On Groups with Countably Many Maximal Subgroups.

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Abstract. The object of this work is to find classes of groups which possess only countably many maximal subgroups. Modules with countably many maximal submodules and group rings having countably many maximal right ideals are also investigated. Examples of soluble groups with uncountably many maximal subgroups are described.

1 Introduction

This article is a study of groups that have only countably many maximal subgroups, a property that is denoted here by **CG**. There are many groups with **CG**; indeed a group with no maximal subgroups at all has the property and little can be said about such groups in general. On the other hand, non-trivial finitely generated groups always have maximal subgroups and clearly the cardinality of the set of maximal subgroups is at most $2^{\aleph_0}$. In fact finitely generated groups with uncountably many maximal subgroups exist even in the soluble case, as has been shown by Y. de Cornulier [3]: thus for a finitely generated group to be a **CG**-group is a real restriction on the group structure.

As examples of **CG**-groups we cite virtually soluble groups with finite abelian ranks and finitely generated nilpotent-by-polycyclic-by-finite groups; in each case the reason is that in these groups maximal subgroups have finite index and there are finitely many subgroups of each finite index. (For the background here see [8: §5, §7]). We mention as well the celebrated theorem of G.A. Margulis and G.A. Soifer [9] that a finitely generated linear group over a field which has **CG** is virtually soluble. The converse statement is also true since virtually soluble linear groups are nilpotent-by-abelian-by-finite by a well known theorem of Mal'cev.

In a **CG**-group the maximal subgroups always have countable index (Lemma 2). In the present work we will find many diverse types of **CG**-groups with maximal subgroups of infinite index. In the final section we describe methods for constructing finitely generated soluble groups with uncountably many maximal subgroups.

Results

We begin with a discussion of extension properties of the class **CG**. The first significant result states that a finite extension of a **CG**-group is a **CG**-group (Theorem 1). A further extension theorem which allows us to expand our list of groups with **CG** is:

Theorem 2. *Let G be a group with a normal subgroup N which has countably many subgroups and let G/N be finitely generated. Then all but a countable number of maximal subgroups of G contain N.*

Thus, if in addition G/N has **CG**, then G has **CG**. For example, a finitely generated extension of a group with max by a soluble group of finite rank has **CG**. In general a direct product of **CG**-groups need not be a **CG**-group; however, we are able to establish:

Theorem 3. *Let H and K be groups with* **CG** *and assume that simple quotients of K are finitely generated. Then $H \times K$ has* **CG**.

We are particularly interested in classes of locally nilpotent groups and soluble groups which have **CG**. In the case of periodic hypercentral groups there is a precise characterization.

Theorem 5. *Let G be a periodic hypercentral group. Then G has* **CG** *if and only if it is an extension of a central divisible subgroup by a direct product of finite p-groups for distinct primes p.*

The relation between soluble groups satisfying chain conditions on normal subgroups and the property **CG** is especially interesting. It is shown in Theorem 6 that *a soluble group satisfying max-n has* **CG**. On the other hand, the status of soluble groups with min-n is unknown, mainly due to our lack of knowledge of such groups. Our strongest result in this direction is the technical Theorem 7 below. A special case of interest is: *metanilpotent groups satisfying min-n have* **CG**. This seems to be the only structural property of metanilpotent groups with min-n that has been found since the work of B. Hartley and D. McDougall over forty years ago ([6], [7], [10]).

Further investigations of **CG**-groups are aided by introducing the corresponding module property: a module has the property **CM** if it has countably many maximal submodules. In the same spirit we will say that a ring has the property **CI** if it has countably many maximal right ideals. It is an easy exercise to show that a countable ring R with identity has **CI** if and only if there are only countably many R-isomorphism types of simple modules. The case that concerns us here is that of group rings; of course a group ring $\mathbb{Z}G$ has the property **CI** precisely when it has **CM** as a G-module.

While the structure of soluble groups with **CG** seems beyond reach in the general case, there is a characterization for metanilpotent groups.

Theorem 8. *Let G be a metanilpotent group and let $A \triangleleft G$ where A and G/A are nilpotent. Then G has* **CG** *if and only if the following conditions hold.*

(i) G/G^pG' is finite for each prime p.

(ii) Each maximal G-submodule of A^{ab} has countable index in A^{ab}.

(iii) There are countably many maximal G-submodules of A^{ab} whose quotients in A^{ab} are not G-trivial.

A large class of finitely generated **CG**-groups is given by:

Theorem 9. *Let G be a finitely generated group with a normal subgroup N. Assume that N has an ascending G-invariant series in which all the infinite factors are abelian and have the property that their finitely generated G-submodules satisfy* **CM**. *Then all but a countable number of maximal subgroups of G contain N.*

Of course, if in addition G/N has **CG**, then so does G. The property **CI** is clearly of interest in its own right. It has drastic structural consequences for soluble groups, as the next result shows.

Theorem 10. *If G is a soluble group whose group ring $\mathbb{Z}G$ satisfies* **CI**, *then G is a minimax group with no sections of type $p^\infty \times p^\infty$.*

It is interesting that the groups described in Theorem 10 are exactly the soluble groups with countably many subgroups, as was shown by G. Cutolo and H. Smith [4]. The converse of Theorem 10 is established for abelian-by-finite groups (Theorem 11). Theorems 10 and 11 have a noteworthy consequence: *if G is a locally finite* *group, then* $\mathbb{Z}G$ *satisfies* **CI** *if and only if* G *is a Černikov group whose maximum divisible subgroup is locally cyclic.* This is Corollary 4.

The methods of proof used in the article are mostly standard, a common tool being the uncountable pigeonhole principle. However, derivations play an important part, while the proofs of Theorems 7 and 8 call for the use of cohomological vanishing theorems of the third author.

Notation

(i) **CG**: a group has countably many maximal subgroups.

(ii) **CM**: a module has countably many maximal submodules.

(iii) **CI**: a ring has countably many maximal right ideals.

