) - \mu(y)| = 1$ and $\lambda(x) = \lambda(y) = 0$ (i.e., $x = \bar{u}$ and $y = \bar{v}$ for some adjacent vertices u, v of P).*

(ii) *$d_\Psi(x, y) \geqslant |\mu(x) - \mu(y)|$; equality holds if and only if either $x = y$ or $x = \bar{u}$ and $y = \bar{v}$ for some vertices u, v of P.*

Proof. Let $I = \operatorname{supp}(x)$ and $J = \operatorname{supp}(y)$. Note that $|i - j| = d_\Delta(a_i, a_j)$ for all $(i, j) \in I \times J$ and let $\ell = \max\{|i - j| \mid (i, j) \in I \times J\}$. In order to prove (i), assume $x \sim y$, hence $x \neq y$ and $\ell > 0$. If $\ell > 1$ we deduce from Lemma 2.1 that there are at least two different pairs $(i, j) \in I \times J$ such that $|i - j| = \ell$. The only possible such pairs are $(m(x), M(y))$ and $(M(x), m(y))$; also, at least one of I and J has more than one element. Thus we obtain $M(y) - m(x) = \ell = M(x) - m(y)$ and hence $\mu(x) = \mu(y)$. Therefore we may assume $\ell = 1$. Then $|\mu(x) - \mu(y)| \leqslant 1$ and $\lambda(x), \lambda(y) \leqslant 2$. Without loss of generality, assume $\lambda(x) \geqslant \lambda(y)$. If $\lambda(x) = 2$ then $I = \{i - 1, i + 1\}$ or $I = \{i - 1, i, i + 1\}$ where $i = \mu(x)$, and $y = \bar{a}_i$. Then $\mu(x) = \mu(y)$ in this case as well. If $\lambda(x) = 1$ then $I = \{i, i + 1\}$, for some $i \in I_n$, and $x \neq y$ implies that J is either $\{i\}$ or $\{i + 1\}$. If $\lambda(x) = 0$ then $I = \{\mu(x)\}$ and $J = \{\mu(y)\}$, now $x \sim y$ gives $|\mu(x) - \mu(y)| = 1$. This proves (i). Part (ii) follows easily from (i) and from the graph embedding of P in Ψ noted earlier. □

An immediate consequence of the last claim in Lemma 3.2 is:

Corollary 3.3. *P is isometrically embedded in Ψ.*

Lemma 3.4. *Let $x \in V(\Psi)$ and assume $\lambda(x) > 1$. Then there exists $y \in V(\Psi)$ such that $x \sim y$, $m(x) < m(y) \leqslant M(y) < M(x)$ and $\mu(y) = \mu(x)$.*

Proof. Let $\lambda = \lambda(x)$, $m = m(x)$ and $M = M(x)$. Also consider two subgroups of G, namely $H = \langle a_i \mid m \leqslant i < M \rangle Z$ and $K = H\langle a_M \rangle$; note that $x \in K/Z$. Fix $g \in x$. We know that $K' = [a_m, K] = \langle [a_m, a_i] \mid m < i \leqslant M \rangle$ has rank $\lambda - 1$, by the definition of $\mathcal{G}(P)$, hence $|K/C_K(g)| \leqslant 2^{\lambda-1}$. On the other hand H/Z has rank λ, by Lemma 3.1. It follows that $Z < C_H(g)$. Let y be a nontrivial element of $C_H(g)/Z$. Then $M(y) < M$ because $y \in H/Z$. Also $\mu(y) = \mu(x)$ by Lemma 3.2 (i) and, as a consequence, $m(y) > m$. □

In the next proofs we will use the fact that, for all $x, y \in V(\Psi)$, the condition $\mu(x) = \mu(y)$ is equivalent to $\zeta(x) = \zeta(y)$ and implies $\lambda(x) \equiv_2 \lambda(y)$. For the meaning of the θ symbol, see Section 2.

Lemma 3.5. *Every $x \in V(\Psi)$ is connected to $\zeta(x)$ in Ψ, and $d_\Psi(x, \zeta(x)) \leqslant \lceil \lambda(x)/2 \rceil - \theta_{\lambda(x) \notin \{0,2\}}$.*

Proof. The proof is by induction on $\lambda = \lambda(x)$. A direct check shows that x and $\zeta(x)$ are connected and $d_\Psi(x, \zeta(x)) = \lceil \lambda/2 \rceil - \theta_{\lambda \notin \{0,2\}}$ if $\lambda \leqslant 4$, because if $0 < \lambda \leqslant 4$ then $x = \bar{a}_i \bar{c} \bar{a}_{i+\lambda}$ for some $i \in I_n$ and $c \in \langle a_{i+1}, \dots, a_{i+\lambda-1} \rangle \leqslant C_G(\zeta(x))$, and also $[a_i a_{i+\lambda}, \zeta(x)] = 1$. Now assume $\lambda > 4$. By Lemma 3.4 and the observation following it, x is adjacent to a vertex y of Ψ such that $\zeta(y) = \zeta(x)$ and $\lambda(y) = \lambda - 2k$ for some positive integer k. Since $\lambda > 4$ we have $\lceil \lambda(y)/2 \rceil - \theta_{\lambda(y) \notin \{0,2\}} = \lceil \lambda/2 \rceil - k - \theta_{\lambda(y) \notin \{0,2\}} \leqslant \lceil \lambda/2 \rceil - 2$. Therefore, by induction hypothesis, we have $d_\Psi(x, \zeta(x)) \leqslant 1 + d_\Psi(y, \zeta(y)) \leqslant 1 + \lceil \lambda(y)/2 \rceil - \theta_{\lambda(y) \notin \{0,2\}} \leqslant \lceil \lambda/2 \rceil - 1 = \lceil \lambda/2 \rceil - \theta_{\lambda \notin \{0,2\}}$, as required. □

Lemma 3.6. *Let $x, y \in V(\Psi)$. Then $\zeta(x)$ and $\zeta(y)$ are connected and:*

(i) *either $\zeta(x) = \zeta(y)$ or $d_\Psi(\zeta(x), \zeta(y)) = \lceil |\mu(x) - \mu(y)| \rceil + \theta_{\lambda(x)\lambda(y) \equiv_2 1}$;*

(ii) *if $\mu(x) > \mu(y)$, then $d_\Psi(\zeta(x), \zeta(y)) = M(x) - m(y) - \lfloor \lambda(x)/2 \rfloor - \lfloor \lambda(y)/2 \rfloor$.*

Proof. If $\lambda(x)$ and $\lambda(y)$ are even, (i) follows from Corollary 3.3. If, say, $\lambda(x)$ is even and $\lambda(y)$ is odd, we have $\zeta(y) = \bar{a}_{i-1}\bar{a}_i$ where $i = \mu(y) + 1/2$. If $\mu(y) \leqslant \mu(x)$, then $d_\Psi(\zeta(x), \bar{a}_i) = \mu(x) - i$. Since $\bar{a}_i \sim \zeta(y)$ we obtain $d_\Psi(\zeta(x), \zeta(y)) \leqslant \mu(x) - i + 1 = \mu(x) - \mu(y) + 1/2 = \lceil \mu(x) - \mu(y) \rceil$. Now Lemma 3.2 (ii) gives the equality in (i), since $\mu(\zeta(x)) = \mu(x)$ and $\mu(\zeta(y)) = \mu(y)$. The remaining cases where $\lambda(x) \not\equiv_2 \lambda(y)$ are treated similarly. Finally, assume that both $\lambda(x)$ and $\lambda(y)$ are odd. Without loss of generality, again assume $\mu(y) \leqslant \mu(x)$ and let $i = \mu(y) + 1/2$. If $\zeta(x) \neq \zeta(y)$ then $\mu(y) < \mu(x)$ and $i < \mu(x)$. Then $d_\Psi(\zeta(x), \bar{a}_i) = \mu(x) - \mu(y)$ by the previous case and (i) follows from Lemma 3.2 (ii) because $\lambda(x) \neq 0$.

