Groups with cyclic outer automizers

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Abstract. We consider groups G such that $N_G(H)/HC_G(H)$ is cyclic for all $H \leq G$. More specifically, we characterize locally nilpotent and supersoluble groups with this property.

1. Introduction

Given a group G and $H \leq G$ the group $\mathrm{Aut}_G(H) = N_G(H)/C_G(H)$ has been called the *automizer* of H in G, with an obvious reference to the faithful conjugacy action that it has on H. A number of papers already in the literature deal with groups in which restictions are imposed on the automizers of subgroups (see [1, 2, 3, 4, 5, 8, 13]). Not all restrictions give rise to new interesting group classes. For instance, since all automizers of subgroups of a group G are isomorphic to sections of $\mathrm{Aut}_G(G) = G/Z(G)$, requiring that all subgroup automizers in G are, say, finite, or abelian, is equivalent to the requirement that $G/Z(G)$ has the same property.

To the contrary, various group classes can be defined if automizers are replaced by what we call outer automizers. If G is a group and $H \leq G$, the *outer automizer* of H in G is the group $\mathrm{Out}_G(H) := N_G(H)/HC_G(H)$, which embeds in $\mathrm{Out}\,H$. Thus $\mathrm{Out}_G(H)$ can be roughly described as the group of outer automorphisms induced on H by conjugation in G. The property of having trivial outer automizer is what has been called the property of having 'small automizer' in [2]; here the starting point was the result, proved by Zassenhaus in [14] as early as 1952, that finite groups in which all abelian subgroups have small automizers are themselves abelian. For further results in the same thread also see [3, 5]. Here we consider the weaker property of having cyclic outer automizer. Let us say that $G \in (\mathrm{CO})$ if and only if $\mathrm{Out}_G(H)$ cyclic for all $H \leq G$. Easy examples of (CO)-groups include all finite groups whose size is the product of at most three (non necessarily different) primes and dihedral groups, or more generally all groups with a cyclic subgroup of prime index, but also insoluble groups like the alternating group on five symbols or Tarski groups. In this paper we shall characterize locally nilpotent (CO)-groups; these groups form a very restricted class of hypercentral groups. We provide two alternative descriptions of such groups in Theorems 3.5 and 3.11. Our other major result is a classification of finite supersoluble (CO)-groups (Theorem 5.5). As a matter of fact, also infinite supersoluble groups are characterized by this result, because every polycyclic-by-finite group is in (CO) if and only if all of its finite quotients are in (CO), as we prove in Proposition 4.5.

2. Generalities

Our first remark is that the class of (CO)-groups is section-closed.

Lemma 2.1. *Let $K \leq G$ and $N \lhd G$, where $G \in (\mathrm{CO})$. Then K and G/N are in* (CO).

Proof. Let $H \leq K$. Then $HC_K(H) = HC_G(H) \cap K$, hence $\mathrm{Out}_K(H) = (N_G(H) \cap K)/(HC_G(H) \cap K)$ is isomorphic to a subgroup of $\mathrm{Out}_H(G)$. Hence $\mathrm{Out}_K(H)$ is cyclic; thus $K \in (\mathrm{CO})$. Next, let $H/N \leq G/N$. Then $C_G(H)N/N \leq C_{G/N}(H/N)$ and $N_{G/N}(H/N) = N_G(H)/N$. So $\mathrm{Out}_{G/N}(H/N)$ is an epimorphic image of $\mathrm{Out}_G(H)$. It follows that $G/N \in (\mathrm{CO})$. □

We have a couple of partial inverses.

Lemma 2.2. *Let G be a group and let $B \lhd G$ be such that $B \cap G' = 1$. Then $G \in (\mathrm{CO})$ if and only if $G/B \in (\mathrm{CO})$.*

Proof. We only have to prove that $G \in (\mathrm{CO})$ on assuming that $G/B \in (\mathrm{CO})$. If $H \leq G$ we have $N_G(H) \leq N_G(HB)$ and $B \leq Z(G) \leq C_G(H) = C_G(HB)$. Hence $\mathrm{Out}_G(H) \leq \mathrm{Out}_G(HB)$. Thus, to show that $G \in (\mathrm{CO})$ we only have to prove that $\mathrm{Out}_G(H)$ is cyclic for all $H \leq G$ such that $B \leq H$. For any such H, if $g \in G$ satisfies $[H, g] \leq B$ then $[H, g] = 1$, because $B \cap G' = 1$. Then $C_G(H)/B = C_{G/B}(H/B)$. Of course, $N_G(H)/B = N_{G/B}(H/B)$, hence $\mathrm{Out}_G(H) \simeq \mathrm{Out}_{G/B}(H/B)$, and the latter is cyclic, because $G/B \in (\mathrm{CO})$. The lemma follows. □

An obvious consequence is that if $G \in (\mathrm{CO})$ and B is any abelian group then also $G \times B$ is in (CO). Therefore the study of (CO)-groups could be reduced to the case of purely nonabelian groups. More generally, one can disregard abelian factors in central products, as is shown by the following lemma.