(iv) max-G, min-G: the maximal and minimal condition for G-invariant subgroups.

(v) G^{ab}: the abelianization of G.

(vi) $G^{(i)}$, $\gamma_i(G)$: terms of the derived series and lower central series of G.

(vii) H_G: the G-core of H.

(viii) All modules are right modules. If G is a group, a G-module is a $\mathbb{Z}G$-module.

(ix) $\mathrm{Der}(G, A)$, $\mathrm{Inn}(G, A)$: sets of derivations and inner derivations.

(x) $I_G, \bar{I}_G$: an augmentation ideal and a relative augmentation ideal.

2 Some groups with countably many maximal subgroups

We begin with two elementary results.

Lemma 1. *Let* G *be a group whose maximal subgroups have finite index and is such that* G/G^n *is finite for all* $n > 0$. *Then* G *is a* **CG**-*group*

Proof. A maximal subgroup contains G^n for some $n > 0$, so there are only countably many of them. □

It follows that finitely generated nilpotent-by-polycyclic groups and soluble groups with finite abelian ranks are **CG**-groups. On the other hand, finitely generated soluble **CG**-groups with maximal subgroups of infinite index exist; for example, the soluble group with max-n constructed by P. Hall in [5] – see also [8: 4.5.1].

The next result is almost the only property shared by all **CG**-groups.

Lemma 2. If G is a **CG**-group, every maximal subgroup of G has countable index.

Proof. Let M be a maximal subgroup of G. If $M \triangleleft G$, then $|G : M|$ is finite. Otherwise $M = N_G(M)$, so that M has $|G : M|$ conjugates in G. Since each conjugate is maximal in G, the result follows. □

In the remainder of this section we investigate extension properties of the class of **CG**.

Theorem 1. *Let G be a group with a normal subgroup N of finite index. If N has* **CG***, then so does G.*

Proof. Assume the result is false; then there are uncountably many maximal subgroups M of G such that $N \nleq M$. Also $G = MN$, $D = M \cap N \triangleleft M$ and M/D is finite. In addition D is a maximal M-invariant subgroup of N. For suppose that $D < L \leq N$ and $L = L^M$; then $L \nleq M$, so $G = ML$ and $N = (ML) \cap N = DL = L$.

Next let T be a transversal to D in M; thus $M = DT$. Let $x \in N \backslash D$; then $\langle D, x^t \mid t \in T \rangle$ is M-invariant and contains D properly, so that $N = \langle D, x^t \mid t \in T \rangle$ by maximality of D. Thus N is generated by D and finitely many elements, which implies that D is contained in a maximal subgroup X of N. Moreover $D = X_M = X_T$ by maximality of D. Therefore D is the intersection of finitely many maximal subgroups of N, each of which has countable index in N by Lemma 2. Consequently $|G : D|$ is countable. Since N has **CG**, there are countably many possible subgroups D. Finally, M is a finite union of cosets of D, so there can be only countably many subgroups M, a contradiction. □

On the other hand, it is uncertain whether the property **CG** is inherited by normal subgroups of finite index.

Theorem 2. *Let G be a group with a normal subgroup N which has only countably many subgroups and let G/N be finitely generated. Then all but a countable number of the maximal subgroups of G contain N.*

Proof. Assume the result is false, so there are uncountably many maximal subgroups M of G such that $N \not\leq M$. Thus $G = MN$ and $M \cap N \lhd M$. By hypothesis there are countably many possibilities for $M \cap N$. Hence $M \cap N = D$ is fixed for uncountably many M's. Notice that $D \lhd \langle M, M_1 \rangle = G$ where M_1 is another maximal subgroup with the properties of M. Now consider $G/D = (M_0/D) \ltimes (N/D)$ with M_0 a fixed maximal subgroup. It is well known that complements of N/D in G/D correspond to elements of $\mathrm{Der}(M_0/D, N/D)$. A derivation is determined by its effect on the generators of M_0/D. Since $M_0/D \simeq G/N$ is finitely generated and N is clearly countable, there are only countably many M/D's and hence M's, a contradiction. □

Corollary 1. *A finitely generated extension of a group with max by a* **CG***-group is a* **CG***-group.*

Nevertheless, the class **CG** is not closed under extensions. For example, consider the metabelian group $G = H \ltimes K$ where K is the additive group and H the multiplicative group of the field of complex numbers, with the natural action of H on K. Both H and K have **CG** since they are divisible and abelian. However, H is maximal in G and has uncountable index; thus it G not a **CG**-group by Lemma 2.

We turn next to direct products of **CG**-groups.

Theorem 3. *Let H and K be* **CG***-groups and assume that all simple quotients of K are finitely generated. Then $H \times K$ is a* **CG***-group.*

Proof. Write $G = H \times K$ and assume G does not have **CG**. Then there are uncountably many maximal subgroups M that contain neither H nor K; hence $G = MH = MK$ and $M \cap H \lhd G, M \cap K \lhd G$. Also $M \cap H$ and $M \cap K$ are maximal normal in H and K respectively. By hypothesis $K/M \cap K$ is finitely generated, so $M \cap K$ is contained some maximal subgroup $K(M)$ of K. If $\bar{M}$ is another maximal subgroup like M and $K(M) = K(\bar{M})$, then $M \cap K \leq K(\bar{M})$ and, since $K/\bar{M} \cap K$ is simple, it follows that $M \cap K \leq \bar{M} \cap K$ and hence $M \cap K = \bar{M} \cap K$ by maximality. There are only countably many $M \cap K$'s because K is a **CG**-group. It follows that there are uncountably many maximal subgroups M with $M \cap K = L$ fixed. Factoring out by L we can assume that $M \cap K = 1$ for uncountably many M. Thus K is finitely generated.