(ii) is an alternative formulation of (i). Indeed, $\mu(x) - \mu(y) = (M(x) - \lambda(x)/2) - (m(y) + \lambda(y)/2)$, hence, on assuming $\mu(x) > \mu(y)$, the second equation in (i) can be rewritten as

$$d_\Psi(\zeta(x), \zeta(y)) = \left\lceil M(x) - m(y) - \frac{\lambda(x) + \lambda(y)}{2} \right\rceil + \theta_{\lambda(x)\lambda(y) \equiv_2 1} = M(x) - m(y) - \left\lfloor \frac{\lambda(x) + \lambda(y)}{2} \right\rfloor + \theta_{\lambda(x)\lambda(y) \equiv_2 1}$$

and clearly $\lfloor \lambda(x)/2 \rfloor + \lfloor \lambda(y)/2 \rfloor = \lfloor (\lambda(x) + \lambda(y))/2 \rfloor - \theta_{\lambda(x)\lambda(y) \equiv_2 1}$. □

Lemma 3.7. *Ψ is connected and for all $x, y \in V(\Psi)$ such that $\mu(x) \geqslant \mu(y)$, we have*

$$d_\Psi(x,y) \leqslant M(x) - m(y) - (\theta_{2<\lambda(x)\equiv_2 0} + \theta_{2<\lambda(y)\equiv_2 0}) \leqslant M(x) - m(y).$$

Proof. Ψ is connected by Lemmas 3.5 and 3.6, or by [2]. Let $x, y \in V(\Psi)$ and assume $\mu(x) \geqslant \mu(y)$. Then $d_\Psi(x,y) \leqslant d_\Psi(x,\zeta(x)) + d_\Psi(\zeta(x),\zeta(y)) + d_\Psi(\zeta(y),y)$. Even in the case $\mu(x) = \mu(y)$ it holds $M(x) - m(y) - \lfloor \lambda(x)/2 \rfloor - \lfloor \lambda(y)/2 \rfloor \geqslant 0 = d_\Psi(\zeta(x),\zeta(y))$, hence Lemmas 3.5 and 3.6 give:

$$d_\Psi(x,y) \leqslant M(x) - m(y) + \left(\left\lceil \frac{\lambda(x)}{2} \right\rceil - \left\lfloor \frac{\lambda(x)}{2} \right\rfloor - \theta_{\lambda(x)\notin\{0,2\}} \right) + \left(\left\lceil \frac{\lambda(y)}{2} \right\rceil - \left\lfloor \frac{\lambda(y)}{2} \right\rfloor - \theta_{\lambda(y)\notin\{0,2\}} \right)$$

and the result follows from the fact that $\lceil \lambda/2 \rceil - \lfloor \lambda/2 \rfloor - \theta_{\lambda\notin\{0,2\}} = \theta_{\lambda\equiv_2 1} - \theta_{\lambda\notin\{0,2\}} = -\theta_{2<\lambda\equiv_2 0} \leqslant 0$ for all $\lambda \in \mathbb{N}$. $\square$

We have essentially proved the assertion made in the main theorem in the case of paths.

Theorem 3.8. *Let P be a path of (finite) length $n > 2$. Then $\mathcal{KG}(P)$ is connected and has diameter n.*

Proof. Lemma 3.7 shows that $\mathcal{KG}(P)$ is connected and $\operatorname{diam}(\mathcal{KG}(P)) \leqslant n$, while $\operatorname{diam}(\mathcal{KG}(P)) \geqslant n$ by Corollary 3.3. $\square$

4. Proof of the main theorem

Let Δ be a connected graph of finite diameter $n > 2$. Let $a, b \in V(\Delta)$ such that $d_\Delta(a,b) = n$ and fix a path P of length n joining a and b. Let $H = \mathcal{G}(P)$, $G = \mathcal{G}(\Delta)$ and $\Gamma = \mathcal{K}(G)$. Since P is isometrically embedded in Δ, Lemma 2.2 shows that the inclusion $V(P) \hookrightarrow V(\Delta)$ induces a monomorphism $\iota \colon H \rightarrowtail G$, and, furthermore, a graph embedding σ of $\Psi := \mathcal{K}(H) = \mathcal{KG}(P)$ into Γ. We use ι to identify H with the subgroup $\langle V(P) \rangle$ of G. Now, since $\operatorname{diam}(P) = n = \operatorname{diam}(\Delta)$ we have $H' = G'$, and Lemma 3.1 implies that $H/Z(H)$ has rank $n+1$ and $Z(H) = G' = H \cap Z(G)$; then σ defines a bijection from $V(\Psi)$ to $HZ(G)/Z(G) \smallsetminus \{Z(G)\}$. If $g \in G \smallsetminus Z(G)$, then $\operatorname{br}_G(g) \leqslant \operatorname{rk}(G') = n-1$, hence $|H/C_H(g)| \leqslant 2^{n-1} < |H/Z(H)|$ and $C_H(g) \nleqslant Z(G)$. This means that $\bar{g}$ is adjacent (in Γ) to some vertex of the image of Ψ under σ. Since the latter graph is connected (by Theorem 3.8) we have proved:

Lemma 4.1. *Γ is connected.*

Our next aim will be that of showing that the diameter of Γ is at most n.

Lemma 4.2. *In the notation introduced in this section, let $r = \operatorname{rk}(G/Z(G))$. Then*
(i) *$r \geqslant n+1$, and $r = n+1$ if and only if $G = HZ(G)$;*
(ii) *if $r > 2(n-1)$ then $\operatorname{diam}(\Gamma) = 2$.*

Proof. (i) is an immediate consequence of Lemma 3.1, which yields $\operatorname{rk}(HZ(G)/Z(G)) = n+1$. Next, let $g, h \in G$. Since $|G'| = 2^{n-1}$ we have $\operatorname{br}_G(g), \operatorname{br}_G(h) \leqslant n-1$ and so $\operatorname{rk}\left(G/C_G(\{g,h\})\right) \leqslant 2(n-1)$. Now assume $r > 2(n-1)$. Then $Z(G) < C_G(\{g,h\})$ and there exists $x \in C_G(\{g,h\}) \smallsetminus Z(G)$; as a consequence, in Γ, we have $\bar{g} \sim \bar{x} \sim \bar{h}$ and so $d_\Gamma(\bar{g},\bar{h}) \leqslant 2$. This completes the proof. $\square$

Lemma 4.3. *Still in same notation, if $n \in \{3,4\}$ then $\operatorname{diam}(\Gamma) \leqslant n$.*

Proof. We may assume $\operatorname{diam}(\Gamma) > 2$. If $G = HZ(G)$ then $\Gamma \simeq \mathcal{K}(H) = \Psi$ by Lemma 2.2, hence $\operatorname{diam}(\Gamma) = n$ by Theorem 3.8. So we may assume $HZ(G) < G$. In view of Lemma 4.2 this leaves us with just one possibility: $n = 4$ and $\operatorname{rk}(G/Z(G)) = 6$, which also yields $|G/HZ(G)| = 2$. For all $g \in G$ we have $\operatorname{br}_G(g) \leqslant \operatorname{rk}(G') = 3$. The vertex a_2 of P at distance 2 from a (and b) satisfies $[H, a_2] = \langle [a, a_2] \rangle$ and hence $\operatorname{br}_G(a_2) \leqslant \operatorname{br}_{HZ(G)}(a_2) + 1 \leqslant 2$. As a consequence, for all $g \in G \smallsetminus Z(G)$ we have $Z(G) < C_G(\{g, a_2\})$ and it follows that $d_\Gamma(\bar{g}, \bar{a}_2) \leqslant 2$. Therefore $\operatorname{diam}(\Gamma) \leqslant 4$. $\square$