Lemma 2.3. *Let $G = XZ$, where $X \leq G$ and $Z \leq Z(G)$. Then $G \in (\mathrm{CO})$ if and only if $X \in (\mathrm{CO})$.*

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Proof. Suppose that $X \in$ (CO) and $H \leq G$. Consider the projection $H^* := HZ \cap X$ of H in X. Since $HZ = H^*Z$ it is clear that $C_G(H) = C_G(HZ) = C_G(H^*)$ and $HC_G(H) = H^*C_G(H^*)$. On the other hand $N_G(H) \leq N_G(H^*)$, so that $\mathrm{Out}_G(H) \leq \mathrm{Out}_G(H^*)$, but $\mathrm{Out}_G(H^*) \simeq \mathrm{Out}_X(H^*)$, hence $\mathrm{Out}_G(H)$ is cyclic. The lemma follows. □

Normal abelian subgroups play a major rôle in our discussion on (CO)-groups, often considered together with the concepts of $\mathfrak{ZC}$-action and $\mathfrak{ZC}$-embedding. If the group X acts on the group N we say that its action is $\mathfrak{ZC}$ if and only if $H/[H,X]$ is cyclic for all X-invariant subgroups H of N. A normal subgroup N of a group G is $\mathfrak{ZC}$-embedded in G if and only if G acts (by conjugation) $\mathfrak{ZC}$ on $N/N \cap Z(G)$. This latter condition means that all central factors U/V of G such that $N \cap Z(G) \leq V < U \leq N$ are cyclic. (Reference to the factor $N/N \cap Z(G)$ rather than to N in this definition is admittedly artificial, but allows to state our results in a more compact form.) It is clear that if N is $\mathfrak{ZC}$-embedded in G and $M \leq N$ is such that $M \lhd G$ then N/M is $\mathfrak{ZC}$-embedded in G/M.

Lemma 2.4. *Suppose that A is a normal abelian subgroup of a* (CO)*-group G. Then:*
- (i) *$A \leq C_G(G')$;*
- (ii) *A is $\mathfrak{ZC}$-embedded in G.*

Proof. Let $C = C_G(A)$. Then $\mathrm{Out}_G(A) = G/C$, thus $G = C\langle x\rangle$ for some $x \in G$ (hence $G' \leq C$ and (i) follows) and $A \cap Z(G) = C_A(x)$. Let U/V be a central factor of G such that $A \cap Z(G) \leq V < U \leq A$, and let $H = V\langle x\rangle$. Then $U \leq N_G(H)$. Moreover $C_{A\langle x\rangle}(H) = C_A(x)C_{\langle x\rangle}(V) \leq H$. Therefore $\mathrm{Out}_{A\langle x\rangle}(H) \geq UH/H \simeq U/U \cap H$. Now we have $U \cap H = V(U \cap \langle x\rangle) = V$, because $A \cap \langle x\rangle \leq C_A(x) \leq V$. It follows that U/V embeds in $\mathrm{Out}_{A\langle x\rangle}(H)$, hence it is cyclic. Thus (ii) is proved. □

In the case when A has prime index in G also the converse holds:

Lemma 2.5. *Let G be a group with a normal abelian subgroup A of prime index. Then $G \in$ (CO) if and only if A is $\mathfrak{ZC}$-embedded in G.*

Proof. The 'only if' part was proved in Lemma 2.4 (ii); we have to prove the 'if' part. Assume that A is $\mathfrak{ZC}$-embedded in G and let $H \leq G$. If $H \leq A$ then $A \leq C_G(H)$ and $\mathrm{Out}_G(H)$ is plainly cyclic. So we may assume $H \not\leq A$, hence $G = AH$. Let $U = N_A(H)$ and $V = A \cap HC_G(H)$. Then $A \cap Z(G) \leq V \leq U \leq A$ and $[U,G] = [U,H] \leq V$, so U/V is cyclic. Finally $U/V \simeq \mathrm{Out}_G(H)$, therefore $\mathrm{Out}_G(H)$ is cyclic. □

In accordance with notation in [2], call (SA) the class of all groups G such that $\mathrm{Out}_G(H) = 1$ for all $H \leq G$.

