Residually finite subgroups of some countable McLain groups

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Abstract. The McLain groups are groups of finitary linear transformations that, besides being of considerable interest in their own right, constitute a most useful source of counterexamples to conjectures arising in the investigation of locally nilpotent groups. In this article it is shown that certain torsion-free McLain groups have subgroups "of periodic index" that are residually finite. This has a bearing on the question as to which McLain groups might embed in simple locally soluble-by-finite groups.

Introduction

In [2] it was proved that a countable torsion-free locally nilpotent group G that is also soluble has a residually finite subgroup J "of periodic index" in G, that is, whose isolator in G is G itself. (In fact, a little more than this was shown, but the question at hand soon reduced to consideration of torsion-free groups.) In [1] it was shown that any countable group G that embeds in a simple group that is locally (soluble-by-finite) must at least have a residually finite subgroup of periodic index, and this observation provided the motivation for the discussion in [2]. A natural next step in this investigation is to examine some of the well-known McLain groups, and especially perfect torsion-free ones, as these are so far from being soluble that thay might provide counterexamples to the conjecture that countable locally nilpotent groups always contain such residually finite subgroups J of periodic index. We describe the basic properties of McLain groups below, though a more thorough presentation will be found in Chapter 6 of [5]. Also, we refer the reader to [4] for the fundamental properties of isolators in locally nilpotent groups; in particular, if every element of a generating set for the locally nilpotent group G has a non-zero power lying in some fixed subgroup H of G then the same holds for *every* element of G. We decided that the first McLain group to examine was $M(\mathbb{Q},\mathbb{Z})$, but during the course of showing that this group does indeed have a residually finite subgroup of the required type we realised that much the same argument holds for certain other rings besides $\mathbb{Z}$, and this allows us to present our main result in a somewhat more general setting. We remark that every countable ordering embeds in that of the rational numbers, and so the further requirement that the basis elements of the underlying vector space be indexed by $\mathbb{Q}$ is not really a restriction at all.

Theorem. *Let A be a commutative ring with identity whose additive group has finite torsion-free rank and let R be a ring embeddable in a polynomial ring on a (possibly infinite) set of commuting indeterminates over A. Then the McLain group $M(\mathbb{Q},R)$ has a residually finite subgroup of periodic index.*

A possible choice for A is, for instance, any finite extension of the rational field, or, more generally, every commutative ring generated by $\mathbb{Q}$ and finitely many algebraic elements.

We have been unable to decide what happens for arbitrary (countable) rings, and it does appear that a definitive result here would require the introduction of further techniques. Of course, there are several intermediate questions that present themselves, and we might ask, for example, what happens if R is the ring of all algebraic integers, or perhaps a finitely generated ring.

1. Notation and basic properties of McLain groups

We start by fixing some notation and recalling some basic facts about McLain groups. Given a ring R with identity 1, we can consider "matrices" with entries in R whose rows and columns are indexed by $\mathbb{Q}$, ordered by the standard ordering of the rationals. The McLain group $M(\mathbb{Q},R)$ consists of all such matrices which are upper unitriangular and have finitely many nonzero entries off the "main diagonal". Let P be the set of all ordered pairs (i,j) of rational numbers such that $i<j$ and, for all $(i,j)\in P$, let e_{ij} denote the matrix whose only nonzero entry is the (i,j)-entry, which is 1. As we confirm below, $M(\mathbb{Q},R)$ is generated by all elements of the form ${}^a t_{ij} := 1+ae_{ij}$, where $a\in R$ and $(i,j)\in P$ (and 1 is here the identity matrix); we call each such element an *elementary generator*. We extend this terminology, along with the following, to the case when R is a ring without identity: in this case R embeds in a ring R_1 with identity and we consider only matrices over R_1 whose entries off the main diagonal are in R; thus $M(\mathbb{Q},R)$ will be the subgroup of $M(\mathbb{Q},R_1)$ generated by the elementary generators of the form ${}^a t_{ij}$ with $a\in R$; it is clear that this group is essentially independent of the choice of R_1.