Next $G/M \cap H = H/(M \cap H) \times (M \cap H)K/(M \cap H)$, so that

$$H/M \cap H \simeq G/(M \cap H)K = MK/(M \cap H)K \simeq M/M \cap H \simeq K.$$

As a consequence $H/M \cap H$ is finitely generated. Arguing as we did above for K, we may also suppose that $M \cap H = 1$ and H is finitely generated. Now the situation is that $G = M \ltimes H = K \times H$. The complements of H in G correspond to elements of $\mathrm{Der}(K, H) = \mathrm{Hom}(K, H)$, which is countable since H and K are finitely generated. This gives the contradiction that there are countably many subgroups M. □

In particular, the direct product of two **CG**-groups is a **CG**-group if at least one of them is soluble. On the other hand, the direct product of two insoluble **CG**-groups need not be a **CG**-group. Indeed suppose that S is a simple group without maximal subgroups: for example, we could take S to be Ol'šanskiĭ's uncountable group with all its proper subgroups countable [11: 35.2 – 35.4]. Set $G = S \times S$ and observe that the diagonal subgroup is maximal and has uncountable index in G; therefore G cannot be a **CG**-group by Lemma 2.

3 Locally nilpotent groups and soluble groups with the property CG

In this section we investigate the property **CG** in the context of locally nilpotent groups and soluble groups. For locally nilpotent groups there is a simple – if not too informative – characterization.

Lemma 3. *Let G be a locally nilpotent group. Then G has* **CG** *if and only if $G^{ab}/(G^{ab})^p$ is finite for all primes p.*

Proof. Recall that a maximal subgroup of G is normal with prime index and hence contains some $G'G^p$. Also an infinite elementary abelian p-group has uncountably many subgroups of index p. The result now follows. □

The structure of even abelian groups with **CG** is not known in general. Clearly an abelian group with finite ranks and the direct product of a divisible abelian group and finite abelian p-groups for distinct primes p are **CG**-groups. But there are uncountable, torsion-free reduced abelian groups of infinite rank with **CG**, for example the cartesian product of the additive groups of rational numbers with denominators indivisible by a prime p for distinct p.

Turning to soluble groups satisfying **CG**, we will find several uses for the next result.

Theorem 4. *Let G be a virtually soluble group of finite exponent. If G has* **CG***, then it is finite.*

Proof. Let N be a soluble normal subgroup with G/N finite and let d be the derived length of N; we argue by induction on $d > 0$. Write $A = N^{(d-1)}$; then by induction G/A is finite. Assume the result is false; then A is infinite and hence A/A^p is infinite for some prime p, since A is reduced. Thus we may suppose that A is an infinite elementary abelian p-group. Next $G = XA$ where X is a finite subgroup and $X \cap A \lhd XA = G$, so we can factor out by $X \cap A$ and assume that $G = X \ltimes A$. If M is a maximal X-submodule of A, then XM is maximal in G. It follows that there are only countably many such XM's and therefore A has countably many maximal submodules, i.e., it has **CM** as an X-module.

To reach a contradiction, the first step is to show that there are uncountably many X-submodules with finite index in A. Certainly there are uncountably many subgroups B such that $|A : B| = p$. Then B_X is a submodule of finite index in A. Suppose that there are only countably many of these B_X. Then $B_X = C$ is fixed for uncountably many B. But A/C is finite and $C \leq B < A$, which leads to the contradiction that there are finitely many B's.

Let $\mathcal{S}_n$ denote the set of all submodules S such that A/S is finite with X-composition length n. By hypothesis there is a least $n > 0$ such that $|\mathcal{S}_n| > \aleph_0$. If $S \in \mathcal{S}_n$, then A/S has composition length n and hence S is a maximal submodule of some submodule belonging to $\mathcal{S}_{n-1}$. Since $\mathcal{S}_{n-1}$ is countable whereas $\mathcal{S}_n$ is uncountable, there exists $B \in \mathcal{S}_{n-1}$ with uncountably many maximal submodules. Also A/B is finite, so we have $A = BD$ for some finite X-submodule D.

Let $I = B \cap D$ and consider $A/I = B/I \times D/I$. If S/I is a maximal X-submodule of B/I, the submodule SD is maximal in A, so there are countably many SD's. It follows that there are uncountably many maximal X-submodules M of B not containing I. For such an M we have $B = MI$ and, as I is finite, there are uncountably many such M's with $M \cap I = J$ fixed. Since $B/J = M/J \times I/J$, we see that $H = \mathrm{Hom}_X(B/I, I/J)$ must be uncountable. Let $0 \neq \theta \in H$. Now $I/J \overset{X}{\simeq} B/M$, which is a simple module. Hence $(B/I)/\mathrm{Ker}(\theta) \overset{X}{\simeq} B/M$. Also $B/I \overset{X}{\simeq} A/D$, which satisfies **CM**. Consequently there are only countably many submodules $\mathrm{Ker}(\theta)$. It follows that there are uncountably many $\theta \in H$ with $\mathrm{Ker}(\theta) = K/I$ fixed. But $\mathrm{Hom}_X(B/K, I/J)$ is even finite since B/K is simple and I/J is finite. Thus we have reached a contradiction. □

Corollary 2. *Let G be a virtually hypocentral group of finite exponent. If G satisfies* **CG***, then it is finite.*

Proof. First of all recall that a group is hypocentral if its lower central series reaches the identity subgroup when continued transfinitely. Let $H \lhd G$ where H is a hypocentral group and G/H is finite. Then G/H' is finite by Theorem 4, so $H = XH'$ where X is a finitely generated subgroup. Hence $H = X\gamma_i(H)$ for all positive integers i. Thus $H/\gamma_i(H)$ is finitely generated and hence is finite, so by the solution of the Restricted Burnside Problem it has boundedly finite order for all i. Therefore $\gamma_i(H) = \gamma_{i+1}(H)$ for some i, which implies that $\gamma_i(H) = 1$, so H, and hence G, is finite. □

Our next aim is to determine the structure of periodic hypercentral groups with **CG**.