Lemma 2.6. *Let $G = \mathrm{Dr}_{i\in I}\, G_i$ be a group.*
- (a) *If $G \in$ (CO) then $G_i \in$ (SA) for all but finitely many $i \in I$.*
- (b) *Assume that G is periodic and $\pi(G_i) \cap \pi(G_j) = \varnothing$ for all distinct $i, j \in I$. Then $G \in$ (CO) if and only if $G_i \in$ (CO) for all $i \in I$ and $G_i \in$ (SA) for all but finitely many $i \in I$.*

Proof. For all $i \in I$ let $H_i \leq G_i$, and let $H = \mathrm{Dr}_{i\in I}\, H_i$. Then $N_G(H) = \mathrm{Dr}_{i\in I}\, N_{G_i}(H_i)$ and $HC_G(H) = \mathrm{Dr}_{i\in I}\, H_iC_{G_i}(H_i)$, hence $\mathrm{Out}_G(H) \simeq \mathrm{Dr}_{i\in I}(\mathrm{Out}_{G_i}(H_i))$. Part (a) follows easily from this remark: let J be the set of all $i \in I$ such that G_i is not an (SA)-group. For each $i \in J$ choose a subgroup H_i such that $\mathrm{Out}_{G_i}(H_i) \neq 1$ and let $H = \mathrm{Dr}_{i\in J}\, H_i$. Since $\mathrm{Out}_G(H) \simeq \mathrm{Dr}_{i\in J}\, \mathrm{Out}_{G_i}(H_i)$ is cyclic J is finite, as required. Also (b) follows, because in the hypothesis of (b) all subgroups H of G are factorised as $H = \mathrm{Dr}_{i\in I}(H \cap G_i)$. □

Finite (SA)-groups are abelian, by Zassenhaus' result from [14] cited in the introduction. Extensions of Zassenhaus' result to classes of infinite groups are in [3], but Tarski groups provide examples of (infinite) nonabelian (SA)-groups.

The last lemma of this section is less general in scope than the previous ones. The classes considered here include those that we shall deal with in the rest of the paper.

Lemma 2.7. *Let G be a locally supersoluble group in* (CO).
- (i) *If A is a maximal abelian normal subgroup of G then G/A is cyclic;*
- (ii) *if $G \in$ (SA) then G is abelian (see [3] for more general results);*
- (iii) *G is metabelian and hypercyclic;*
- (iv) *if G is locally nilpotent then it is hypercentral.*

Proof. Assume first that G is hypercyclic. Let A be a maximal abelian normal subgroup of G. Then $A = C_G(A)$, hence $G/A = \mathrm{Out}_G(A)$ is cyclic and G is metabelian (also, $G = A$ if $G \in$ (SA)). If G is merely assumed to be locally supersoluble, this conclusion applies to all finitely generated subgroups of G, and it follows that G itself is metabelian in this case too. To prove parts (i–iii) we have to prove that G is hypercyclic, and to this end we only need to show that G has a nontrivial cyclic normal subgroup, since all quotients of G satisfy the hypothesis of the lemma. Let $C = C_G(G')$. Then $G = C\langle x\rangle$ for some $x \in G$. Let $a \in G'$ and $H = \langle a, x\rangle$, so that $H = B\langle x\rangle$ where $B = \langle a\rangle^{\langle x\rangle}$. Since H is supersoluble it has a cyclic normal subgroup X such that $1 \neq X \leq B$. Clearly $X \lhd G$. Thus (i–iii) are proved. The proof of (iv) goes along similar lines: if G is locally nilpotent and H and B are defined as above, H is nilpotent, hence $Z(H) \cap B \neq 1$, but $Z(H) \cap B \leq Z(G)$. Thus $Z(G) \neq 1$ and, as above, this is enough to prove that G is hypercentral. □

3. Locally nilpotent groups

In this section we shall describe locally nilpotent groups in (CO). Our first result is that such groups are abelian if they are torsion-free.

