Extra: some 2-groups with s-breadth 1

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This note adds some bits of information to our paper *On finite p-groups with subgroups of breadth 1*, Bull. Aus. Math. Soc., 82 (2010), pp. 84–98. There, in the remarks following Lemma 2.8, we make a couple of unproven statements describing the groups for which 'case (iii)' occurs. Here we prove these statements.

First, suppose that G is a group as in Lemma 2.8, with an involution a contained in a nonabelian subgroup $H = \langle a, b\rangle$. Further suppose that $C := C_G(H)$ is not a Dedekind group. Lemma 2.8 and its proof provide the following information. $H = V \rtimes \langle b\rangle$, where $V = \langle a, a^b\rangle \simeq V_4$, the Klein four-group, and b has order 8. Moreover, $|C/Z(C)| = 4$. If $g \in C \smallsetminus Z(C)$ then g must have order 4 and either $g^2 = u := aa^b$ (if $\langle g\rangle \ntriangleleft C$) or $g^2 = ub^4$ is the generator of C' (if $\langle g\rangle \triangleleft C$). Theorem B yields $|G/Z(G)| \leq 16$. But $Z(G) \leq Z(C)$ and $|HC/Z(C)| = |H/Z(H)|\,|C/Z(C)| = 16$, therefore $Z(G) = Z(C)$ and $G = HC$.

We have $C = \langle c, d\rangle Z(C)$, for some noncommuting elements c, d (of order 4). Also, $b^2 \in Z(C)$. Since $(cb^2)^2 = c^2b^4 \neq c^2$ one of $\langle c\rangle$ and $\langle cb^2\rangle$ is normal in C and the other isn't. The same holds for $\langle d\rangle$ and $\langle db^2\rangle$. So we may assume $\langle c\rangle \ntriangleleft C$ and $\langle d\rangle \triangleleft C$. Let $K = \langle c, d\rangle$, hence $K = \langle d\rangle \rtimes \langle c\rangle$ is the nonabelian semidirect product of two cyclic groups of order 4, and $C = KZ(C)$. As $\exp C = 4 = \circ(b^2)$, we also have $Z(C) = \langle b^2\rangle \times E_0$ for some $E_0 \leq C$. If E_0 has an element e of order 4 then c, cb^2 and ce are three elements of $C \smallsetminus Z(C)$ with different squares, which cannot exist. Therefore $E_0^2 = 1$. Now $Z(C) \cap (HK) = \langle u\rangle \times \langle b^2\rangle$, and there is no loss of generality in assuming $u \in E_0$. Then $\langle u\rangle$ has a complement E in E_0, and it follows that $G = HC = HK \times E$. Thus the structure of G is almost completely described, the only parameter which is not prescribed being the rank of E. In fact H and K are described, $[H, K] = 1$ and the amalgamation in $H \cap K$ is also given by Lemma 2.8: $c^2 = aa^b$ and $d^2 = aa^bb^4$.

Conversely, now we shall prove that a group G with this structure has subgroup breadth 1. First we describe the elements of order 2 in G. Let $g = hke \in G$, where $h \in H$, $k \in K$ and $e \in E$. Then $g^2 = 1$ if and only if $h^2k^2 = 1$. Since $\exp K = 4$ this condition means $h^2 = k^2$. It is easy to check that the sets of all squares in H and K are $S(H) = \langle b^2\rangle \cup \{aa^bb^2, aa^bb^{-2}\}$ and $S(K) = \{1, c^2, d^2\}$ respectively, and $S(H) \cap S(K) = \{1\}$. Therefore, if $g^2 = 1$ then $h^2 = k^2 = 1$ and $k \in H$ because $\Omega_1(K) \leq H$. It follows that $\Omega_1(G) = \Omega_1(H) \times E$. Now $\Omega_1(H) = \langle a, a^b, b^4\rangle$ where $b^4 \in Z(G)$ and $V_4 \simeq \langle a, a^b\rangle \triangleleft G$, hence $C_G(\Omega_1(G)) = C_G(\langle a, a^b\rangle)$ has index 2 in G. Therefore

$$\text{if } U \leq G \text{ and } \exp U \leq 2 \text{ then } \mathrm{sbr}_G(U) \leq 1. \tag{1}$$