Proposition 4.4. *Let Δ be a connected graph of finite diameter $n > 2$. Then $\operatorname{diam}(\mathcal{KG}(\Delta)) \leqslant n$.*

Proof. In view of Lemma 4.3 we may assume $n > 4$. We still use the notation introduced at the beginning of this section. With reference to vertices v of Ψ, we define $m(v)$, $M(v)$ and $\lambda(v)$ as in the previous section, and we also extend this notation to vertices of Γ in the image of σ by letting $m(v^\sigma) = m(v)$, $M(v^\sigma) = M(v)$ and $\lambda(v^\sigma) = \lambda(v)$ for all $v \in V(\Psi)$. Let $H_a = Z(H)\langle V(P) \smallsetminus \{a\} \rangle$ and $H_b = Z(H)\langle V(P) \smallsetminus \{b\} \rangle$; both are maximal subgroups of H. Since $|G'| = 2^{n-1}$, for all $g \in G \smallsetminus Z(G)$ we have $|H/C_H(g)| \leqslant 2^{n-1}$; it follows that $Z(H) < C_{H_a}(g)$ and $Z(H) < C_{H_b}(g)$.

Arguing by contradiction, assume $\operatorname{diam}(\mathcal{KG}(\Delta)) > n$ and fix $r, s \in G \smallsetminus Z(G)$ such that $d_\Gamma(\bar{r}, \bar{s}) > n$. Let X (resp. Y) be the set of nontrivial cosets in $C_H(r)Z(G)/Z(G)$ (resp. $C_H(s)Z(G)/Z(G)$). Then $d_\Gamma(x,y) > n-2$ for all $x \in X$ and $y \in Y$. Keeping in mind that $Z(H) = H \cap Z(G)$, the concluding remark in the previous paragraph shows that there exist $x_a, x_b \in X$ and $y_a, y_b \in Y$ such that $m(x_a), m(y_a) > 0$ and $M(x_b), M(y_b) < n$. If $\mu(x_b) \geqslant \mu(y_a)$ then Lemma 3.7 yields (with a slight abuse of notation) $d_\Gamma(x_b, y_a) \leqslant d_\Psi(x_b^{\sigma^{-1}}, y_a^{\sigma^{-1}}) \leqslant M(x_b) - m(y_a) \leqslant n-2$, a contradiction. Therefore

$$\mu(x_b) < \mu(y_a) \qquad \text{and, similarly,} \qquad \mu(y_b) < \mu(x_a);$$

furthermore, still by looking at $d_\Gamma(x_b, y_a)$ and $d_\Gamma(y_b, x_a)$ and using Lemma 3.7 again, we also see that $M(x_a), M(y_a) \in \{n, n-1\}$ and $m(x_b), m(y_b) \in \{0, 1\}$. Next, since also $d_\Gamma(x_a, y_a)$ and $d_\Gamma(x_b, y_b)$ are greater than $n-2$, the same lemma makes sure that one of $\big(M(x_a), m(y_a)\big)$ and $\big(M(y_a), m(x_a)\big)$ is $(n,1)$, while one of $\big(M(x_b), m(y_b)\big)$ and $\big(M(y_b), m(x_b)\big)$ is $(n-1,0)$, and also that each of $\lambda(x_a)$, $\lambda(x_b)$, $\lambda(y_a)$ and $\lambda(y_b)$ is either 0 or 2 or an odd integer.

As a first case, assume $n = M(x_a) = M(y_a)$. Since one of $m(x_a)$ and $m(y_a)$ is 1, one of $\lambda(x_a)$ and $\lambda(y_a)$ is $n-1$. As $n > 4$, this implies that $n-1$ is odd, so n is even. If $0 = m(x_b) = m(y_b)$, then $n = M(x_a x_b) = M(y_a y_b)$ and $0 = m(x_a x_b) = m(y_a y_b)$, hence $\lambda(x_a x_b) = \lambda(y_a y_b) = n$ is even and greater than 2, and so $d_\Gamma(x_a x_b, y_a y_b) \leqslant n-2$ by Lemma 3.7, a contradiction. Otherwise, without loss of generality, $m(x_b) = 0$ and $m(y_b) = 1$, but in this case the above constraints give $M(y_b) = n-1$ and $2 < n-2 = \lambda(y_b) \equiv_2 0$, again a contradiction.

Therefore $M(x_a)$ and $M(y_a)$ are not both n, say $M(x_a) = n$ and $M(y_a) = n-1$. It follows: $m(y_a) = 1$ and $2 < n-2 = \lambda(y_a)$, hence n is odd. As $n-1 \leqslant d_\Gamma(x_b, y_a) \leqslant M(y_a) - m(x_b)$, we also have $m(x_b) = 0$. Now $M(x_b) = \lambda(x_b)$ is either odd or less than 3, then $M(x_b) < n-1$. As a consequence, $M(y_b) = n-1$ and so $M(y_a y_b) < n-1$. This leads to the final contradiction $d_\Gamma(x_b, y_a y_b) < n-1$. Now the proof is complete. □

To complete the proof of the main theorem we only need to deal with the case of connected graphs of infinite diameter. The following lemma settles the special case of those graphs which, even after the reduction suggested by Lemma 2.3, are not locally finite.

Lemma 4.5. *Assume that Δ is a connected graph with a vertex a of infinite degree such that $d_\Delta(a,x) > 1$ for at least one $x \in V(\Delta)$. Also assume $\text{Þ}(Y) \notin Z(\mathcal{G}(\Delta))$ for all $Y \subseteq V(\Delta)$ such that $|Y| = 2$. Then $\mathcal{KG}(\Delta)$ is connected (more precisely, $d(\bar{a}, v) \leqslant 2$ for all vertices v of $\mathcal{KG}(\Delta)$, hence $\operatorname{diam}(\mathcal{KG}(\Delta)) \leqslant 4$).*

Proof. Let $v \in V(\mathcal{KG}(\Delta))$; then $v = \overline{\text{Þ}}(X)$ for some finite subset X of $V(\Delta)$. By hypothesis $N := N^+(a) = \{x \in V(\Delta) \mid d_\Delta(a,x) \leqslant 1\}$ is infinite, while $\{d_\Delta(x,u) \mid u \in N \text{ and } x \in X\}$ is finite because X is finite. Then there exist $b, c \in N$ such that $b \neq c$ and $d_\Delta(b,x) = d_\Delta(c,x)$ for all $x \in X$. Let $Y = \{b,c\}$, and note that $\overline{\text{Þ}}(Y)$ and $\bar{a}$ are vertices of $\mathcal{KG}(\Delta)$ by hypothesis. Lemma 2.1 shows that, in $\mathcal{KG}(\Delta)$, we have $v = \overline{\text{Þ}}(X) \sim \overline{\text{Þ}}(Y) \sim \bar{a}$. Thus v is connected to $\bar{a}$ and $d_{\mathcal{KG}(\Delta)}(v, \bar{a}) \leqslant 2$. The result follows. □

It is well-known ([1, Proposition 8.2.1]) that every infinite, locally finite connected graph Δ has a ray, i.e., a connected subgraph in which one vertex has degree 1 and all remaining vertices have degree 2; the vertex of degree 1 is called the endvertex of the ray. For each $i \in \mathbb{N}$ a ray has exactly one vertex at distance i from its endvertex. Let's say that a vertex a and a ray R with endvertex r_0 of a connected graph Δ are *close* iff R has a vertex r such that $d_\Delta(a,r) \leqslant d_R(r_0,r)$.