Lemma 3.1. *Let G be a locally nilpotent group in* (CO)*. Then G' is periodic.*

Proof. We have to prove that G/T is abelian, where T is the torsion subgroup of G. Hence we may suppose that G is torsion-free and prove that it is abelian in this case. Assume false. Then G has a nonabelian finitely generated subgroup S. Of course S is nilpotent, of class c, say, and $S/Z_{c-2}(S)$ still is a counterexample. Hence we may replace G with this section of its; in other words we may assume that G is a (torsion-free) nilpotent group of class 2. Let p be a prime and let $H = Z(G)G^p$. Since G is torsion-free $C_G(G^p) = Z(G)$, hence $C_G(H) = Z(G) \leq H$ and $\mathrm{Out}_G(H) = G/H$. But $G/Z(G)$ is a noncyclic free abelian group, so G/H is not cyclic. This is a contradiction, because $G \in$ (CO), so the proof is complete. □

Thus it is the case of periodic groups that needs more attention. In this case we are essentially reduced to considering p-groups.

Lemma 3.2. *For a prime p, let G be a nonabelian locally finite p-group in the class* (CO) *and let A be a maximal abelian normal subgroup of G. Then $|G/A| = p$.*

Proof. First assume that G is nilpotent. By Lemma 2.7 (i), $G = A\langle x\rangle$ for some $x \in G$. Of course $Z(G) \leq A$. Suppose, by contradiction, $x^p \notin A$ or, equivalently, $x^p \notin Z(G)$. At the expense of replacing G by $\langle a, x\rangle$ for some $a \in A$ such that $[a, x^p] \neq 1$, we may assume that G is finite. Let $H = \langle x^p\rangle A^p[A, x]$. Then $H \lhd G$, as $[A, x] = G'$. Moreover $C_G(H) = C_{\langle x\rangle}(A^p[A, x])C_A(x^p)$. If x centralizes $A^p[A, x] = A^pG'$ then G has nilpotency class 2 and so $1 = [A^p, x] = [A, x^p]$. But this is a contradiction, since $x^p \notin Z(G)$. Thus $C_{\langle x\rangle}(A^p[A, x]) \leq \langle x^p\rangle$, so that $HC_G(H) = \langle x^p\rangle A^p[A, x]C_A(x^p)$. Also, $C_A(x^p) < A$, so $B := A^p[A, x]C_A(x^p) < A$, because $A^p = \Phi(A)$ and x acts nilpotently on A. It follows that $\mathrm{Out}_G(H) = G/\langle x^p\rangle B \simeq (A/B) \times (\langle x\rangle/\langle x^p\rangle)$ is not cyclic. This is a contradiction, hence $|G/A| = p$.

Now consider the general case. By the previous part of the proof every finitely generated subgroup of G has a (normal) abelian subgroup of index at most p. By theorem due to Mal'cev (see [9], Theorem 2.5.10) it follows that G has a (normal) abelian subgroup B of index p. If $A \leq B$ then $A = B$ and so $|G/A| = p$. Otherwise, $G = AB$, hence G is nilpotent and the result follows from the previous case. □

A consequence of Lemma 3.2 is that a locally finite p-group G in (CO) must either be nilpotent of class 2 at most (and satisfy $|G/Z(G)| \leq p^2$) or have exactly one maximal abelian normal subgroup. Another, more relevant, consequence is that, together with Lemmas 2.5, 2.6 and 2.7 (ii), this lemma provides a characterization of periodic locally nilpotent (CO)-groups.

Corollary 3.3. *Let G be a periodic locally nilpotent group. Then $G \in$* (CO) *if and only if G has an abelian normal $\mathfrak{ZC}$-embedded subgroup A such that G/A is cyclic of square-free order.*

Corollary 3.3 shows how relevant in our context $\mathfrak{ZC}$-actions are. In the case of (locally finite) p-groups the idea of $\mathfrak{ZC}$-action can be considered as a generalisation of the concept of uniserial action, discussed, for instance in [6], Chapter 4. As a matter of fact, in the case of locally nilpotent (CO)-groups, our $\mathfrak{ZC}$-actions essentially reduce to uniserial actions, as we shall see soon. Beforehand, however, we remark that a standard, elementary argument shows that one of the key properties usually associated to uniserial actions of finite p-groups (see [6], Propositions 4.1.7 and 4.1.8) is even more precisely expressed in terms of $\mathfrak{ZC}$-actions. By saying that a group X acts on a group N hypercentrally we mean that N has an ascending normal series of X-invariant subgroups, on all factors of which X acts trivially.

