Designs and graph decompositions over finite fields

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A 2-(v, K, λ) design over the finite field $\mathbb{F}_q$ is a collection $\mathcal{S}$ of subspaces of the vector space $\mathbb{F}_q^v$ with dimensions from the set K and the property that any 2-dimensional subspace of $\mathbb{F}_q^v$ is contained in exactly λ members of $\mathcal{S}$. Of course it can be also viewed as a collection $\overline{\mathcal{S}}$ of subspaces of the projective space $\mathrm{PG}(v-1, q)$ with dimensions from $\{k-1 \mid k \in K\}$ such that any two distinct points belong to exactly λ members of $\overline{\mathcal{S}}$. It is trivial when $K = \{2\}$. When $K = \{k\}$ is a singleton, one simply writes "k" rather than "$\{k\}$".

To construct these designs seems to be quite hard. Indeed, in spite of the fact that the topic received a considerable amount of attention over the years, the well-celebrated 2-$(13, 3, 1)$ design over $\mathbb{F}_2$ recently obtained by Braun et al. [1] with the use of the Kramer-Mesner method is the only non-trivial example having $\lambda = 1$ known at this moment. Also, for $\lambda > 1$ only few theoretical constructions are known. Among them, we have the existence of a 2-$(v, 3, 7)$ design over $\mathbb{F}_2$ for any $v \equiv \pm 1 \pmod 6$ obtained by Thomas [2].

In the first part of my talk I will show how we used difference methods in order to extend Thomas result to the case $v \equiv 3 \pmod 6$ and to multiple dimension sizes.

In the second part, starting from the very well-known remark that a classic 2-(v, k, λ) design can be viewed as a decomposition of the λ-fold of the complete graph of order v into cliques of size k, I will propose the new notion of a *graph decomposition over a finite field* presenting several concrete constructions such as a decomposition of the complete graph on the points of $\mathrm{PG}(6, 2)$ into heptagons each of which has vertex-set coinciding with the point-set of a plane.

Bibliography

[1] M. Braun, T. Etzion, P.R.J. Ostergard, A. Vardy, A. Wassermann, *Existence of q-analogs of Steiner Systems*, *Forum of Mathematics, Pi*, 4 (2017).

[2] S. Thomas, Designs over finite fields, *Geom. Dedicata*, 93 (1987), 237–242.