We say that an elementary generator ${}^a t_{ij}$ has *position* (i,j) and *level* j. If ${}^b t_{kl}$ is another elementary generator we have ${}^a t_{ij}\,{}^b t_{kl} = 1+ae_{ij}+be_{kl}$ if $j\neq k$, because $e_{ij}e_{kl}=0$ under this condition (while ${}^a t_{ij}\,{}^b t_{jl} = 1+ae_{ij}+be_{jl}+abe_{il}$

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if $i < j < l$). In particular, elementary generators of the same level commute; moreover, every matrix $(a_{ij}) \in M(\mathbb{Q}, R)$ can be written as

$$1 + \sum_{(i,j)\in F} a_{ij} e_{ij} = \prod_{(i,j)\in F} {}^{a_{ij}}t_{ij} \tag{1}$$

where F is the (finite) set of all $(i,j) \in P$ such that the entry a_{ij} is not zero and the factors of the product on the right hand side are ordered in such a way that ${}^{a_{uv}}t_{uv}$ comes before ${}^{a_{rs}}t_{rs}$ if $v > s$. In other words, any element of $M(\mathbb{Q}, R)$ can be written as a product of elementary generators of non-increasing levels. Also, if the additive group of R is generated by a set X then

$$(a_{ij}) = 1 + \sum_{(i,j)\in F} \Big(\sum_{x\in X_0} \lambda_{xij} x\Big) e_{ij} = \prod_{(i,j)\in F} \Big(\prod_{x\in X_0} {}^{x}t_{ij}^{\lambda_{xij}}\Big) \tag{2}$$

for a finite subset X_0 of X and suitable integers λ_{xij} (note that ${}^{\lambda x}t_{ij} = {}^{x}t_{ij}^{\lambda}$ for all $\lambda \in \mathbb{Z}$ and for all elementary generators ${}^{x}t_{ij}$). Thus $M(\mathbb{Q}, R) = \langle {}^{x}t_{ij} \mid x \in X,\ (i,j) \in P \rangle$.

Other useful relations are: for all $(i,j),(k,l) \in P$ and $a, b \in R$,

$$\big[{}^{a}t_{ij}, {}^{b}t_{jk}\big] = {}^{ab}t_{ik} \quad \text{if } j < k, \qquad \text{and} \qquad \big[{}^{a}t_{ij}, {}^{b}t_{kl}\big] = 1 \quad \text{if } j \neq k \text{ and } i \neq l, \tag{3}$$

which also imply $[{}^{a}t_{ij}, {}^{b}t_{jk}, {}^{a}t_{ij}] = 1$ in any case. We shall use these remarks to obtain a more explicit version of what we noticed after equation (1). Namely, if $g = {}^{a}t_{ij}\,{}^{b}t_{kl}$ is a product of two elementary generators and $j < l$ then we can rewrite g as follows: either $g = {}^{b}t_{kl}\,{}^{a}t_{ij}$ (if $j \neq k$) or $g = {}^{b}t_{kl}\,{}^{ab}t_{il}\,{}^{a}t_{ij}$ (if $j = k$), since ${}^{ab}t_{il} = [{}^{a}t_{ij}, {}^{b}t_{jl}]$ commutes with ${}^{a}t_{ij}$; in either case g is now a product of elementary generators *of the same levels as before*, but rearranged in non-increasing order. We shall use this remark often.

We shall deal with *saturated* subgroups of $M(\mathbb{Q}, R)$, that is, subgroups L with the property that, for each $(i,j) \in P$ and $a \in R$, there exists an element in L whose (i,j)-entry is a if and only if ${}^{a}t_{ij} \in L$. If L is a subgroup of $M(\mathbb{Q}, R)$ and $(i,j) \in P$ it is easy to check that $H_{ij} := \{a \in R \mid {}^{a}t_{ij} \in L\}$ is a subgroup of (the additive group of) R; moreover, the first part of (3) yields:

$$(\forall i,j,k \in \mathbb{Q})\,(i < j < k \Rightarrow H_{ij}H_{jk} \subseteq H_{ik}). \tag{$\mathfrak{Q}$}$$

The following lemma shows that the reverse construction provides saturated subgroups.