Lemma 3.4. *Suppose that a group X acts hypercentrally on the periodic locally nilpotent group N. The action of X on N is $\mathfrak{ZC}$ if and only if $\pi(N)$ is finite and, for every primary component P of N, the set of all X-invariant subgroups of P, ordered by inclusion, is a chain.*

Proof. Assume that X acts $\mathfrak{ZC}$ on N. Then $C_N(X)$ is cyclic, hence finite. The hypothesis that X acts hypercentrally on N implies that $\pi(N) = \pi(C_N(X))$, hence $\pi(N)$ is finite. Let U and V be two X-invariant subgroups in the same primary component of N and let $D = U \cap V$. If $D < U$ and $D < V$ then there exist X-invariant subgroups U^* and V^* such that $D < U^* \leq U$ and $D < V^* \leq V$ and X acts trivially on both U^*/D and V^*/D. Let $H = \langle U^*, V^*\rangle$, then $[H, X] \leq D$, but H/D is not cyclic. This is a contradiction, thus the necessity of the condition is proved. To prove sufficiency, assume that N has the structure required (as an X-group) and suppose, by contradiction, that it has a noncyclic factor which is centralised by X. Then there is a prime p such that the projection of this factor in the p-component of N contains an elementary abelian p-subgroup of rank 2, say $(U/D)\times(V/D)$, where $|U/D| = |V/D| = p$. Then U and V are X-invariant and not comparable by inclusion. This is a contradiction, and the proof is complete. □

Thus, apart from the few exceptions given in [6], Propositions 4.1.7 and 4.1.8, $\mathfrak{ZC}$-actions between finite p-groups are nothing else than uniserial actions.

We are in position to prove the first major result of this section.

Theorem 3.5. *Let G be a locally nilpotent group. Then $G \in$ (CO) if and only if $G/Z(G)$ is periodic and G has an abelian normal subgroup A such that G/A is cyclic of finite square-free order and any of the following equivalent (under the previous requirements) conditions holds, where A^* denotes $A/A \cap Z(G)$:*

(a) *A is $\mathfrak{ZC}$-embedded in G;*
(b) *every central factor U/V of G such that $A \cap Z(G) \leq V$ and $U \leq A$ is finite of square-free order;*
(c) *$(Z_2(G) \cap A)/(Z(G) \cap A)$ is finite of square-free order;*
(d) *$\pi(A^*)$ is finite and, for every primary component P of A^*, the set of all G-invariant subgroups of P, ordered by inclusion, is a chain;*
(e) *$\pi(A^*)$ is finite and G acts uniserially on every primary component of A^*.*

Moreover, if $G \in$ (CO) then the above description still is correct if A is replaced by any maximal normal abelian subgroup A, and the order of each central factor U/V in (b) divides $|G/A|$.

Proof. Suppose that $G \in$ (CO) and let A be a maximal normal abelian subgroup of G. We know that $G = A\langle x\rangle$ for some element x and that G is hypercentral (Lemma 2.7). The mapping $a \in A \mapsto [a, x] \in G'$ is an epimorphism with kernel $C_A(x) = Z(G)$, hence $A^* = A/Z(G) \simeq G'$ is periodic by Lemma 3.1. Moreover, A is $\mathfrak{ZC}$-embedded in G (Lemma 2.4), hence Lemma 3.4 shows that $\pi(A^*)$ is finite.

Suppose that G/A is finite. Then $G/Z(G)$ is periodic and we may choose $B \leq Z(G)$ such that G/B is periodic and B is torsion-free. Then A/B is maximal abelian in G/B, for, if U/B is an abelian subgroup of G/B then $U' \leq B \cap G' = 1$ and U is abelian—recall that G' is periodic. Therefore $|G/A|$ is square-free, as a consequence of Lemma 3.2. Thus, to prove that G/A has finite, square-free order we only have to show that it is finite. This amounts to saying that x^n centralises A for some positive integer n, since $C_{\langle x\rangle}(A) = \langle x\rangle \cap Z(G) \leq A$.