Lemma 4.6. *Let Δ be an infinite, locally finite connected graph, and let X be a finite subset of $V(\Delta)$. Then Δ has an isometrically embedded ray which is close to all vertices in X.*

Proof. Let R be a ray in Δ constructed as in the proof of [1, Proposition 8.2.1], and let r_0 be its endvertex. The construction actually shows $d_R(r_0,r) = d_\Delta(r_0,r)$ for all $r \in V(R)$, and it easily follows that R is isometrically embedded in Δ. Now, among all isometrically embedded rays in Δ choose one, call it R, such that the set C of all vertices in X close to R has maximal cardinality. Arguing by contradiction, assume that there exist some $a \in X \smallsetminus C$. Let r_0 be the endvertex of R and, more generally, for all $i \in \mathbb{N}$ let r_i be the vertex of R such that $d_R(r_0, r_i) = i$. For all $v \in V(\Delta)$ and $i \in \mathbb{N}$, let $\underline{v}_i = i - d_\Delta(v, r_i)$. The sequence $(\underline{v}_i)_{i\in\mathbb{N}}$ is increasing and assumes nonnegative values iff v is close to X. Therefore there exists $j \in \mathbb{N}$ such that $\underline{c}_j \geqslant 0$ for all $c \in C$ and $\underline{a}_j = \underline{a}_{j+i}$ for all $i \in \mathbb{N}$, hence $d_\Delta(c, r_j) \leqslant j$ for all $c \in C$ and $d_\Delta(a, r_{j+i}) = k + i$ for all $i \in \mathbb{N}$, where $k = d_\Delta(a, r_j)$. This latter property shows the following: if P is a path of length k joining a to r_j, then substituting P for the path joining r_0 and r_j in R produces a new ray S with endvertex a which is isometrically embedded in Δ; more explicitly, for all $i \in \mathbb{N}$, the vertex of S at distance i from a is the vertex of P with the same property if $i \leqslant k$ and r_{j+i-k} otherwise. For all $c \in C$ we have $d_\Delta(c, r_j) \leqslant j < k = d_\Delta(a, r_j)$, hence c is close to S, but S is also trivially close to a. This contradicts the maximality of $|C|$, and this contradiction gives $C = X$, thus proving the lemma. □

Lemma 4.7. *Let Δ be a connected graph of infinite diameter. Then $\mathcal{KG}(\Delta)$ is connected.*

Proof. As usual, let $G = \mathcal{G}(\Delta)$, $Z = Z(G)$ and $\Gamma = \mathcal{KG}(\Delta)$. Since $\operatorname{diam}(\Delta)$ is infinite $a \notin Z$ for all $a \in V(\Delta)$. At the expense of replacing Δ with the subgraph Δ_ρ referred to in Lemma 2.3, we may assume $\text{Þ}(Y) \notin Z$ for all subsets Y of Δ such that $|Y| = 2$. By Lemma 4.5 we may also assume that Δ is locally finite.

If $a, b \in V(\Delta)$, then $\bar{a}$ and $\bar{b}$ are connected in Γ, because the mapping $a \mapsto \bar{a}$ defines a graph embedding from Δ to Γ. Therefore, to complete the proof it will be enough to show that every vertex of Γ is connected to $\bar{a}$ for some $a \in V(\Delta)$.

Let $v \in V(\Gamma)$. Then $v = \overline{\text{Þ}}(X)$ for some finite, nonempty subset X of $V(\Delta)$. By Lemma 4.6, Δ has an isometrically embedded ray R close to every vertex in X. For all $i \in \mathbb{N}$ let r_i be the vertex of R at distance i from the endvertex of R (which is therefore r_0). Since R is close to the vertices of X there exists $j \in \mathbb{N}$ such that $d_\Delta(x, r_j) \leqslant j$ for all $x \in X$; we may also assume $j > 2$. Let ℓ be the diameter of $X \cup \{r_i \mid j \geqslant i \in \mathbb{N}\}$ (as a metric subspace of $V(\Delta)$) and let $X_1 := X \cup \{r_i \mid j + \ell \geqslant i \in \mathbb{N}\}$. For all $x \in X$ and $i \in \mathbb{N}$ we have $d_\Delta(x, r_{j+i}) \leqslant d_\Delta(x, r_j) + i \leqslant j + i$. As R is isometrically embedded in Δ, it follows that the diameter of X_1 is $j + \ell$. Let H be the subgroup $\langle X_1 \rangle$ of G and $K = \langle r_i \mid j + \ell \geqslant i \in \mathbb{N} \rangle \leqslant H$; note that $v = hZ$ for some $h \in H$. Then $\operatorname{rk}\big(H/C_H(h)\big) \leqslant \operatorname{rk}(H') = j + \ell - 1$. On the other hand KZ/Z has rank $j + \ell + 1$ by Lemmas 2.2 and 3.1. It follows that $C_K(h) \nleqslant Z$ and so, as a consequence of Theorem 3.8, the vertex v is connected to $\bar{r}_0$ in Γ. As observed in the previous paragraph, this is enough to draw the conclusion that Γ is connected. □

In view of Propositions 2.4 and 4.4 and Theorem 3.8 this result completes the proof of the main theorem.

5. Further remarks and examples

We collect here some examples of diameter computation and add some extra information supplementing the main result of this paper. Some of the proofs make use of the following simple (and quite possibly known) lemma.

Lemma 5.1. *Let p be a prime and G a p-group. Assume that $|G/Z(G)| > p^2$ and G has two elements u, v of breadth 1 such that one of the following holds:*

(i) $[u, v] \neq 1$;

(ii) $C_G(u) \neq C_G(v)$ *and* $[G, u] = [G, v] \leqslant Z(G)$.

Then $\mathcal{K}(G)$ is connected of diameter at most 3.

Proof. In either case $C_G(u) \neq C_G(v)$. Both $C_G(u)$ and $C_G(v)$ have index p in G, hence they are normal and $H := C_G(\{u, v\}) = C_G(u) \cap C_G(v) \lhd G$; also $G'G^p \leqslant H$ and G/H is elementary abelian of rank 2, hence $Z(G) < H$. In case (i) let $x = u$ and $y = v$, in case (ii) choose $x \in C_G(u) \smallsetminus C_G(v)$ and $y \in C_G(v) \smallsetminus C_G(u)$; in either case let $T = \{y, x, xy, xy^2, \dots, xy^{p-1}\}$. For all $g \in G$ there exists $t_g \in T$ such that $g \in H\langle t_g\rangle$.

Assume (i). Then $[H, T] = 1$ and so $t \in Z(H\langle t\rangle)$ for all $t \in T$. We claim that $Z(G) < C_{H\langle t\rangle}(g)$ for all $t \in T$ and $g \in G$. Indeed, assume that, for some choice of t and g, we have $Z(G) = C_{H\langle t\rangle}(g)$. Since $t_g \in Z(H\langle t_g\rangle)$ and $T \cap Z(G) = \varnothing$, we have $t_g \in C_G(g) \smallsetminus H\langle t\rangle$. But $|G/H\langle t\rangle| = p$, hence $C_G(g) = Z(G)\langle t_g\rangle$, so that $g \in Z(G)\langle t_g\rangle$. Then $C_G(g) \geqslant C_G(t_g) \geqslant H$, yielding the contradiction $H \leqslant Z(G)$. This establishes our claim. Now let $a, b \in G$. We have just proved that there exists $c \in C_{H\langle t_b\rangle}(a) \smallsetminus Z(G)$. As $c, b \in H\langle t_b\rangle$ and $t_b \in Z(H\langle t_b\rangle)$, we see that $[a, c] = 1 = [c, t_b] = [t_b, b]$ and the result follows in this case.