Lemma 1.1. *Let $\mathbf{h} = (H_{ij})_{(i,j)\in P}$ be a family of subgroups of the additive group of R and let*

$$L_{\mathbf{h}} = \{(a_{ij}) \in M(\mathbb{Q}, R) \mid (\forall (i,j) \in P)(a_{ij} \in H_{ij})\}.$$

Then $L_{\mathbf{h}} \leq M(\mathbb{Q}, R)$ if and only if ($\mathfrak{Q}$) holds, in which case $L_{\mathbf{h}}$ is saturated. Moreover, if $L_{\mathbf{h}} \leq M(\mathbb{Q}, R)$ and each H_{ij} is generated by a set X_{ij} then $L_{\mathbf{h}} = \langle {}^{x}t_{ij} \mid (i,j) \in P,\ x \in X_{ij} \rangle$.

Proof. It is obvious that $L_{\mathbf{h}}$ is saturated if it is a subgroup. Let $T = \{{}^{x}t_{ij} \mid (i,j) \in P,\ x \in H_{ij}\}$, the set of the elementary generators in $L_{\mathbf{h}}$. Equation (1) and equalities like that in (2) show that $L_{\mathbf{h}} \subseteq \langle T \rangle = \langle {}^{x}t_{ij} \mid (i,j) \in P,\ x \in X_{ij} \rangle$. To complete the proof we need only show that $\langle T \rangle \subseteq L_{\mathbf{h}}$ if ($\mathfrak{Q}$) holds. In view of (3), ($\mathfrak{Q}$) implies that the commutator of two elements of T is still in T. Therefore the rewriting procedure following equation (3) makes it easy to write any element of $\langle T \rangle$ as the product of elements of T of non-increasing levels. Now (1) shows that $\langle T \rangle \subseteq L_{\mathbf{h}}$ and the proof is complete. □

Remark 1.2. The saturated subgroups of $M(\mathbb{Q}, R)$ are precisely those generated by elementary generators, and they all arise from the construction in Lemma 1.1. Indeed, if X is a set of elementary generators in $M(\mathbb{Q}, R)$, let Y be the closure of X under the commutator operation in the set of elementary generators, defined by letting $(u,v) \mapsto [u,v]$. Then $\langle X \rangle = \langle Y \rangle$ and the rewriting trick used in the proof of Lemma 1.1 shows that all elements of $\langle X \rangle$ are products of powers of elements of Y of non-increasing levels. Now equation (1) shows that $\langle X \rangle$ is saturated. Conversely, if L is a saturated subgroup of $M(\mathbb{Q}, R)$ then it is clear from equation (1) that L is generated by the set T of all elementary generators belonging to L. More precisely, as in the proof of Lemma 1.1, we note that all elements of L are actually products of elements of T of non-increasing levels, and it follows from (1) that $L = \{(a_{ij}) \in M(\mathbb{Q}, R) \mid (\forall (i,j) \in P)(a_{ij} \in H_{ij})\}$, where, as before, $H_{ij} = \{a \in R \mid {}^{a}t_{ij} \in L\}$ for all $(i,j) \in P$.

Yet another application of the usual rewriting trick allows us to find factorisations of saturated subgroups.