First we consider the case when G is nilpotent and show that $G/Z(G)$ is finite in this case. If G is nilpotent the fact that A^* is periodic and $\mathfrak{ZC}$-embedded in $G/Z(G)$ easily yields that A^* is finite. Then there exists a positive integer k such that $[A, x^k] \leq Z(G)$, so x^k stabilises the series $1 \leq Z(G) \leq A$. The stability group of this series is isomorphic to $\mathrm{Hom}(A^*, Z(G))$ and so has finite exponent. Hence $[x^n, A] = 1$ for some positive integer n. Then G/A is finite, so that $G/Z(G)$ is finite, as we claimed. Now consider the general case. By earlier remarks we may assume that x has infinite order. Let p be a prime and P the p-component of $T := \operatorname{tor} A$. If F is a finitely generated subgroup of $P\langle x\rangle$ containing x then $F_0 := F \cap P$ is finite and $F = F_0\langle x\rangle$. Let $C = C_{\langle x\rangle}(F_0)$; then F/C is a finite p-group and therefore Lemma 3.2 shows that $[x^p, F_0] = 1$. It follows that $[P, x^p] = 1$. Since $\pi(A^*)$ is finite we have that $[x^k, T] = 1$ for some positive integer k. As $G' \leq T$ it follows that $G_0 := A\langle x^k\rangle$ is nilpotent. Therefore $G_0/Z(G_0)$ is finite, and this shows that $C_{\langle x\rangle}(A) \neq 1$. Hence $G/Z(G)$ is periodic and so $|G/A|$ is finite and square-free.

Now we prove sufficiency, together with the equivalence of the conditions listed. Suppose that $G = A\langle x\rangle$, where the abelian normal subgroup A has finite square-free index n, and $G/Z(G)$ is periodic. Then $A \leq Z_\omega(G)$ by [10], Theorem 4.38, and G is hypercentral. Lemma 3.4 shows that (a) and (d) are equivalent. If U/V is a central factor of G such that $A \cap Z(G) \leq V$ and $U \leq A$ then, modulo $[V, x]$, we have $[U^n, x] \equiv [U, x^n] = 1$, hence $[U^n, x] \leq [V, x]$. As $a \in A \mapsto [a, x] \in A$ is an endomorphism whose kernel $A \cap Z(G)$ is contained in V, we conclude that $U^n \leq V$. This shows that (a) implies (b), and also justifies the final statement in the theorem. That (b) implies both (a) and (c) is obvious. Hypercentrality of G and an almost immediate extension of [6], Lemma 4.1.6 to the case of locally finite p-groups show that (c) implies (e)—note that $(Z_2(G) \cap A)/(Z(G) \cap A)$ is the centralizer of x in A^*. Finally, (b) certainly follows from (e); thus we have proved that conditions (a)–(e) are pairwise equivalent. Now suppose that any of them holds. Choose again $B \leq Z(G)$ such that B is torsion-free and $Z(G)/B$ is periodic. Then $G/B \in$ (CO) by Corollary 3.3 and hence $G \in$ (CO) by Lemma 2.2. The proof is complete. □

Corollary 3.6. *Let $G \in$ (CO). Then the Fitting subgroup of G is nilpotent and coincides with the Baer radical of G.*

Proof. Let F be the Baer radical of G. Then F is abelian-by-finite, hence it is generated by an abelian normal subgroup and finitely many cyclic subnormal subgroups. Therefore F is nilpotent. The result follows. □

Corollary 3.7. *Let G be a locally nilpotent (CO)-group. Then G is nilpotent if and only if $G/Z(G)$ is finite. In this case, $G/Z(G)$ is two-generator, unless G is abelian. If G is not nilpotent then it is hypercentral of length $\omega + 1$; moreover $\operatorname{Fit} G$ coincides with $Z_\omega(G)$ and with the FC-centre of G.*

Proof. Theorem 3.5 shows that $Z_{i+1}(G)/Z_i(G)$ is finite for all positive integers i. Hence G is nilpotent if and only if $G/Z(G)$ is finite. If $G/Z(G)$ is finite then, in the notation of the same theorem, all primary components of A^* are cyclic as G-modules, hence the same is true of A^* and it follows that $G/Z(G)$ is two-generator. Still in the same notation, but now in the case when G is not nilpotent, $G = A\langle x\rangle$ for some x, hence $F := \operatorname{Fit} G = A\langle x^\lambda\rangle$ for some $\lambda \in \mathbb{N}$. By the previous corollary F is nilpotent, hence F' is finite by the first part of the proof. But $F' = [x^\lambda, A] = [x^\lambda, G]$; it follows that F lies in the FC-centre of G, hence in $Z_\omega(G)$ (see [10], Theorem 4.38). On the other hand, it is obvious that $Z_\omega(G) \leq F$. Thus $Z_\omega(G) = F < G$; since G/F is cyclic the hypercentral length of G is $\omega + 1$, and the proof is complete. □

We shall see (Corollary 4.2) that in every (CO)-group the Fitting subgroup is contained in the FC-centre.