Theorem 5. *Let G be a periodic hypercentral group. Then G has* **CG** *if and only if it is an extension of a central divisible subgroup by a direct product of finite p-groups for distinct primes p.*

Proof. It suffices to prove the result when G a p-group. Since divisible subgroups of G lie in the center ([2]; see also [13: 9.2]), we can assume there are no non-trivial divisible subgroups. Form the lower Frattini series $G = F_0 \geq F_1 \geq \cdots$; thus $F_{i+1} = \phi(F_i) = F_i^p F_i'$. By Theorem 4 each G/F_i is finite and $G = XF_1$ for some finite subgroup X, which implies that $G = XF_i$ for all i. Since X is finite, it follows that G/F_i has boundedly finite order and $F_i = F_{i+1}$ for some i, that is $F_i = F_i^p F'$. Now a hypercentral group cannot have a non-trivial perfect factor; hence $F_i = F_i^p$, so that F_i is divisible and hence is trivial. Therefore G is finite. Conversely, a group with the structure in the statement has **CG** because the divisible subgroup is contained in every maximal subgroup. □

By way of contrast we note that McLain's characteristically simple p-group (see [13: 6.2]) has no maximal subgroups and its centre is trivial; of course it is a Fitting group. A further limitation to progress in this direction is indicated by a construction of M. Vaughan-Lee and J. Wiegold [16] of countably infinite, locally nilpotent groups of finite exponent that have no maximal subgroups.

Next we discuss soluble groups satisfying chain conditions. First comes a useful result concerning the module property **CM**.

Lemma 4. *Let R be a ring and A a countable R-module. If A is either noetherian or artinian, then it has* **CM**.

Proof. If A is noetherian, each submodule is finitely generated and, since A is countable, it has countably many submodules. Hence A has **CM**. Next let A be artinian. The intersection I of all the maximal submodules of A is the intersection of finitely many of them, which implies that A/I is noetherian. By the first part, A has **CM**. □

Theorem 6. *A soluble group satisfying max-n has* **CG**.

Proof. Let G be a soluble group with derived length d satisfying max-n. Recall that G is finitely generated by a theorem of P. Hall [5]. The result is clearly true when $d \leq 1$: we argue by induction on $d > 1$. Let $A = G^{(d-1)}$. Suppose there are uncountably many maximal subgroups M of G. Since the result holds for G/A, there are uncountably many M such that $A \not\leq M$; thus $G = MA$ and $M \cap A \lhd G$. Now A satisfies max-G, so there are countably many possibilities for $M \cap A$. Hence there are uncountably many maximal subgroups M with $M \cap A = B$ fixed. Consider $G/B = M/B \ltimes A/B$ and observe that $\mathrm{Der}(G/A, A/B)$ is countable because G/A is finitely generated and A/B is countable. Therefore there are countably many complements of A/B in G/B and hence countably many maximal subgroups M, a contradiction. □

Theorem 6 can also be deduced from the much more general Corollary 3 below. It is natural to ask if the dual theorem for soluble groups with min-n holds; this is a difficult problem because nothing is known about the structure of soluble groups with min-n, except for the theorem of R. Baer [1] that they are periodic and hence locally finite. It was shown by D. McDougall [10] that metabelian groups with min-n are countable, and from this it follows readily that the same holds for metanilpotent groups with min-n. On the other hand, B. Hartley [6] constructed examples of uncountable soluble groups of derived length 3 with min-n.

The most general result about soluble groups with min-n we have found is the following criterion for **CG**.

Theorem 7. *Let G be a countable, periodic soluble group. Assume there is a normal series $G = L_0 \geq L_1 \geq \cdots \geq L_\ell = 1$, $\ell > 0$, with nilpotent factors such that L_i/L_{i+1} satisfies min-L_{i-1} for $i = 0, 1, 2, \ldots, \ell - 1$. Then G has* **CG**.

When $i = 0$, the condition on the factor is to be interpreted as requiring G/L_1 to satisfy min, that is, to be a Černikov group.

Proof of Theorem 7. If $\ell = 1$, G is a Černikov group and the result is clearly true; we proceed by induction on $\ell \geq 2$. Set $A = L_{\ell-1}$ and $B = L_{\ell-2}$. Since A is nilpotent, $A' \leq \phi(G)$; thus we can pass to G/A' and assume that A is abelian. Assume the result is false; since G/A has **CG** by induction, there exist uncountably many maximal subgroups M not containing A; thus $G = MA$, $M \cap A \triangleleft G$ and $M \cap A$ is a maximal G-submodule of A. Since A has min-B, it has min-G, so by Lemma 4 it satisfies **CM** as a G-module. Consequently there are uncountably many maximal subgroups M with $M \cap A = A_0$ fixed. By passing to G/A_0, we can assume that $M \cap A = 1$ for all such M. Thus $G = M \ltimes A$ and A is a simple G-module.

Next $C_A(B) = 1$ or A. In the latter event B is nilpotent and there is a series of shorter length with the same properties, so the result follows by induction on ℓ. Assume therefore that $C_A(B) = 1$. By hypothesis A satisfies min-B and also B/A is nilpotent; therefore $H^1(B/A, A) = 0$, (see [8: 10.3.2]). From $1 \to B/A \to G/A \to G/B \to 1$ and the standard 5-term cohomology sequence we obtain

$$0 \to H^1(G/B, C_A(B)) \to H^1(G/A, A) \to H^1(B/A, A).$$

Hence $H^1(G/A, A) = 0$ and $\mathrm{Der}(G/A, A) = \mathrm{Inn}(G/A, A)$. Since A is countable, so is $\mathrm{Der}(G/A, A)$ and it follows that there are countably many maximal subgroups M. □

Taking $\ell \leq 2$ in Theorem 7, we deduce that *a metanilpotent group satisfying min-n has* **CG**. When $\ell = 3$, the statement of the theorem is that if G has min-n and L_2 has min-L_1, then G has **CG**.