Lemma 1.3. *Let L be a saturated subgroup of $M(\mathbb{Q}, R)$ and let $F \subseteq P$. Then $L = L_{(\bar{F})}L_{(F)}$, where $L_{(F)}$ (resp. $L_{(\bar{F})}$) is a subgroup generated by elementary generators whose positions are in F (resp. in $P \smallsetminus F$).*

Proof. For the scope of this proof let us say that a *red token* (of level k) is any product of elementary generators belonging to L and of the same level k whose positions are in F, and a *blue token* is an elementary generator belonging to L whose position is not in F. Equation (1) and the fact that elementary generators of the same level commute show that every $g \in L$ can be written as a product $t_1 t_2 \cdots t_n$ of (red or blue) tokens in such a way that if t_u is a red token of level k appearing in this product then all tokens t_v such that $v > u$ have level less than k. We shall rewrite g as a product of tokens still satisfying the same property and such that red tokens appear to the right of all blue tokens, that is: for some nonnegative integer $m \leq n$, the first m factors are blue tokens and the remaining ones are red. Suppose that $r = t_u$ and $b = t_{u+1}$ are two consecutive factors in the given expression for g, where r is a red token and b is a blue one. If $rb = br$ then we can obviously rewrite g as $t_1t_2\cdots t_{u-1}brt_{u+2}\cdots t_n$. Otherwise, suppose that $c := [r, b] \neq 1$. Let (i, j) be the position of b and k be the level of r. Our requirements on the factorization yield $j < k$; it follows that all elementary generators of level k and position different from (j, k) commute with b. Now r is the product of (pairwise commuting) elementary generators of level k; since $c \neq 1$ at least one of them has position (j, k). As the product of two elementary generators of position (j, k) still has position (j, k) we may assume that r is the product of one elementary generator $x \in L$ of position (j, k) and other elementary generators of different positions (all of level k). Thus $c = [x, b]$ has position (i, k). Now $rb = bcr$, since c commutes with r. If $(i, k) \in F$ then $r' := cr$ is a red token of level k and we may write g as $t_1t_2\cdots t_{u-1}br't_{u+2}\cdots t_n$; otherwise c is a blue token and $g = t_1t_2\cdots t_{u-1}bcrt_{u+2}\cdots t_n$. In each of the three cases considered we have rewritten g as a product with the same number of red tokens in which one of the red tokens has been pushed one place to the right (if we count places from the right). Moreover the property that every red token has level greater than that of every token following it in the product has been preserved. It is clear that by iterating this procedure we can write g as the product of a product of blue tokens and a product of red tokens. Hence $g \in L_{(\bar{F})}L_{(F)}$, where $L_{(\bar{F})}$ and $L_{(F)}$ are the subgroups generated by the blue and red tokens respectively. Thus the proof is complete. □

2. The construction

Every subgroup L of periodic index in $M(\mathbb{Q}, R)$ contains a saturated subgroup of periodic index. For, if t is an elementary generator then $t^{n_t} \in L$ for some positive integer n_t and the subgroup generated by all these elements t^{n_t} is saturated (see Remark 1.2) and of periodic index. Thus, aiming at finding a residually finite subgroup of periodic index there is no loss in restricting the search to saturated subgroups.

In view of Lemma 1.1, the first step in our strategy is to define a family of subgroups of the additive group of R (indexed in P) and use it to define a saturated subgroup L of $M(\mathbb{Q}, R)$. Then we shall consider families of subgroups of R that differ from the former slightly and at just finitely many points in P. These latter determine subgroups of L, and we shall prove that, under suitable hypotheses, L has periodic index in $M(\mathbb{Q}, R)$ and these subgroups of L have finite index and trivial intersection.

To start with, fix a subring H of R. We assume that R has an identity, but note that we do not require that subrings likewise have one, in particular, all ideals of R are subrings. Let δ be any mapping $\mathbb{Q} \to \mathbb{N}$ and, for every $(i, j) \in P$, let $H_{ij} = H^{i^\delta + j^\delta}$. It is clear that the family $\mathbf{h} = (H_{ij})_{(i,j)\in P}$ of subgroups defined thus satisfies $(\mathcal{Q})$ and so, as in Lemma 1.1, $\mathbf{h}$ gives rise to a saturated subgroup $L_{\mathbf{h}}$. Next, for all $\lambda \in \mathbb{N}$ and $(u, v) \in P$ we define the family of subgroups $\mathbf{h}^{(\lambda,u,v)} = (H_{ij}^{(\lambda,u,v)})_{(i,j)\in P}$ of R by letting $H_{ij}^{(\lambda,u,v)} = H_{ij}$ unless the pair (i, j) is *exceptional* (with respect to δ, λ, u and v), that is, $u \leq i < j \leq v$ and $i^\delta + j^\delta < \lambda$; in this case we let $H_{ij}^{(\lambda,u,v)} = H_{ij}H^\lambda$.