The following remark is of some interest in the light of Lemma 2.3. If G is a nilpotent (CO)-group then $G = HZ(G)$ for a two-generator subgroup H of G, by Corollary 3.7. Thus, by Lemma 2.3, the study of nilpotent (CO)-groups is in some sense equivalent to the study of nilpotent two-generator (CO)-groups. Groups of the latter kind are of course rather much restricted. For instance, they may have torsion-free rank 2 at most, since they have finite derived subgroups.

Central factorisations of a different kind are also useful in the context of locally nilpotent (CO)-groups, for such groups can be described as central products of certain groups in which the quotient modulo the centre involves one prime only and this fact leads to a description of them which is more explicit than that obtained as Theorem 3.5. Indeed, since it has periodic central factor group, any locally nilpotent (CO)-group has a central factorisation in 'primary components modulo the centre'. More precisely, let G be a locally nilpotent (CO)-group and $\pi := \pi(G/Z(G))$. For each $p \in \pi$, let $G_p/Z(G)$ be the primary p-component of $G/Z(G)$. Then $[G_p, G_q] = 1$ for all different $p, q \in \pi$ (for, every periodic automorphism of G_p stabilising $1 \le Z(G) \le G_p$ has p-power order), hence G is a central product of the subgroups G_p, with p ranging over the (finite) set π. Clearly $Z(G) = Z(G_p)$ for all $p \in \pi$. Conversely, it is an immediate consequence of Theorem 3.5 that if G is a central product of finitely many (CO)-groups $G_1, G_2, \dots, G_k$ such that, for all i, $G_i/Z(G_i)$ is a locally finite p_i-group, where $p_1, p_2, \dots, p_k$ are pairwise distinct primes, then $G \in$ (CO).

For instance, a corollary to Theorem 3.5, providing a characterization of nilpotent (CO)-groups can be phrased as follows.

Corollary 3.8. *Let G be a nilpotent group. Then $G \in$ (CO) if and only if either G is abelian or there are finitely many, pairwise distinct primes p_1, $p_2, \dots, p_k$ such that G has a central decomposition $G = G_1G_2\cdots G_k$, where, for each i, the group G_i has an abelian subgroup of index p_i and $|G_i/G_i'Z(G_i)| = p_i^2$.*

Proof. For every nontrivial primary component $G_p/Z(G)$ of $G/Z(G)$ we have $|G_p/G_p'Z(G)| > p$, because G_p is nilpotent and not abelian. This and Theorem 3.5 show that the condition is necessary. Conversely, let G be as specified, but not abelian, and fix $i \in \{1, 2, \dots, k\}$. Let bars denote images modulo $Z(G_i) = G_i \cap Z(G)$. Since $\bar{G}_i/\bar{G}_i'$ is finite, also $\bar{G}_i$ is finite. Let A_i be an abelian subgroup of index p_i in G_i and let $x \in G_i \smallsetminus A_i$. Then $\bar{G}_i' = [\bar{A}_i, \bar{x}] \simeq \bar{A}_i/C_{\bar{A}_i}(\bar{x}) = \bar{A}_i/\bar{A}_i \cap Z(\bar{G}_i)$. But $\bar{A}_i$ is finite, hence $p_i = |\bar{A}_i/\bar{G}_i'| = |\bar{A}_i \cap Z(\bar{G}_i)|$. It follows that condition (c) in Theorem 3.5 is satisfied if we let $A = A_1A_2\cdots A_k$, and $G \in$ (CO). □

Theorem 3.5 and the remarks preceding Corollary 3.8 make clear that, in order to give an explicit description of locally nilpotent (CO)-groups, it is enough to concentrate on the case when the central factor group is a p-group. Our next aim will be determining the structure of such central factor groups in detail.

Let p be a prime and G be (CO)-group such that $G/Z(G)$ is a (nontrivial, locally finite) p-group. It follows from Theorem 3.5 that $G/Z(G)$ is the split extension of the abelian group $A^* = A/Z(G)$ (where A is abelian itself) by a group $\langle y\rangle$ of order p, where y acts uniserially on A^* and induces on A^* a splitting automorphism (that is: $uu^yu^{y^2}\cdots u^{y^{p-1}} = 1$ for all $u \in A^*$; in other words, $aa^xa^{x^2}\cdots a^{x^{p-1}} \in Z(G)$ for all $a \in A$ and $x \in G \smallsetminus A$). Groups with this structure can be described as quotients of easily defined wreath products. Let C_{p^n} denote either a cyclic group of order p^n, if n is a positive integer, or a Prüfer p-group, if n is the symbol ∞. In either case let $W_{p,n}$ be the (standard, regular) wreath product of C_{p^n} by a group of order p, and let $\overline{W}_{p,n} = W_{p,n}/Z(W_{p,n})$. Recall that, if finite (and not too small), $\overline{W}_{p,n}$ is a p-group of maximal class; thus the base group of $W_{p,n}$ is $\mathfrak{ZC}$-embedded in $W_{p,n}$.