Now assume (ii) and, as we may, $[u, v] = 1$. Then $u, v \in Z(H)$. For all $t \in T$ we shall find an element $h(t) \in (H \cap Z(H\langle t\rangle)) \smallsetminus Z(G)$. If $t = y$ then we can let $h(t) = v$. Otherwise $t = xy^i$ for some integer i and (ii) gives $[v, x] = [u, y]^j$ for some integer j not divisible by p; choose $k \in \mathbb{Z}$ such that $jk \equiv_p -i$, then $[uv^k, t] = [u, y]^i[v, x]^k = 1$, so we can let $h(t) = uv^k$. For all $a, b \in G \smallsetminus Z(G)$ we have $[a, h(t_a)] = 1 = [h(t_a), h(t_b)] = [h(t_b), b]$. The lemma follows. □

In applying this lemma is can be useful to note that if $G = \mathcal{G}(\Delta)$ for a connected graph Δ and $u, v \in V(\Delta) \smallsetminus Z(G)$, then $\mathrm{br}_G(u) + 1$ is the eccentricity of u in Δ, and $\mathrm{br}_G(u) = \mathrm{br}_G(v)$ implies $[G, u] = [G, v]$. This follows from the fact that, for all $w \in V(\Delta)$, if $D_w = \{d_\Delta(w, z) \mid z \in V(\Delta)\} \smallsetminus \{0, 1\}$, that is the set of all integers greater than 1 and not exceeding the eccentricity of w, then a basis for $[G, w]$ is the set $\{c_i \mid i \in D_w\}$, where, for each $i \in D_w$, c_i is a commutator $[x, y]$ where x and y are vertices of Δ at distance i; recall that c_i depends only on i by definition of $\mathcal{G}(\Delta)$.

Remark 5.2. Let G be the class-2 nilpotent group with generators a, b, c subject to the extra relations $1 = a^2 = b^2 = c^2 = [b, c]$. Then $|G| = 2^5$ and $|G/Z(G)| = 2^3$, and both b and c have breadth 1 in G, but $\mathcal{K}(G)$ is not connected since the vertex $aZ(G)$ is isolated.

On the other hand, it can be worth noting that if G is a p-group with an element u of breadth 1 and $\mathcal{K}(G)$ is connected (or at least has no isolated vertices), then $\mathrm{diam}(\mathcal{K}(G)) \leqslant 4$. Indeed, every $g \in G \smallsetminus Z(G)$ must satisfy $Z(G) < C_G(g) \cap C_G(u)$, otherwise $C_G(g)/Z(G)$ would have order p, hence $C_G(g) = \langle g\rangle Z(G)$ and $gZ(G)$ would be isolated in $\mathcal{K}(G)$; therefore $uZ(G)$ has eccentricity 2 in $\mathcal{K}(G)$. The case when $\mathrm{diam}(\mathcal{K}(G)) = 4$ actually occurs for the group $\mathcal{G}(P)$ where P is a path of length 4, which is a finite 2-group of class 2 with an element of breadth 1.

If Δ is a finite connected graph it is possible that $\mathrm{diam}(\mathcal{K}\mathcal{G}(\Delta))$ is much smaller than $\mathrm{diam}(\Delta)$. We provide a few examples of this behaviour.

Proposition 5.3. *If Δ is a cycle on more than 3 vertices, then* $\mathrm{diam}(\mathcal{K}\mathcal{G}(\Delta)) = 2$.

Proof. Let $n = |V(\Delta)|$, $G = \mathcal{G}(\Delta)$ and $\Gamma = \mathcal{K}\mathcal{G}(\Delta)$. Then $\mathrm{diam}(\Delta) = \nu := \lfloor n/2 \rfloor$, so that $\mathrm{br}_G(g) \leqslant \mathrm{rk}(G') = \nu - 1$ for all $g \in G$. It will be enough to show that $\mathrm{rk}\left(G/Z(G)\right) > 2(\nu - 1)$, since in this case $Z(G) < C_G(\{g, h\})$ and so $d_\Gamma(\bar{g}, \bar{h}) \leqslant 2$ for all $g, h \in G \smallsetminus Z(G)$.

Let $X \subseteq V(\Delta)$ and assume $\mathrm{Þ}(X) \in Z(G)$. If n is even, that is, $n = 2\nu$, we claim that $X = \varnothing$. For, let $a \in X$. There exists exactly one vertex a' of Δ such that $d_\Delta(a, a') = \nu$, and no vertex of Δ except a has distance ν from a'. Since $\nu > 1$ this contradicts Lemma 2.1 if this latter is applied to X and $\{a'\}$. Then $Z(G) = G'$ if n is even, and this proves the result in this case since $\mathrm{rk}(G/G') = n$. Now assume that n is odd, that is, $n = 2\nu + 1$. Let again $a \in X$. Then there exist exactly two vertices b_1, b_2 in Δ at distance ν from a. There are exactly two vertices of Δ at distance ν from b_1, namely a and a vertex a_1 adjacent to a. On applying Lemma 2.1 to X and $\{b_1\}$, since $a \in X$ we deduce $a_1 \in X$. Also, a and the vertex a_2 adjacent to a and different from a_1 are the only two vertices of Δ at distance ν from b_2, therefore $a_2 \in X$. We have shown that every vertex of Δ which is adjacent to an element of X is in X as well.

Therefore either $X = \varnothing$ or $X = V(\Delta)$, and so $|Z(G)/G'| \leqslant 2$ (as a matter of fact, equality holds here). Therefore $\mathrm{rk}\big(G/Z(G)\big) \geqslant n-1 = 2\nu$; this completes the proof. □

In the next example we consider a class of starlike trees. In a way, this example could be considered more interesting, given that the embeddings of subtrees in trees are always isometric. For all positive integers $\rho > 2$ and $d > 1$, we let $\Sigma_{\rho,d}$ be the finite tree in which one vertex r (the root) has degree ρ, all remaining vertices have degree 1 or 2 and those of degree 1 (the leaves) have distance d from r. In other words, $\Sigma_{\rho,d}$ is the union of ρ paths of length d which have one endpoint (r) and (pairwise) no other vertices in common. Of course, $\mathrm{diam}(\Sigma_{\rho,d}) = 2d$.

Proposition 5.4. *For all integers $d > 1$ we have* $\mathrm{diam}\big(\mathcal{KG}(\Sigma_{3,d})\big) = 3$ *and* $\mathrm{diam}\big(\mathcal{KG}(\Sigma_{\rho,d})\big) = 2$ *for all integers $\rho > 3$.*

Proof. Let $\Sigma = \Sigma_{\rho,d}$ and $\Gamma = \mathcal{KG}(\Sigma)$. For each integer $\rho > 2$ let L be the set of all leaves in Σ; for all $a \in L$ let B_a (the a-branch) be the path in Σ joining a to the root r, and let a' be the vertex of B_a adjacent to a. Next, let $L' = \{a' \mid a \in L\}$.