Lemma 2.1. *With the notation just established, all families* $\mathbf{h}^{(\lambda,u,v)}$ *satisfy* $(\mathcal{Q})$.

Proof. Fix $\lambda \in \mathbb{N}$ and $(u, v) \in P$, and let i, j, k be rational numbers such that $i < j < k$. That $H_{ij}^{(\lambda,u,v)}H_{jk}^{(\lambda,u,v)} \subseteq H_{ik}^{(\lambda,u,v)}$ is clear unless (i, k) is exceptional while (i, j) and (k, j) are not. In this case $i^\delta + j^\delta$, $j^\delta + k^\delta \geq \lambda$, and $i^\delta + k^\delta < \lambda$, hence $2j^\delta > \lambda$ and $H_{ij}^{(\lambda,u,v)}H_{jk}^{(\lambda,u,v)} = H^{i^\delta+2j^\delta+k^\delta} \subseteq H^{i^\delta+k^\delta+\lambda} = H_{ik}^{(\lambda,u,v)}$. This shows that $\mathbf{h}^{(\lambda,u,v)}$ satisfies $(\mathcal{Q})$. □

Thus Lemma 1.1 provides subgroups $L_{\mathbf{h}^{(\lambda,u,v)}}$ (each defined by the associated family $\mathbf{h}^{(\lambda,u,v)}$), which clearly are contained in $L_{\mathbf{h}}$. It is easy to decide when these subgroups have periodic index in $M(\mathbb{Q}, R)$. First we make a very elementary remark:

Lemma 2.2. *Let* Y *be a subring of the ring-with-unit* S. *If the additive group* $(S/Y, +)$ *is periodic then* $(S/Y^n, +)$ *is periodic for all positive integers* n.

Proof. Let $x \in S$ and let m and r be the additive orders modulo Y of 1_S and x, respectively. Let n be a positive integer. Then $(m^{n-1}r)x = (m1_S)^{n-1}(rx) \in Y^n$; hence x is periodic modulo Y^n. □

Lemma 2.3. *With the notation employed in this section thus far, the following conditions are equivalent:*

(i) *the additive group* R/H *is periodic;*
(ii) $L_{\mathbf{h}}$ *has periodic index in* $M(\mathbb{Q}, R)$;
(iii) *for all* $\lambda \in \mathbb{N}$ *and* $(u, v) \in P$, $L_{\mathbf{h}^{(\lambda,u,v)}}$ *has periodic index in* $M(\mathbb{Q}, R)$.

Proof. Suppose that $(R/H, +)$ is periodic. Fix an elementary generator $g = {}^xt_{ij}$ of $M(\mathbb{Q}, R)$. For all $\lambda \in \mathbb{N}$ and $(u, v) \in P$, there exists $m \in \mathbb{N}$ such that $g^m = {}^{mx}t_{ij} \in L_{\mathbf{h}^{(\lambda,u,v)}}$, because of Lemma 2.2. Since the elementary generators generate $M(\mathbb{Q}, R)$ this proves that (iii) holds. Of course (iii) implies (ii). Finally, suppose that $L_{\mathbf{h}}$ has periodic index in $M(\mathbb{Q}, R)$ and let $x \in R$ and $(i, j) \in P$. Then ${}^{nx}t_{ij} = {}^xt_{ij}^n \in L_{\mathbf{h}}$ for some $n \in \mathbb{N}$, hence $nx \in H^{i^\delta+j^\delta} \subseteq H$. This shows that R/H is periodic. The proof is complete. □