The proof of Theorem 7 shows that in order to prove that soluble groups of nilpotent length 3 satisfying min-n have **CG** what has to be established is this: $H^1(Q,B)$ is countable whenever Q is a metanilpotent group with min-n and B is a simple Q-module. In the light of this observation it is worthwhile noting the following.

Remark: Hartley's uncountable soluble groups with min-n and derived length 3 *have* **CG**.

We will briefly describe the construction. Let p,q,r be distinct primes such that $p \nmid q-1$, but $q \mid r-1$. Let H be a Čarin (p,q)-group, i.e., the split extension of an infinite elementary abelian q-group E by a group of type p^∞ acting faithfully and irreducibly on E. Hartley constructed an H-module A which is artinian and uniserial with length Ω, the first uncountable ordinal. This means that the set of proper submodules of A is well ordered by inclusion, so its members form an ascending series of ordinal type Ω. Then $G = H \ltimes A$ is a soluble group of derived length 3 satisfying min-n. Observe that A has no maximal submodules since Ω is a limit ordinal, so that all the maximal subgroups of G contain A. Hence G satisfies **CG** since G/A does.

The section ends with a characterization of metanilpotent groups with **CG**.

Theorem 8. *Let G be a metanilpotent group and let $A \triangleleft G$ where A and G/A are nilpotent. Then G has* **CG** *if and only if the following conditions hold.*

(i) G/G^pG' is finite for each prime p.

(ii) Each maximal G-submodule of A^{ab} has countable index in A^{ab}.

(iii) There are countably many maximal G-submodules of A^{ab} with non G-trivial quotients in A^{ab}.

Proof. In the first place $A' \leq \phi(G)$, so that G has **CG** if and only if G/A' does; thus we may assume that A is abelian. Assume that G satisfies **CG**. Then G/G^pG' has **CG** and hence it is finite for all primes p. Next let B denote a maximal G-submodule of A. If A/B is trivial as a G-module, $|A:B|$ is finite. Now assume that A/B is a non-trivial G-module. Since G/A is nilpotent and A/B is simple and non-trivial as a G-module, we have $H^2(G/A, A/B) = 0$, (see [8: 10.3.2]). This shows that G/B splits over A/B, so $G = X(B)A$ where $X(B) \leq G$ and $X(B) \cap A = B$. Clearly $X(B)$ is maximal in

G, so there are only countably many $X(B)$'s. Since $X(B) \cap A = B$, it follows that there are countably many B's. In addition $|G : X(B)| = |A : B|$, which implies that $|A : B|$ is countable. Thus the conditions (i)-(iii) are necessary.

Conversely, assume that the three conditions are satisfied, but G does not have **CG**. Now G/A has **CG** since it is nilpotent and each $G/G^pG'A$ is finite. Hence there are uncountably many maximal subgroups M of G not containing A. Thus $G = MA$, $M \cap A \lhd G$ and $M \cap A$ is a maximal G-submodule of A. If $A/M \cap A$ were G-trivial, then $M \lhd G$ and $|G : M| = p$, a prime. Consequently, $G^pG' \leq M$ and, as G/G^pG' is finite, there can be only countably many such M's.

From now on we assume that $A/M \cap A$ is non-trivial as a G-module. By (iii) there are countably many possibilities for $M \cap A$. Consequently, there must exist an uncountable number of maximals M such that $C = M \cap A$ is fixed and A/C is a simple, non-trivial G-module. Therefore $H^1(G/A, A/C) = 0$, whence $\mathrm{Der}(G/A, A/C) = \mathrm{Inn}(G/A, A/C)$. Moreover, A/C is countable by (ii), which means that $\mathrm{Der}(G/A, A/C)$ is countable. However, $G/C = (M/C) \ltimes (A/C)$, so we reach the contradiction that there are countably many M's. □

4 Modules with countably many maximal submodules

In this section we undertake a more detailed study of the module condition **CM**. Unlike **CG** this property is closed under forming extensions, for countable rings at least.

Lemma 5. *Let R be a countable ring and let A be an R-module with a submodule B. If B and A/B have* **CM***, then A has* **CM***.*

Proof. Assume the result is false, so there are uncountably many maximal submodules M of A not containing B. Thus $A = M + B$ and $M \cap B$ is a maximal submodule of B because $B/M \cap B \overset{R}{\simeq} A/M$. Since B has **CM**, there are uncountably many M's with $M \cap B = C$ fixed. Then, with M fixed, we have $A/C = M/C \oplus B/C$ and $M/C \overset{R}{\simeq} A/B$, which has **CM**. Let $\bar{M} \neq M$ be some other maximal submodule of A containing C but not B. Then $(M \cap \bar{M})/C$ is a maximal submodule of M/C since $M/M \cap \bar{M} \overset{R}{\simeq} A/\bar{M}$. Hence there are uncountably many $\bar{M}$'s with $M \cap \bar{M} = D$ fixed. However, A/M

and $A/\bar{M}$ are simple and countable, which implies that A/D is a countable noetherian R-module. This leads to the contradiction that there are just countably many $\bar{M}/D$'s. □

The next result is very simple, but it features a property that will be used to construct a large class of finitely generated groups with **CG**.

Lemma 6. *Let R be a countable ring with identity and let A be an R-module. Then the following are equivalent.*

(i) Every finitely generated submodule of A has **CM**.

(ii) Every cyclic submodule of A has **CM**.

Proof. Assume that each cyclic submodule has **CM** and let $B = a_1R + a_2R + \cdots + a_nR$ be a finitely generated submodule of A with $a_i \in A$. By hypothesis each a_iR has **CM** and by Lemma 5 this property is extension closed; hence B has **CM**. The converse is obvious. □

Notice the consequence: *if G is a countable group and $\mathbb{Z}G$ satisfies* **CI**, *then every finitely generated G-module has* **CM**. Next comes a key lemma involving the property characterized in Lemma 6.

