A remark about central automorphisms of groups

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To Guido Zappa on his 90th birthday

The aim of this paper is to prove and discuss the following simple result on central automorphisms, that, although elementary and not surprising at all, and therefore possibly known, seems to have not been recorded in the literature yet.

Proposition. *Let Z be a subgroup of the centre of a group G, and suppose that Z has finite exponent n. Also suppose that G has no nontrivial direct factor contained in Z. Let Γ be the group of those automorphisms of G acting trivially on G/Z. Then*

- (i) *Γ has finite exponent dividing n;*
- (ii) *Γ acts nilpotently on G. More precisely, if k is the least positive integer such that n is not divisible by the $(k+1)$th power of any prime, then $[Z, {}_{2k-1}\Gamma] = 1$;*
- (iii) *Γ is nilpotent of class at most $2k-1$, where k is defined as in (ii).*

Of course the word ‘abelian’ in the statement can be equivalently replaced by ‘cyclic’. As it always happens when dealing with central automorphisms, some hypothesis to exclude the presence of certain kinds of abelian direct factors is necessary here, the reason being that every automorphism of a direct factor of a group obviously extends to an automorphism of the group acting trivially modulo the factor itself. It is worth recalling that to every $\gamma \in \Gamma$ there corresponds the homomorphism $\gamma^* : g \in G \mapsto [g, \gamma] \in Z$; if, as we are assuming, no nontrivial direct factor of G is contained in Z, in the further hypothesis that Z is finite Fitting Lemma shows that the mapping $\gamma \mapsto \gamma^*$ defines a bijection between Γ and $\mathrm{Hom}(G, Z)$ (see [1, 3]).

Part (i) of the Proposition can be compared to a theorem by Dixon and Evans [2], according to which if G is a group and $Z(G)$ has finite exponent then G has a central automorphism of infinite order if and only if G has an infinite abelian direct factor (in which case G admits an uncountable torsion-free abelian group of central automorphisms). Easy examples show that if $Z(G)$ is only assumed to be periodic then G may have nonperiodic central automorphism even if it has no notrivial abelian direct factors. For instance, for any periodic abelian group A of infinite exponent, one such example is the well-known class-2 nilpotent group $G = (A \times Z) \rtimes \langle x \rangle$, where the assignment $a \mapsto [a, x]$ gives an isomorphism from A to $Z = Z(G)$; if, instead, $n = \exp A$ is supposed to be finite, then the same group provides an instance where the exponent of Γ, as in part (i) of the Proposition, is exactly n.

Next we shall prove the Proposition. Throughout we will assume that G and Z are chosen as in the hypothesis. The first remark is that if p is a prime dividing n and $Z_{p'}$ is the p'-component of Z then $G/Z_{p'}$ and $Z/Z_{p'}$ inherit the hypothesis from G and Z. In fact, if $A/Z_{p'}$ is a direct factor of $G/Z_{p'}$ contained in $Z/Z_{p'}$ then $A = Z_{p'} \times B$ for some subgroup B, and it follows that B is a direct factor of G contained in Z, hence $B = 1$ and so $A = Z_{p'}$.

This remark allows us to reduce the proof of the Proposition to the case when Z is a p-group. The statement will be a consequence of the property established in the following Lemma, which is in fact equivalent to the absence of nontrivial direct factors of G contained in Z.

Lemma. *In the given hypotheses, further suppose that Z is a p-group for some prime p. Then $\Omega_1\big(Z^{p^i}\big) \le G'G^{p^{i+1}}$ for every natural number i.*

Proof. Let $x \in Z$ and suppose that $y := x^{p^i}$ has order p. If $y \notin N := G'G^{p^{i+1}}$ then $\langle x \rangle \cap N = 1$ and xN has order $p^{i+1} = \exp(G/N)$. Hence $\langle xN \rangle$ is a direct factor of G/N, and it follows that $\langle x \rangle$ is a direct factor of G. This is excluded by the hypotheses, hence $y \in N$. □

From now on we will assume that Z is a nontrivial p-group and let $n = p^\lambda$. To prove that $\Gamma^n = 1$ we can argue by induction on λ. Let $m = p^{\lambda-1}$. Then $Z^m \le G'G^n$ by the Lemma. Now Γ, as a group of central automorphisms, centralizes G', and $[G^n, \Gamma] = [G, \Gamma]^n = 1$, because $[G, \Gamma] \le Z(G)$. Therefore Γ centralizes Z^m, and so it acts trivially on Z/Z_1, where $Z_1 = \Omega_{\lambda-1}(Z)$. Therefore $\Gamma/C_\Gamma(G/Z_1)$ can be embedded in $\mathrm{Hom}(G/Z, Z/Z_1)$, that has exponent p at most, hence $[G, \Gamma^p] \le Z_1$. By applying the induction hypothesis to G and Z_1 we see that the exponent of Γ^p divides $m = \exp Z_1$, hence $\Gamma^n = 1$. This proves part (i) of the Proposition.