It is clear that $\bigcap_{\lambda\in\mathbb{N},\,(u,v)\in P} L_{\mathbf{h}^{(\lambda,u,v)}} = 1$ if and only if $\bigcap_{\lambda\in\mathbb{N}} H^\lambda = 0$. Thus our main concern will be understanding which additional conditions allow us to conclude that the subgroups $L_{\mathbf{h}^{(\lambda,u,v)}}$ have finite index in $L_{\mathbf{h}}$. A convenient hypothesis is the following condition on δ:

$$(\forall (i,j)\in P)\,(\forall \lambda\in\mathbb{N})\ \{k\in\mathbb{Q}\mid i\le k\le j \text{ and } k^\delta\le\lambda\} \text{ is finite.} \tag{$\mathfrak{D}$}$$

(For instance, this is satisfied if δ is an injective mapping, or the "denominator mapping" defined by setting q^δ, for each $q\in\mathbb{Q}$, to be the additive order of q modulo $\mathbb{Z}$.) Indeed, ($\mathfrak{D}$) is precisely the condition ensuring that, for all $\lambda\in\mathbb{N}$ and $(u,v)\in P$, with respect to δ, λ, u and v the set $\{(i,j)\in P\mid u\le i<j\le v \text{ and } i^\delta+j^\delta<\lambda\}$ of all exceptional pairs is finite.

Proposition 2.4. *Let R be a ring with identity and suppose that H is a subring of R such that:*

(i) *the additive group R/H is periodic;*
(ii) $\bigcap_{n\in\mathbb{N}} H^n = 0$;
(iii) *for all $n\in\mathbb{N}$ the additive group H/H^n is finitely generated.*

If δ is a mapping $\mathbb{Q}\to\mathbb{N}$ satisfying ($\mathfrak{D}$), then, in the notation employed thus far, $|L_{\mathbf{h}} : L_{\mathbf{h}^{(\lambda,u,v)}}|$ is finite for all $\lambda\in\mathbb{N}$ and $(u,v)\in P$. Thus $L_{\mathbf{h}}$ is residually finite and of periodic index in $M(\mathbb{Q},R)$.

Proof. For all $\lambda\in\mathbb{N}$ and $(u,v)\in P$ condition ($\mathfrak{D}$) implies that the set F of all exceptional pairs with respect to δ, λ, u and v is finite; therefore also the set F^* of all $i\in\mathbb{Q}$ appearing as the first or the second coordinate of some pair in F is finite. Lemma 1.3 yields a factorisation $L_{\mathbf{h}} = AB$, where A is generated by the elementary generators in $L_{\mathbf{h}}$ with positions in $F^*\times F^*$ and B is generated by the elementary generators in $L_{\mathbf{h}}$ with positions not in F. Then $B\le L_{\mathbf{h}^{(\lambda,u,v)}}$. Moreover, if $\mu=\lambda+2\max\{i^\delta\mid i\in F^*\}$ then $L_{\mathbf{h}^{(\lambda,u,v)}}$ also contains the normal subgroup N of A consisting of all matrices in A all of whose non-diagonal entries lie in H^μ. Now A embeds in a linear group (of degree $|F^*|$) over R; since H/H^μ is finitely generated (by (iii)) and hence finite, by (i) and Lemma 2.2, it follows that A/N is finite. Therefore $|L_{\mathbf{h}} : L_{\mathbf{h}^{(\lambda,u,v)}}| = |A : A\cap L_{\mathbf{h}^{(\lambda,u,v)}}|$ is finite. Now $\bigcap_{\lambda\in\mathbb{N},\,(u,v)\in P} L_{\mathbf{h}^{(\lambda,u,v)}} = 1$ by (ii), and $L_{\mathbf{h}}$ has periodic index in $M(\mathbb{Q},R)$ by Lemma 2.3. The proposition is proved. □

Some remarks on condition (iii) of Proposition 2.4 are in order. Firstly, as one sees from the proof, *finitely generated* is equivalent here to *finite*. Secondly, condition (iii) is certainly satisfied if H is finitely generated as a ring; indeed, if $n\in\mathbb{N}$ and H is generated as a ring by the set X, then H is generated as a group by all products of elements of X and hence, modulo H^n, by all products of at most $n-1$ elements of X.