Lemma 7. *Let G be a finitely generated group with a normal subgroup A. Assume that A is either finite or abelian with every finitely generated G-submodule satisfying* **CM**. *Then all but a countable number of maximal subgroups of G contain A.*

Proof. Assume the result is false. Then A must be infinite by Theorem 2, so it is abelian. By assumption there are uncountably many maximal subgroups of G not containing A; let M be one of them. Choose $a \in M \backslash A$ and set $A(M) = a^G$. Then $M \cap A(M) \lhd MA(M) = G$ and also $M \cap A(M)$ is a maximal G-submodule of $A(M)$. There are only countably many possibilities for $A(M)$ since it is a cyclic G-module and A is countable. By hypothesis $A(M)$ satisfies **CM**, so there are only countably many possible submodules $M \cap A(M)$. Consequently, there is an uncountable set $\mathcal{M}$ of maximal subgroups M of G such that $A(M) = B$ and $M \cap A(M) = C$ with B and C fixed for all $M \in \mathcal{M}$.

Let $M \in \mathcal{M}$ and consider the group $G/C = (M/C) \ltimes (B/C)$. Since G is finitely generated and B/C is countable, $\mathrm{Der}(G/B, B/C)$

is countable, showing that there are only countably many complements of B/C in G/C. Hence there are countably many $M \in \mathcal{M}$, a contradiction. □

We are now in a position to produce a large class of finitely generated groups with **CM**.

Theorem 9. *Let G be a finitely generated group with a normal subgroup N. Assume that N has an ascending G-invariant series in which each infinite factor is abelian with the property that all its finitely generated G-submodules satisfy* **CM**. *Then all but a countable number of maximal subgroups of G contain N.*

Proof. Let $\{N_\alpha \mid \alpha \leq \beta\}$ be the given ascending G-invariant series in N. If the result is false, there is a least ordinal α for which there is an uncountable set $\mathcal{M}$ of maximal subgroups of G none of which contains N_α.

Suppose first that α is not a limit ordinal. Then all but a countable number of subgroups in $\mathcal{M}$ contain $N_{\alpha-1}$. By hypothesis either $N_\alpha/N_{\alpha-1}$ is finite or it is abelian and its finitely generated G-submodules satisfy **CM**. Thus we can apply Lemma 7 to the group $G/N_{\alpha-1}$, concluding that all but a countable number of subgroups in $\mathcal{M}$ contain N_α, a contradiction that shows α to be a limit ordinal. Let $\gamma < \alpha$; then all but a countable number of subgroups in $\mathcal{M}$ contain N_γ. Keeping in mind that N is countable, so that there are countably many such N_γ's, we conclude that all but a countable number of subgroups in $\mathcal{M}$ contain every N_γ for $\gamma < \alpha$, and hence contain N_α, a final contradiction. □

Of course, if in addition G/N has **CM**, the same is true of G.

Corollary 3. *Let G be finitely generated group with a normal subgroup N. Assume that N has an ascending G-invariant series in which each infinite factor is abelian and its finitely generated G-submodules satisfy max-G or min-G. Then all but a countable number of maximal subgroups of G contain N.*

Proof. This follows at once from Theorem 9 and Lemma 4. □

From Corollary 3 one can read off many types of finitely generated **CG**-groups. For example, N could be hypercentral in G or even hypercyclically embedded in G and G/N nilpotent-by-polycyclic or soluble with finite rank.

Finally, here is another simple result which shows connections between the properties **CG**, **CM** and **CI**.

Proposition 1. *Let Q be a finitely generated group with* **CG**. *Then the following statements are equivalent.*

(i) Every extension of a finitely generated Q-module by Q has **CG**.

(ii) Every finitely generated Q-module has **CM**.

(iii) The group ring $\mathbb{Z}Q$ has **CI**.

Proof. (i) ⇒ (ii). Let A be a finitely generated Q-module and write $G = Q \ltimes A$, the natural semidirect product. By hypothesis G has **CG**. If M is a maximal submodule of A, then QM is a maximal subgroup of G, so there are countably many QM's. This implies that there are countably many M's because $M = A \cap QM$.

(ii) ⇒ (i). Let G be an extension of a finitely generated Q-module A by Q. By hypothesis every finitely generated submodule of A has **CM**. Since G is finitely generated, Lemma 7 shows that G has **CG**.

(ii) ⇒ (iii). This is obvious.

(iii) ⇒ (ii). As has already been observed, this is a consequence of Lemma 6. □

5 Group rings with countably many maximal right ideals

We move on to study the property **CI** for a group ring $\mathbb{Z}G$. This is a very strong property since it is preserved on passing to subgroups of G. The basis for this is:

Lemma 8. *Let R be a ring with identity and let G be a group with a subgroup H. If RG has* **CI**, *the same is true of RH.*

This follows from a standard result about group rings: *if H is a subgroup of a group G and I is a maximal right ideal of RH, there is a maximal right ideal J of RG such that $I = J \cap \mathbb{Z}H$* – for this see [12: 6.1.2].

The next result may be compared with Theorem 1.

Proposition 2. *Let G be a countable group with a subgroup H of finite index. Then $\mathbb{Z}G$ has* **CI** *if and only if $\mathbb{Z}H$ has* **CI**.

Proof. In the first place Lemma 8 shows that $\mathbb{Z}H$ will have **CI** if $\mathbb{Z}G$ does. Assume that $\mathbb{Z}H$ has **CI**. By Lemma 8 again we can replace H by H_G, so we may as well assume that $H \triangleleft G$. Let T be a transversal to H in G; then $\mathbb{Z}G = \bigoplus_{t\in T}(\mathbb{Z}H)t$ and consequently $\mathbb{Z}G$ is finitely generated as an H-module. Since cyclic H-modules have **CM**, we conclude from Lemma 6 that $\mathbb{Z}G$ has countably many maximal H-submodules.