Now decompose Z as the direct product of homocyclic factors: $Z = A_1 \times A_2 \times \cdots \times A_t$, where each of the A_i is homocyclic of exponent p^{e_i} and the integers e_i are such that $k \ge e_1 > e_2 > \cdots > e_t > 0$. For each i we have $\Omega_1(A_i) = A_i^{p^{e_i-1}} \le G'G^{p^{e_i}}$, by the Lemma, hence $[\Omega_1(A_i), \Gamma] \le [G'G^{p^{e_i}}, \Gamma] = [G, \Gamma]^{p^{e_i}} \le Z^{p^{e_i}}$, so that $[\Omega_1(A_i), \Gamma] \le \Omega_1(A_1 A_2 \cdots A_{i-1})$. As a further consequence, for every positive integer $j \le e_i$, since $\Omega_j(A_i) \le Z^{p^{e_i-j}}$,

$$\begin{aligned}
[\Omega_j(A_i), \Gamma] &\le \Omega_j(A_1 A_2 \cdots A_{i-1}) \times \Omega_{j-1}(A_i A_{i+1} \cdots A_t) \cap Z^{p^{e_i-j}} \\
&= \Omega_j(A_1 A_2 \cdots A_{i-1}) \times \Omega_{j-1}(A_i) \times \prod_{s=i+1}^{t} \Omega_{j-(e_i-e_s)}(A_s),
\end{aligned} \tag{$*$}$$

where $\Omega_r(A_s)$ is 1 if $r < 0$. For every $z \in Z$ define z^τ as follows: if $z = 1$ let $z^\tau = 2$; if $1 \ne z \in A_i$ for some i, let $z^\tau = i + 2n$, where p^n is the order of z and finally, if $z = \prod_{i=1}^t a_i$, where $a_i \in A_i$ for all i, let $z^\tau = \max\{a_i^\tau \mid 1 \le i \le t\}$. We can also extend to scope of τ to subgroups of Z, by setting $H^\tau = \max\{h^\tau \mid h \in H\}$ for all $H \le Z$. Now $(*)$ shows that $[H, \Gamma]^\tau < H^\tau$ for all nontrivial subgroups H of Z (note that $z^\tau > 2 = 1^\tau$ for all $z \in Z \smallsetminus 1$). Since $Z^\tau = 2e_1 + 1 \le 2k + 1$ it easily follows that $[Z, {}_{2k-1}\Gamma]^\tau \le 2$, hence $[Z, {}_{2k-1}\Gamma] = 1$, as stated in (ii). It also follows that $[G, {}_{2k}\Gamma] = 1$, which yields (iii). Thus the Proposition is proved. □

It would have also been possible to prove part (i) by using the equality $[g, {}_n\gamma] = \prod_{i=1}^n [g, {}_i\gamma]^{\binom{n}{i}}$, true for all $g \in G$ and $\gamma \in \Gamma$, where $n = p^\lambda = \exp Z$, and an argument like that employed for part (ii) to deal with the commutators of the form $[g, {}_i\gamma]$.

The bounds $2k - 1$ in parts (ii) and (iii) of the Proposition are actually attained, at least in the case when Z is not a 2-group. An example is the following. Let p be a prime, let k be an integer greater than 1 and let

$$G = \left\langle u_1, u_2, g_1, g_2, x \;\middle|\; \begin{matrix} u_i^p = g_i^{p^k} = x^{p^k} = 1, & [u_i, g_i] = x^{p^{k-1}} & \forall i, j \in \{1, 2\} \\ [x, g_i] = [x, u_i] = [g_1, g_2] = [u_1, u_2] = [u_i, g_j] = 1, & & \text{if } i \ne j \end{matrix} \right\rangle.$$