3. Proof of the theorem

Proposition 2.4 is sufficient to prove that, for instance, $M(\mathbb{Q},\mathbb{Z})$ has a residually finite subgroup of periodic index, since every nonzero ideal of Z has the property required for H in the hypotheses of the proposition. We shall prove the theorem stated in the Introduction by slightly refining the argument. Indeed, to establish the result it is enough to show that the given ring R has a subring H of periodic index and a set of ideals with trivial intersection such that H satisfies the conditions of Proposition 2.4 modulo each of these ideals.

Here is the proof of the main theorem. Fix a mapping $\delta\colon\mathbb{Q}\to\mathbb{N}$ satisfying ($\mathfrak{D}$). Also fix a commutative ring-with-unit A whose additive group has finite torsion-free rank. An argument due to Beaumont and Peirce (see [3], Proposition 121.5) shows that A has a subring B such that the additive group $(A/B,+)$ is periodic and $(B,+)$ is free abelian.

Consider first the case when $R=A[x_1,\dots,x_t]$ is the polynomial ring on finitely many commuting indeterminates over A. We have $m\in B$ for some integer $m>1$; let H be the ideal of $B[x_1,\dots,x_t]$ generated by $\{m,x_1,x_2,\dots,x_t\}$. It is clear that the additive group R/H is periodic. For all $n\in\mathbb{N}$ the subring H^n consists of all the polynomials in which the coefficients of monomials of degree $i<n$ are divisible in B by m^{n-i}. Then H/H^n is finite and $\bigcap_{n\in\mathbb{N}} H^n=0$, because the additive group of $B[x_1,\dots,x_n]$ is free abelian and B has finite rank. Thus, in this case, the result follows from Proposition 2.4; more precisely, if $L_{\mathbf{h}}$ is the subgroup of $M(\mathbb{Q},R)$ defined as in the previous section by using H and δ then $L_{\mathbf{h}}$ has periodic index and is residually finite.

Consider now the general case. There is no loss in assuming that R is the whole polyomial ring in the statement, so $R=A[x\mid x\in X]$ is the polynomial ring on a certain set X of commuting indeterminates over A. Again, let H be the the ideal of $B[x\mid x\in X]$ generated by a fixed integer $m>1$ and X, and let $L=L_{\mathbf{h}}$ be the subgroup of $M(\mathbb{Q},R)$ defined as in the previous section by using H and δ. It is clear that the additive group R/H is periodic and L has periodic index in $M(\mathbb{Q},R)$ by Lemma 2.3. For every finite subset Y of X let J_Y be the ideal of R generated by $X\smallsetminus Y$. By repeating our usual construction starting from the ring $R/J_Y\simeq A[x\mid x\in Y]$ and its subring $H+J_Y/J_Y$ (and δ) we obtain a saturated subgroup L_Y of $M(\mathbb{Q},R/J_Y)$. By the previous case L_Y is residually finite. Now, the natural epimorphism $R\twoheadrightarrow R/J_Y$ induces an epimorphism $M(\mathbb{Q},R)\twoheadrightarrow M(\mathbb{Q},R/J_Y)$, whose kernel is $M_Y:=M(\mathbb{Q},J_Y)$, which, despite the fact that J_Y has no multiplicative identity, is defined in the natural way as explained in the Introduction. L_Y is precisely the image of L under this epimorphism. Thus LM_Y/M_Y is residually finite. But clearly $\bigcap\{M_Y\mid Y \text{ is a finite subset of } X\}=1$, hence L is residually finite. Thus the proof of the theorem is complete.

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