Now suppose, by contradiction, that U is a subgroup of G whose s-breadth in G is greater than 1. We may choose U minimal with respect to this property; hence we cannot have $U = U_1U_G$ for any $U_1 < U$, because otherwise $N_G(U_1) \leq N_G(U)$ and $\mathrm{sbr}_G(U_1) \geq \mathrm{sbr}_G(U)$. Therefore

$$U_G \leq \Phi(U) = U^2. \tag{2}$$

Also, $\exp U > 2$, by (1), so U contains an element u of order 4. Thus u^2 is a square and an involution. We have $u^2 = h^2k^2$ for some $h \in H$ and $k \in K$. Since $K^4 = 1$ also h^2 must have order 2 at most. The above description of $S(H)$ and $S(K)$ yields $h^2 \in \langle b^4\rangle$ and $u^2 \in \{b^4, c^2, d^2\}$, since $d^2 = b^4c^2$.

$Z(H) = \langle aa^b, b^2\rangle$ and $Z(K) = \langle c^2, d^2\rangle$ have index 4 in H and K respectively. Hence, if N is one of $H' = \langle c^2\rangle$ and $K' = \langle d^2\rangle$, and $\bar{G} = G/N$ then $|\bar{G}/Z(\bar{G})| = 4$, so $\mathrm{sbr}(\bar{G}) \leq 1$. This shows that neither c^2 nor d^2 is contained in U. By the previous paragraph $b^4 = u^2 \in U$.

Now let bars denote images modulo $\langle b^4\rangle$. Note that $\bar{G}$ is a central product $\bar{H}(\bar{K}\bar{E})$, where $\bar{K}\bar{E}$ is hamiltonian and $\bar{H} \cap (\bar{K}\bar{E}) = \bar{H}' = \bar{K}' = \bar{G}'$. Then $\bar{U} \cap \bar{K}\bar{E} \triangleleft \bar{G}$, but $b^4 \in U \cap KE$, hence $U \cap KE \leq U^2$ by (2). So $U \cap KE \leq G^2 \cap KE = K'$. Clearly $\bar{G}' \nleq \bar{U}$; it follows that $\bar{K} \cap \bar{U} = 1$, therefore $\bar{U} \cap \bar{K}\bar{E} = 1$, so $\bar{U}$ is embedded in $\bar{G}/\bar{K}\bar{E} \simeq \mathcal{C}_4 \times \mathcal{C}_2$. Hence $\bar{U} = \langle \bar{x}, \bar{y}\rangle$, where $\bar{x}^2 = 1$ and $[\bar{x}, \bar{y}] = 1$.

We can compute the involutions in $\bar{G}$ as we did for G above. If $h \in H$ and $k \in K$ are such that $\bar{h}^2 = \bar{k}^2 \neq 1$ then k^2 is one of h^2 and $h^2b^4 = (hb^2)^2$, but we already know that neither can happen. Hence $\bar{h}^2 = \bar{k}^2$ if and only if $\bar{h}^2 = \bar{k}^2 = 1$. It follows that $\Omega_1(\bar{G}) = \Omega_1(\bar{H})\bar{E}$, and $C := C_{\bar{G}}(\Omega_1(\bar{G})) = \Omega_1(\bar{H})\bar{K}\bar{E} = \langle \bar{a}, \bar{b}^2\rangle \times \bar{K} \times \bar{E}$ is a hamiltonian group and is maximal in $\bar{G}$. Also note that $|\Omega_1(\bar{G})Z(\bar{G})/Z(\bar{G})| = 2$.

Going back to $\bar{U} = \langle \bar{x}, \bar{y}\rangle$, assume first that $\bar{x} \in Z(\bar{G})$. In this case $C_{\bar{G}}(\bar{U}) = C_{\bar{G}}(\bar{y})$. But $|\bar{G}'| = 2$, hence this latter centralizer has index 2 at most in G, and we have a contradiction (alternatively, we could have used (2) and the fact that $\bar{x} \notin \bar{U}^2$ to obtain a contradiction). Thus $\bar{x} \notin Z(\bar{G})$, so $\Omega_1(\bar{G}) = \langle \bar{x}\rangle\Omega_1(Z(\bar{G}))$ and $C_{\bar{G}}(\bar{x}) = C$, hence $\bar{U} \leq C$. As C is hamiltonian and maximal in $\bar{G}$ we conclude that $\mathrm{sbr}_{\bar{G}}(\bar{U}) \leq 1$. As $b^4 \in U$, this is a contradiction, and the proof is complete.