Let $X \subseteq V(\Sigma)$. If, in $G := \mathcal{G}(\Sigma)$, some $a \in L$ is centralised by $\text{Þ}(X)$, then it follows from Lemma 2.1 that $X \cap V(B_a) \subseteq \{a, a'\}$ and for each $i \in \{1, 2, \dots, d\}$ the number of the leaves $b \in L \smallsetminus \{a\}$ such that the vertex of B_b at distance i from r lies in X is even. Now assume $\text{Þ}(X) \in Z(G)$. From the fact that $\text{Þ}(X)$ centralises all leaves in Σ it follows that $X \subseteq L \cup L'$. Also, for all $a \in L$ the number of leaves in $X \smallsetminus \{a\}$ is even; then either $L \subseteq X$ and ρ is odd or $L \cap X = \varnothing$. In the former case $\text{Þ}(X)$ would not centralise r, therefore $X \subseteq L'$. By a similar argument either $X = L'$ and ρ is odd or $X = \varnothing$. In the former case $[\text{Þ}(X), r] = 1$ implies $d = 2$; conversely, $\text{Þ}(L') \in Z(G)$ if ρ is odd and $d = 2$. We have shown that either $Z(G) = G'$, in which case $\mathrm{rk}(G/Z(G)) = |V(\Sigma)| = 1 + d\rho$, or ρ is odd and $d = 2$, in which case $Z(G) = G'\langle \text{Þ}(L')\rangle$ and $\mathrm{rk}(G/Z(G)) = 2\rho$. On the other hand, $\mathrm{rk}(G') = \mathrm{diam}(\Sigma) - 1 = 2d - 1$. If $\rho > 3$ then we have $\mathrm{rk}(G/Z(G)) > 2\,\mathrm{rk}(G')$ and as in the proof of Proposition 5.3 we deduce $\mathrm{diam}(\Gamma) = 2$ in this case, as required.

From now on we let $\rho = 3$. Let a, b and c be the three leaves in Σ. The 3-cycle $(a\,b\,c)$ extends to an automorphism τ of Σ permuting the three branches. Let $X \subseteq V(\Sigma)$ be such that $\text{Þ}(X) \in C_G(\{a, br\})$. Let $X_a := B_a \cap X$, $X_b := B_b \cap X$ and $X_c := B_c \cap X$. The description of $C_G(a)$ in the previous paragraph shows that $r \notin X$, $X_a \subseteq \{a, a'\}$ and $X_c = X_b^\tau$. Also, $X_a \cup X_c$ is the set of all vertices in X at distance greater than d from either r or b, namely from b, then Lemma 2.1 and $[br, \text{Þ}(X)] = 1$ imply $\text{Þ}(X_a \cup X_c) \in C_G(b)$. So, by the previous paragraph again, $X_a = X_c^\tau$. Therefore $X = X^\tau \subseteq L \cup L'$; in other words $X \in \{\varnothing, L, L', L \cup L'\}$. It follows that $\text{Þ}(X)$ commutes with b and hence with r. As in the previous paragraph $[\text{Þ}(X), r] = 1$ implies $L \nsubseteq X$, hence either $X = \varnothing$ or $d = 2$ and $X = L'$. In both cases $\text{Þ}(X) \in Z(G)$. Therefore $C_G(\{a, br\}) = Z(G)$, so that $d_\Gamma(\bar{a}, \bar{r}\bar{b}) > 2$ and $\mathrm{diam}(\Gamma) \geqslant 3$.

If $d = 2$ then r and $\text{Þ}(L)$ are two noncommuting elements of breadth 1 in G; therefore $\mathrm{diam}(\Gamma) = 3$ by Lemma 5.1. Now assume $d > 2$; then $Z(G) = G'$. The automorphism τ of Σ induces an automorphism φ of G, of order 3, acting trivially on G'. Call symmetric the elements of G which are fixed by φ. If s is one such element, then $\mathrm{br}_G(s) \leqslant d - 1$, because if B is any of the three branches of Σ and $v \in V(B)$, then $s = uu^\varphi u^{\varphi^2}$ for some $u \in \langle V(B)\rangle$ and $[v, u^\varphi u^{\varphi^2}] = 1$, hence $[v, s] = [v, u] \in \langle V(B)\rangle'$ and $\langle V(B)\rangle' = \langle V(B_1)\rangle'$ for any other branch B_1. Now let $g, h \in G \smallsetminus Z(G)$. Then $s := gg^\varphi g^{\varphi^2}$ is symmetric and commutes with g, since $[g, g^\varphi] = [g, g^\varphi]^{\varphi^2} = [g^{\varphi^2}, g]$. But $\mathrm{br}_G(h) \leqslant \mathrm{rk}(G') = 2d - 1$, hence $\mathrm{br}_G(h) + \mathrm{br}_G(s) \leqslant (2d-1) + (d-1) < 1 + 3d = \mathrm{rk}(G/Z(G))$ and $Z(G) < C_G(\{h, s\})$, so that $d_\Gamma(\bar{s}, \bar{h}) \leqslant 2$ and $d_\Gamma(\bar{g}, \bar{h}) \leqslant 3$. We conclude that $\mathrm{diam}(\Gamma) = 3$ in this case as well; now the proof is complete. □

It is easily seen that if Δ is an infinite connected graph, then $\mathrm{diam}(\mathcal{KG}(\Delta))$ may well be infinite. This is for instance the case when Δ is a ray:

Proposition 5.5. *Assume that Δ is a ray. Then Δ is isometrically embedded in $\mathcal{KG}(\Delta)$, hence* $\mathrm{diam}\big(\mathcal{KG}(\Delta)\big)$ *is infinite.*

Proof. Let $G := \mathcal{G}(\Delta)$ and $\Gamma = \mathcal{KG}(\Delta)$, and let $a, b \in V(\Delta)$. Also let $n = d_\Delta(a, b)$ and $m = d_\Gamma(\bar{a}, \bar{b})$. Fix a path Q in Γ of length m joining $\bar{a}$ and $\bar{b}$. There are a finite subset X of $V(\Delta)$ such that each vertex in Q belongs to $\langle X\rangle Z(G)/Z(G)$ and a path P in Δ of length greater than 2 such that $X \subseteq V(P)$. Of course $a, b \in X$ and $n = d_P(a, b)$. Up to the identification of $\mathcal{KG}(P)$ with a subgraph of Γ, Q may be viewed as a path of minimal length joining $\bar{a}$ and $\bar{b}$ in $\mathcal{KG}(P)$, hence Corollary 3.3 yields $m = d_{\mathcal{KG}(P)}(\bar{a}, \bar{b}) = d_P(a, b) = n$. □

Remark 5.6. Assume that Δ is a connected graph, and let $G = \mathcal{G}(\Delta)$. If, after the reduction suggested by Lemma 2.3, $|V(\Delta)|$ is 'much bigger' than $\mathrm{diam}(\Delta)$, then it is often the case that $\mathcal{KG}(\Delta)$ is connected and $\mathrm{diam}\big(\mathcal{KG}(\Delta)\big) = 2$. For instance, this is the case when $V(\Delta)$ is uncountable and $\text{Þ}(S) \notin Z(G)$ for all 2-element subsets S of $V(\Delta)$. For, if $\mathcal{KG}(\Delta)$ is not connected or $\mathrm{diam}\big(\mathcal{KG}(\Delta)\big) > 2$ then there are finite nonempty subsets X, Y of $V(\Delta)$ such that $C_G(\{g, h\}) = Z(G)$, where $g = \text{Þ}(X)$ and $h = \text{Þ}(Y)$. Since $V(\Delta)$ is uncountable there exist uncountably many pairs (a, b) of vertices of Δ such that $a \neq b$ but $d_\Delta(a, x) = d_\Delta(b, x)$ for all $x \in X \cup Y$, and so $ab \in C_G(\{g, h\}) = Z(G)$.

By the same argument, $\mathrm{diam}(\mathcal{KG}(\Delta)) = 2$ if the condition that $V(\Delta)$ is uncountable is replaced by requiring that Δ is infinite but $\mathrm{diam}(\Delta)$ is finite (which implies, by itself, that $\mathrm{diam}\big(\mathcal{KG}(\Delta)\big)$ is finite, according to the main theorem).