Let M be a maximal right ideal of $\mathbb{Z}G$. Now M is contained in a maximal H-submodule L of $\mathbb{Z}G$, since the latter is finitely generated as an H-module. If $t \in T$, then Lt is also a maximal H-submodule of $\mathbb{Z}G$ containing M. Hence $M \leq \bigcap_{t\in T} Lt = \bar{L}$ and $\bar{L}$ is a proper right ideal of $\mathbb{Z}G$, from which it follows that $M = \bar{L}$. Therefore M is the intersection of finitely many maximal H-submodules of $\mathbb{Z}G$. Since $\mathbb{Z}G$ has countably many maximal H-submodules, there are only countably many possibilities for M. □

Next we examine the effect on the structure of a soluble group of imposing the property **CI** on its group ring.

Theorem 10. *Let G be a soluble group such that $\mathbb{Z}G$ has the property* **CI**. *Then G is a minimax group with no sections of type $p^\infty \times p^\infty$.*

As already noted, the groups appearing in Theorem 10 are exactly the soluble groups that have countably many subgroups [4]. In the proof of the theorem we will use the following lemma.

Lemma 9. *Let A be an abelian group which has an uncountable set of subgroups $\{B_\lambda | \lambda \in \Lambda\}$ such that A/B_λ is a periodic, locally cyclic p'_λ-group for some prime p_λ. Then $\mathbb{Z}A$ has uncountably many maximal ideals.*

Proof. It follows from the hypothesis that A/B_λ is isomorphic with a subgroup of the multiplicative group of $\bar{\mathbb{Z}}_{p_\lambda}$, the algebraic closure of $\mathbb{Z}_{p_\lambda}$; moreover, the image of A/B_λ generates a subfield F_λ of $\bar{\mathbb{Z}}_{p_\lambda}$ as a ring. It follows that F_λ is a simple A-module via the natural action of A/B_λ. Hence $F_\lambda \overset{A}{\simeq} \mathbb{Z}A/M_\lambda$ where M_λ is a maximal ideal of $\mathbb{Z}A$. The kernel of the action of A on F_λ is clearly B_λ. Therefore $B_\lambda \neq B_\mu$ implies that $M_\lambda \neq M_\mu$ and consequently $\mathbb{Z}A$ has uncountably many maximal ideals. □

Proof of Theorem 10. Since the property **CI** is inherited by group rings of subgroups and quotients, we may assume that G is abelian. Furthermore, it is sufficient to prove that G cannot be an infinite elementary abelian p-group, an infinite direct product of groups of distinct prime orders or a group of type $p^\infty \times p^\infty$. If G is one of these groups, it is easily verified that there is a set of subgroups with the properties listed in Lemma 9. Therefore $\mathbb{Z}G$ has uncountably many maximal ideals in $\mathbb{Z}G$, and the theorem is established. □

Whether the converse of Theorem 10 is true remains open, but we are able to prove it in the case of abelian-by-finite groups.

Theorem 11. *Let G be a finite extension of an abelian minimax group without factors of type $p^\infty \times p^\infty$ for any prime p. Then $\mathbb{Z}G$ satisfies* **CI**.

Proof. In the first place by Proposition 2 we may assume G to be abelian and minimax. Let M be a fixed maximal ideal of $\mathbb{Z}G$ and write $F = \mathbb{Z}G/M$, which is a simple G-module and also a field. We show that $Q = G/C_G(F)$ is locally (finite cyclic).

There is a monomorphism from Q into F^*, the multiplicative group of F, whose image generates F as a ring; we regard Q as a subgroup of F^*. Since Q is minimax, it has a free abelian subgroup A such that Q/A is periodic. Let $R = Rg\langle A\rangle$, the subring of F generated by A. Since each element of Q has a positive power in A and $F = Rg\langle Q\rangle$, it follows that F is integral over R. But F is a field, so R is also a field. Since R is finitely generated as a ring, it is finite and therefore $A = 1$, showing that Q is periodic. In addition $Q = G/C_G(F)$ acts on F as an irreducible group of automorphisms, so we can apply a theorem of R. Baer ([1]; see also [13: 5.26]) to show that F has characteristic $p > 0$ and hence Q is a locally cyclic p'-group.

Next assume that $\mathbb{Z}G$ has uncountably many maximal ideals M. Clearly we may suppose that $I_G \not\leq M$. There are only countably many subgroups $C_G(\mathbb{Z}G/M)$, from which we deduce that there are uncountably many M's with $C_G(\mathbb{Z}G/M) = C$ fixed and $\mathbb{Z}G/M$ elementary abelian p for some p. Since $\bar{I}_C \leq M$, we can pass to $\mathbb{Z}G/\bar{I}_C \simeq \mathbb{Z}(G/C)$ and assume that $C = 1$. Thus G is a locally cyclic p'-group.

We know that G acts faithfully on $\mathbb{Z}G/M$ and p does not belong to the spectrum of G; also of course M is a prime ideal. We are therefore in a position to apply a result of D. Segal [14: Corollary 1.2] to conclude that M is finitely generated as an ideal of $\mathbb{Z}G$. However, this means that there only countably many M's. $\square$

An immediate application of Theorems 10 and 11 is a characterization of locally finite groups whose group rings satisfy **CI**.

Corollary 4. *Let G be a locally finite group. Then $\mathbb{Z}G$ satisfies* **CI** *if and only if G is a Černikov group whose maximum divisible subgroup is locally cyclic.*

Proof. Assume that $\mathbb{Z}G$ has **CI**. By Lemma 8 and Theorem 10 abelian subgroups of G are Černikov groups. The well known theorem of V.P. Šunkov [15] shows that G is a Černikov group. It follows from Theorem 10 that the maximum divisible subgroup of G is locally cyclic. The converse is true by Theorem 11. $\square$

6 Soluble groups with uncountably many maximal subgroups

One approach to constructing soluble groups that do not have **CG** is to look for group rings which do not satisfy **CI**. Let Q be a group such that $\mathbb{Z}Q$ does not have **CI**. Then the natural semidirect product $G = Q \ltimes \mathbb{Z}Q \simeq \mathbb{Z}\ wr\ Q$ has uncountably many maximal subgroups of the form QM where M is a maximal right ideal of $\mathbb{Z}Q$. If Q is soluble or finitely generated, then so is G.