Then $Z := Z(G) = \langle g_1^p, g_2^p, x \rangle$ has exponent p^k and G has no nontrivial abelian direct factor. Consider the central automorphisms α and β of G defined by:

$$\alpha, \beta\colon \begin{cases} u_1 \mapsto u_1 \\ u_2 \mapsto u_2 \\ g_1 \mapsto g_1 g_2^p \\ g_2 \mapsto g_2 x \end{cases} \qquad \begin{matrix} \alpha\colon x \mapsto g_1^p x \\ \\ \beta\colon x \mapsto g_2^p x. \end{matrix}$$

It is easy to check that $[x, {}_{2(k-1)}\beta] = x^{p^{k-1}} \ne 1$; this shows that the bound in (ii) is sharp. If p is odd it can be shown that $[\alpha, {}_{2(k-1)}\beta] \ne 1$, thus not only $\langle \alpha, \beta \rangle$ has class $2k - 1$, but it is not even a $2(k-1)$-Engel group. We sketch a proof of this fact. Every element of $\Gamma_0 := \langle \alpha, \beta \rangle$ acts trivially on $\langle u_1, u_2 \rangle$, hence it is described by its action on the homocyclic group $\langle g_1, g_2, x \rangle$, and so by (the image mod p^k of) a 3×3 matrix over the p-adic integers, taken with respect to the basis

(g_1, g_2, x). Thus α and β are represented by

$$A = \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & 1 \\ p & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & 1 \\ 0 & p & 1 \end{pmatrix}.$$

Exactly as in the proof of the Proposition, decompose $Z = Z(G)$ as the direct product of the homocyclic groups $A_1 = \langle x \rangle$ (of exponent p^k) and $A_2 = \langle g_1^p, g_2^p \rangle$ (of exponent p^{k-1}), and define a weight map τ accordingly. It follows that $[\langle g_1, x\rangle, \gamma_n(\Gamma_0)]^\tau \leq [\langle g_1, x\rangle, {}_n\,\Gamma_0]^\tau \leq 2k - n + 1$ and $[\langle g_2\rangle, \gamma_n(\Gamma_0)]^\tau \leq [\langle g_2\rangle, {}_n\,\Gamma_0]^\tau \leq 2k - n + 2$ for all positive integers $n < 2k$. Now, for every positive integer $t < k$, the elements $z \in Z$ such that $z^\tau \leq 2(k-t)$ (resp. $z^\tau \leq 2(k-t)+1$) are those in $\Omega_{k-t-1}(Z)$ (resp. $\Omega_{k-t-1}(Z)\langle x^{p^t}\rangle$); hence, by setting $n = 2t+1$ above, we see that a matrix representing $[\alpha, {}_{2t}\beta]$ must have the form

$$S_t = \begin{pmatrix} {}_t\lambda_1 & {}_t\theta_{12} & {}_t\theta_{13} \\ {}_t\theta_{21} & {}_t\lambda_2 & {}_t\theta_{23} \\ {}_t\theta_{31} & {}_t\theta_{32} & {}_t\lambda_3 \end{pmatrix} \qquad \text{where} \qquad \begin{aligned} &{}_t\lambda_1 \equiv {}_t\lambda_2 \equiv {}_t\lambda_3 \equiv 1 \pmod{p^{t+1}} \\ &{}_t\theta_{23} \equiv 0 \pmod{p^t} \\ &{}_t\theta_{ij} \equiv 0 \pmod{p^{t+1}} \text{ for all other values of } i \text{ and } j. \end{aligned}$$

By direct (although not immediate) computation one can show that the $(2,3)$- and the $(3,2)$-entries of these matrices also satisfy the relations ${}_t\theta_{32} \equiv -p\,{}_t\theta_{23} \pmod{p^{t+2}}$; indeed, one sees that the corresponding congruence holds (for the appropriate value of t) for the matrix obtained by commuting twice either of A or S_t with B. Once this has been established, it is not hard to check that ${}_{(t+1)}\theta_{23} \equiv 4pd^{-1}{}_t\theta_{23} \pmod{p^{t+2}}$ for all integers t such that $1 \leq t < k$, where $d = 1 - p$ is the determinant of B. As ${}_1\theta_{23}$ is not divisible by p^2 and $p > 2$, it follows that p^k does not divide ${}_{(k-1)}\theta_{23}$, which amounts to saying that $[\alpha, {}_{2(k-1)}\beta]$ does not fix g_2, and so $[\alpha, {}_{2(k-1)}\beta] \neq 1$, as claimed.

References

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