In either case the hypothesis on the two-element subsets cannot be disposed of. For instance, the graph Σ obtained by joining a ray R with a (disjoint) complete graph K of any uncountable cardinality, and adding edges from each vertex of K to the endvertex of R is an example of an uncountable connected graph such that $\mathcal{KG}(\Sigma) \simeq \mathcal{KG}(R)$ has infinite diameter.

The final result in this paper is a refinement of the main theorem in the case of graphs of small diameter. The proof of Lemma 4.3 shows that if Δ is a connected graph of diameter 3 not only $\operatorname{diam}(\mathcal{KG}(\Delta)) \leqslant 3$ but equality here is only attained in the obvious case, that is when, up to identification, $\mathcal{G}(\Delta) = \mathcal{G}(P)Z(\mathcal{G}(\Delta))$ for a path P of maximal length in Δ. We are going to show that the same is true for graphs of diameter 4, 5 or 6. It might even be the case that this strengthened form of the main theorem actually holds for connected graph of arbitrary finite diameter.

To start with our proof we state two extended versions of Lemmas 3.1 and 4.2.

Lemma 5.7. *Let Δ be a connected graph containing an isometrically embedded path P of length $n > 3$. Let x be a vertex of Δ at distance 1 from P and assume that X is a nonempty subset of $V(P) \cup \{x\}$ such that $\text{Þ}(X) \in Z(\mathcal{G}(\Delta))$. Then $x \in X$ and either $|X| = 2$ or $n = 4$ and the following hold: X is the set whose elements are x and the two vertices of P adjacent to the endvertices a and b of P, the only vertex of P adjacent to x is the one at distance 2 from a (and b) and the subgraph of Δ induced by $V(P) \cup \{x\}$ is isometrically embedded in Δ.*

Proof. $x \in X$ because of Lemma 3.1, and $|X| > 1$ because x cannot be adjacent to both a and b. For all $i \in \{0, 1, \ldots, n\}$ let a_i be the vertex of P at distance i from a. Let $d = d_\Delta(a, x)$. Since $[\text{Þ}(X), a] = 1$, Lemma 2.1 shows that $(X \cap V(P)) \smallsetminus \{a, a_1\}$ is empty if $d = 1$ and is $\{a_d\}$ if $d > 1$. A similar remark applies to b and a_{n-1} in place of a and a_1; this also shows that $|X \cap \{a, a_1\}| \leqslant 1$, and $|X \cap \{b, a_{n-1}\}| \leqslant 1$. It follows that either $|X| = 2$ or X has three elements, namely x, a_d and $a_{n-d'}$, where $d' = d_\Delta(b, x)$, and in this latter case $d, d' \geqslant n-1$. Assume $|X| = 3$, and let f and ℓ respectively be the least and the largest i such that $a_i \sim x$. Then $n-1 \leqslant d \leqslant f+1$ and $n-1 \leqslant d' \leqslant n-\ell+1$, hence $n-2 \leqslant f \leqslant \ell \leqslant 2$. Since $n > 3$ we obtain $n = 4$, $f = \ell = 2$ and $d = d' = 3$. Therefore $X = \{x, a_1, a_3\}$. It is easily seen that the information collected also implies the remaining properties required; now the proof is complete. □

The case $|X| = 3$ in the previous lemma can actually occur, it does for instance in the graph $\Sigma_{3,2}$ discussed in Proposition 5.4. It is plain that the condition $|X| = 3$ uniquely determines the subgraph induced by $V(P) \cup \{x\}$ up to isomorphism.

Lemma 5.8. *In the notation of Lemma 4.2, let $g \in G \smallsetminus Z(G)$. If $\operatorname{diam} \Gamma > 4$ then $r = \operatorname{rk}(G/Z(G)) \leqslant \operatorname{br}_G(g) + n - 1$.*

Proof. Let $h \in G \smallsetminus Z(G)$. Then $\operatorname{br}_G(h) \leqslant n-1$. If $r > \operatorname{br}_G(g) + n - 1$ then $Z(G) < C_G(g) \cap C_G(h)$, hence $d_\Gamma(\bar{h}, \bar{g}) \leqslant 2$. The result follows. □

Proposition 5.9. *Let Δ be a connected graph of diameter $n \in \{3, 4, 5, 6\}$, and let P be a path of length n in Δ joining two vertices whose distance in Δ is n. Then $\operatorname{diam}\big(\mathcal{KG}(\Delta)\big) = n$ if and only if $G := \mathcal{G}(\Delta) = HZ(G)$, where H is the subgroup of G generated by the vertices of P, and so $\mathcal{KG}(\Delta) \simeq \mathcal{KG}(P)$.*

Proof. P is isometrically embedded in Δ, hence Lemma 2.2 (i) gives $H \simeq \mathcal{G}(P)$. By Lemma 2.2 (ii), if $G = HZ(G)$ then $\mathcal{KG}(\Delta) = \mathcal{K}(G)$ is isomorphic to $\mathcal{K}(H) \simeq \mathcal{KG}(P)$ and so has diameter n by Theorem 3.8.

Conversely, assume $\operatorname{diam}(\mathcal{KG}(\Delta)) = n$; we have to show that $G = HZ(G)$. The vertices in P are pairwise not equivalent with respect to the equivalence relation ρ defined on $V(\Delta)$ as in the paragraph preceding Lemma 2.3. As a consequence, the subgraph Δ_ρ defined there can be chosen such that it contains P as a subgraph. Now we have two isometric embeddings: $P \hookrightarrow \Delta_\rho$ and $\Delta_\rho \hookrightarrow \Delta$; according to Lemma 2.2 they induce group monomorphisms $\alpha\colon \mathcal{K}(P) \rightarrowtail G_0 := \mathcal{K}(\Delta_\rho)$ and $\beta\colon G_0 \rightarrowtail G$. Lemma 2.3 gives $\operatorname{diam}(\Delta_\rho) = n$ and $G = G_\rho Z(G)$, where $G_\rho = \operatorname{im}\beta$. Note that $H_0 = \operatorname{im}\alpha$ is the subgroup of G_0 generated by $V(P)$; moreover $H = H_0^\beta$. If $G_0 = H_0 Z(G_0)$, then, on taking images under β, we have $G_\rho = HZ(G_\rho)$ and hence $G = HZ(G)$, because $G = G_\rho Z(G)$ and so $Z(G_\rho) \leqslant Z(G)$. This means that, in order to complete our proof, we may substitute Δ_ρ for Δ. Thus we may (and shall) assume $\text{Þ}(X) \notin Z(G)$ for all subsets X of $V(\Delta)$ such that $|X| = 2$, since, by construction, Δ_ρ satisfies the corresponding property. As $n > 2$ we also have $v \notin Z(G)$ for all $v \in V(\Delta)$.

Arguing by contradiction, assume $HZ(G) < G$. Let a and b be the endvertices of P and, as usual in this paper, for each $i \in \mathbb{N}$ such that $i \leqslant n$ let a_i be the vertex of P at distance i from a. Let $r = \operatorname{rk}(G/Z(G))$. By Lemma 4.2 we have $n + 1 < r \leqslant 2(n-1)$. Then $n > 3$.