Examples.

(i) Let Q be a group of type $p^\infty \times p^\infty$ where p is a prime. Then $\mathbb{Z}Q$ does not have **CI** by Theorem 10, so $G = \mathbb{Z}\ wr\ Q$ is a countable metabelian group that does not have **CG**.

(ii) Let q be a prime and put $Q = \mathbb{Z}_q\ wr\ \mathbb{Z}$. Since the base group of the wreath product is infinite elementary abelian, $\mathbb{Z}Q$ does not have **CI**. Hence $G = \mathbb{Z}\ wr\ Q$ is a finitely generated soluble group of derived length 3 which does not have **CG**. This is the minimum derived length possible here since finitely generated metabelian groups satisfy **CG**.

(iii) Let $H = \langle x, a \mid a^x = a^p \rangle$ where p is a prime, and put $Q = H \times H$. Because Q has a quotient of type $p^\infty \times p^\infty$, its group ring does not have **CI**. Consequently, $G = \mathbb{Z}\ wr\ Q$ is a finitely generated, torsion-free soluble group of derived length 3 that does not satisfy **CG**.

(iv) Let A be a non-trivial countable abelian group. By a well known construction of P. Hall ([5]; see also [8: 4.1.3]), there is a finitely generated centre-by-metabelian group Q whose centre is isomorphic with A. If we choose A so that $\mathbb{Z}A$ does not have **CI**, then neither will $\mathbb{Z}Q$, and the group $\mathbb{Z}\ wr\ Q$ will fail to have **CG**.

(v) The final example is an explicit construction of uncountably many maximal right ideals in a group algebra in case (ii). The idea is due to Y. de Cornulier [3]. Since the construction has not appeared in print form, we present an account of a generalized version.

Proposition 3. *Let p, q be primes such that $p \equiv 1 \mod q$ and let $Q = \mathbb{Z}_q\ wr\ \mathbb{Z}$. Then $\mathbb{Z}_pQ$, and hence $\mathbb{Z}Q$, does not satisfy* **CI**.

Proof. Write $Q = \langle u_0 \rangle\ wr\ \langle t \rangle$ where $u_0^q = 1$. Let M be a $\mathbb{Z}_p$-vector space with a countably infinite basis $\{e_i | i \in \mathbb{Z}\}$. Since q divides $p-1$, there is a primitive qth root of unity ω in $\mathbb{Z}_p$. Denote by F the set of all functions from $\mathbb{Z}$ to $\langle \omega \rangle$. Each choice of ϕ from F turns M into a right Q-module M_ϕ by means of the rules

$$(e_i)u_0 = \phi(i)e_i, \quad (e_i)t = e_{i+1}, \ (i \in \mathbb{Z}).$$

Writing $u_k = u_0^{t^k}$, we have $(e_i)u_k = \phi(i-k)e_i$. Regarding F as the set of all bi-infinite sequences of elements from $\langle \omega \rangle$, we define F_0 to be the subset of all $\phi \in F$ such that every finite sequence of elements of $\langle \omega \rangle$ occurs as a (consecutive) subsequence of ϕ. Notice that F_0 is uncountable.

We show next that M_ϕ is a simple Q-module if $\phi \in F_0$. To prove this let $0 \neq m \in M_\phi$ and write $m = \sum_{i=k}^{\ell} a_i e_i$ where $k \leq \ell, a_i \in \mathbb{Z}_p, a_\ell \neq 0$. By assumption the sequence $\omega, \omega, \dots, \omega, 1$, with $\ell - k$ ω's, occurs as a subsequence of ϕ. Hence there is a conjugate $\bar{u}$ of u_0 such that $(e_i)\bar{u} = \omega e_i$ for $i = k, k+1, \dots, \ell - 1$ and $(e_\ell)\bar{u} = e_\ell$. Therefore the submodule $S = (m)\mathbb{Z}Q$ contains the element

$$m\Big(\sum_{j=0}^{q-1} \bar{u}^j\Big) = \sum_{i=k}^{\ell-1} a_i \Big(\sum_{j=0}^{q-1} \omega^j\Big) e_i + q a_\ell e_\ell = q a_\ell e_\ell.$$

Hence $q a_\ell e_\ell \in S$ and $e_\ell \in S$. It follows that S contains all the e_i, so $S = M_\phi$ and M_ϕ is a simple Q-module.

Next define a Q-module homomorphism $\pi_\phi : \mathbb{Z}_pQ \to M_\phi$ by $(1)\pi_\phi = e_0$. Since M_ϕ is simple, π_ϕ is surjective and $M_\phi \overset{\mathbb{Q}}{\simeq} \mathbb{Z}Q/K_\phi$

where $K_\phi = \mathrm{Ker}(\pi_\phi)$, which is a maximal right ideal of $\mathbb{Z}Q$. Suppose that $K_\phi = K_{\bar\phi}$ where $\phi, \bar\phi \in F_0$. For any j we have $(u_j)\pi_\phi = ((1)\pi_\phi)u_j = (e_0)u_j = \phi(-j)e_0$. Hence $(u_j - b)\pi_\phi = (\phi(-j) - b)e_0$ for any $b \in \mathbb{Z}_p$. Thus $u_j - b \in K_\phi \Leftrightarrow \phi(-j) = b$. Suppose that $\phi \neq \bar\phi$, so that $\phi(-j) = b \neq \bar b = \bar\phi(-j)$ for some $j \in \mathbb{Z}$. Hence $u_j - b \in K_\phi = K_{\bar\phi}$, so that $\bar\phi(-j) = b$ and $b = \bar b$. By this contradiction $\phi = \bar\phi$. It follows that the map $\phi \mapsto K_\phi$ is injective, which shows that $\mathbb{Z}_pQ$ has uncountably many maximal right ideals K_ϕ, $\phi \in F_0$. □

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