Assume $n = 4$, then the same inequalities yield $r = 6$. Lemma 3.1 gives $\operatorname{rk}(HZ(G)Z(G)) = 5$, hence $|G/HZ(G)| = 2$. Choose $x \in V(\Delta) \smallsetminus HZ(G)$ at minimum distance d from P and let $K = H\langle x\rangle$. As $G' = H' \leqslant H$, we have $K \leqslant G$; moreover $G = KZ(G)$, because $HZ(G)$ is maximal, and $\operatorname{rk}(K/G') = 6$, hence $Z(K) = K \cap Z(G) = G'$. Lemma 5.7 and our assumptions show that if $y \in V(\Delta) \cap HZ(G)$ (that is, $\text{Þ}(Y \cup \{y\}) \in Z(G)$ for some $Y \subseteq V(P)$) and $d_\Delta(y, P) = 1$ then a_2 is the only vertex of P adjacent to y. Suppose $d > 1$. By the previous remark, a_2 is a vertex of each path joining x to a vertex of P, hence $d = d_\Delta(x, a_2)$, moreover, if Q is a path of length d from x to a_2 then the subgraph Σ of Δ induced by $V(P) \cup V(Q)$ is isometrically embedded in Δ. Let y be the vertex of Q adjacent to a_2. Then Lemma 5.7 also gives $g := a_1 a_3 y \in Z(G)$, hence $1 = [x, g] = [x, y]$ so that $d = 2$ and Σ is isomorphic to the graph $\Sigma_{3,2}$ of Proposition 5.4. But $G = \langle V(\Sigma)\rangle Z(G)$, hence $\mathcal{KG}(\Sigma) \simeq \mathcal{KG}(\Delta)$ and we obtain the contradiction $\operatorname{diam}(\mathcal{KG}(\Delta)) = 3$. Therefore $d = 1$.

Assume $x \sim a_2$. Then $\operatorname{br}_G(a_2) = 1$ (note that $\operatorname{br}_G(g) = \operatorname{br}_K(g)$ for all $g \in K$). Let $\alpha = d_\Delta(x, a)$ and $\beta = d_\Delta(x, b)$; then $\alpha, \beta \leqslant 3$. If $\alpha, \beta \leqslant 2$ then $\operatorname{br}_G(x) = 1$ and $[x, G] = [a_2, G]$; then Lemma 5.1 implies $C_G(x) = C_G(a_2)$, whence we obtain the contradiction $xa_2 \in Z(G)$. Then, without loss of generality, $\beta = 3$, hence $b \in C_G(xa_1)$. If $\alpha < 3$ then $[xa_1, G] = [a_2, G]$ (note that $xa_1 \notin Z(G)$ by our assumptions), but $b \notin C_G(a_2)$, so Lemma 5.1 yields a contradiction again. If, finally, $\alpha = 3$ then the subgraph induced by $V(P) \cup \{x\}$ is isometrically embedded in Δ, and $xa_1a_3 \in Z(G)$, so $x \in HZ(G)$, yet another contradiction. Therefore $x \nsim a_2$. This also shows that all vertices of Δ adjacent to a_2 lie in $HZ(G)$.

Next we assume that no vertex of Δ outside P is adjacent to either of a_1 and a_3. Then, without loss of generality $x \sim a$, and x cannot be adjacent to any other vertex of P. If there exists $y \in V(\Delta)$ such that $x \sim y \sim a_2$ then $y \notin V(P)$, and $y \in HZ(G)$ by the previous paragraph, hence $ya_1a_3 \in Z(G)$ by Lemma 5.7; this is impossible because $[x, a_1a_3] \neq 1$. Then $d_\Delta(x, a_2) = 3$. Next, $d_\Delta(x, a_3) = 4$, because $d_\Delta(x, b) > 2$ and a_3 has degree 2 in Δ. Since $d_\Delta(x, b) \leqslant 4$ it is now clear that there exists $y \in V(\Delta) \smallsetminus V(P)$ such that $y \sim b$. But $G = KZ(G) = H\langle y\rangle Z(G)$: the latter equality follows from the fact that otherwise, by Lemma 5.7, a_2 would have to be the only vertex of Δ adjacent to y. Then $xy\text{Þ}(S) \in Z(G)$ for some $S \subseteq V(P)$. This is certainly false because x is the only vertex in $V(P) \cup \{x, y\}$ at distance 4 from a_3.

Collecting the information obtained thus far (and invoking Lemma 5.7 again) we see that we may assume, without loss of generality, $x \sim a_1$. Then $d_\Delta(x, a_2) = 2$ and so $\mathrm{br}_2(a_2) = 1$. Since $1 \neq [x, a_2] = [xa_1, a_2]$, Lemma 5.1 gives $\mathrm{br}_G(x) \neq 1 \neq \mathrm{br}_G(xa_1)$. Let $\sigma = d_\Delta(x, a_3)$ and $\tau = d_\Delta(x, b)$. If $\sigma \leqslant 2$, either $\tau \leqslant 2$ (and so $\mathrm{br}_G(x) = 1$) or $\sigma = 2$ and $\tau = 3$ (and so $\mathrm{br}_G(xa_1) = 1$). Therefore $\sigma = 3$. If $\tau = 4$ then $\mathrm{br}_G(xa) = 1$ and $b \in C_G(xa) \smallsetminus C_G(a_2)$, which is again excluded by Lemma 5.1 and the remark following it. Then $\tau \leqslant 3$, and so $y \sim b$ for some $y \in V(\Delta) \smallsetminus V(P)$. As before Lemma 5.7 gives $g := xy\text{Þ}(S) \in Z(G)$ for some $S \subseteq V(P)$; but $[a_3, g] = 1$ implies $a \in S$, which is excluded by $[b, g] = 1$. This contradiction completes the proof in the case $n = 4$.

Now we tackle the cases $n = 5$ and $n = 6$; the argument is largely based on locating $C := C_G(a_3)$. Since $\mathrm{br}_H(a_3) = 2$, Lemma 3.1 yields $\mathrm{rk}\,\big(C \cap HZ(G)/Z(G)\big) = n - 1$. Then $r = \mathrm{br}_G(a_3) + s + n - 1$, where $s = \mathrm{rk}\,\big(C/C \cap HZ(G)\big)$. On the other hand Lemma 5.8 gives $r \leqslant \mathrm{br}_G(a_3) + n - 1$; therefore $s = 0$, that is, $C \leqslant HZ(G)$. If $n = 5$, after exchanging the roles of a and b, the same argument also gives $C_G(a_2) \leqslant HZ(G)$. Let x be any vertex of Δ at distance 1 from P. By Lemma 5.7, $x \notin HZ(G)$. If $n = 5$, it is easily checked that there is some $v \in V(P)$ such that one of a_2 and a_3 has the same distance from v and x and so centralises vx. This gives $C_G(a_2)C_G(a_3) \nleqslant HZ(G)$, a contradiction. Then $n = 6$. Now, since $x \nsim a_3$, the same argument (applied to a_3 only) excludes the possibility that x is adjacent to any vertex in $V(P) \smallsetminus \{a, b\}$. Then for all $i \in \{1, 2, 3, 4, 5\}$, a_i has degree 2 in Δ and we may assume $x \sim a$. If b has degree 1, then $d_\Delta(x, b) = 7$, a contradiction, hence $b \sim y$ for some $y \in V \smallsetminus V(P)$. Also, $d_\Delta(x, a_3) = 4 = d_\Delta(y, a_3)$, hence $xy \in C_G(a_3) \leqslant HZ(G)$. Then, once again, $g := xy\text{Þ}(S) \in Z(G)$ for some $S \subseteq V(P)$, but, for instance, such a g cannot centralise a_1, because $d_\Delta(y, a) \geqslant 5$ and so $d_\Delta(y, a_1) = 6$. Now the proof is complete. $\square$

As a closing remark, we note that even in the case $n = 6$ the argument in the last paragraph of the proof does only use the assumption that $HZ(G) < G$ and $\mathrm{diam}(\mathcal{KG}(\Delta)) > 4$. Therefore it actually shows that *if Δ is a connected graph of diameter 6, then $\mathcal{KG}(\Delta)$ cannot have diameter 5.*

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