Although a unified discussion could also be possible, we deal with the case when $G/Z(G)$ is finite first. In this case, excluding the obvious small exceptions, $G/Z(G)$ is a rather special group of maximal class.

Proposition 3.9. *Let p be a prime. For a finite nontrivial p-group Q the following conditions are equivalent:*

(i) *there exists $G \in$ (CO) such that $G/Z(G) \simeq Q$;*

(ii) *either Q is elementary abelian of order p^2 or $Q = A^* \rtimes \langle y\rangle$, where A^* is abelian, y has order p and acts on A^* uniserially and inducing a nontrivial splitting automorphism;*

(iii) *$Q \simeq \overline{W}_{p,n}/N$, where n is a positive integer and N is a $\overline{W}_{p,n}$-invariant subgroup of $\overline{W}'_{p,n}$.*

Moreover, if Q satisfies these conditions then the rank r of A^* in (ii) is at most $p-1$, and $r = p-1$ unless A^* has exponent p.

Proof. We know that (i) implies (ii). Assume that (ii) holds, and let $p^n = \exp A$. If Q is abelian then $Q \simeq \overline{W}_{p,n}/\overline{W}'_{p,n}$. In the other case, uniseriality implies that A^* is a cyclic $\langle y\rangle$-module, say $A^* = \langle u\rangle^{\langle y\rangle}$. Now $\overline{W}_{p,n}$ has a presentation with two generators, a, x and relations $a^{p^n} = x^p = 1$, $[a^{x^i}, a^{x^j}] = 1$ for all integers i, j and $aa^xa^{x^2}\cdots a^{x^{p-1}} = 1$. These relations are satisfied by u and y (for a and x respectively), hence $Q \simeq \overline{W}_{p,n}/N$ for some $N \lhd \overline{W}_{p,n}$. Since $\overline{W}_{p,n}$ has maximal class and $\overline{W}'_{p,n} \nleq N$ we have $N < \overline{W}'_{p,n}$. Thus (iii) holds. Finally, assume (iii). If $N = 1$ let $G = W_{p,n}$, otherwise let $G = \overline{W}_{p,n}/[N, \overline{W}_{p,n}]$. In either case $Q \simeq G/Z(G)$ and G has an abelian subgroup of index p, thus Corollary 3.8 shows that $G \in$ (CO) and (i) holds. Thus we have proved that the conditions (i), (ii) and (iii) are pairwise equivalent. The final claim can be deduced from the structure of $\overline{W}_{p,n}$. □

As already noted, the requirement that y acts uniserially on A^* in (ii) can be replaced by either of $|C_{A^*}(y)| = p$ and $|A^*/[A^*, y]| = p$. Also note that these conditions imply that A^* is either homocyclic or the direct product of two homocyclic groups, of exponents p^λ and $p^{\lambda+1}$ for some positive integer λ. This follows from part (iii), or also directly from [6], Proposition 4.3.6.

In the case of infinite p-groups, equivalences like those in Proposition 3.9 still hold, but we obtain a sharper and more explicit result, the structure of $G/Z(G)$ being uniquely determined.

Proposition 3.10. *Let G be group such that $G/Z(G)$ is an infinite locally finite p-group for some prime p. Then $G \in (\mathrm{CO})$ if and only if $G/Z(G) \simeq \overline{W}_{p,\infty}$.*

Proof. Assume that $G \in (\mathrm{CO})$ and let $Q := G/Z(G) = A^* \rtimes \langle y\rangle$ as in the discussion preceding Proposition 3.9, and fix $x \in yZ(G)$. If $F/Z(G)$ is a finite, nontrivial y-invariant subgroup of A^* then $Z(F\langle x\rangle) = C_F(x) = Z(G)$, so Proposition 3.9 shows that $F/Z(G)$ has rank $p-1$ at most, and it has rank $p-1$ if it has exponent greater than p. It follows that A^* has rank $p-1$. Since y acts on it uniserially, the characteristic subgroups of A^* form a chain and therefore A^* is divisible. For all positive integers n, let $A^*_n = \Omega_n(A^*)$. What we have just proved and Proposition 3.9 yield $A^*_n\langle y\rangle \simeq \overline{W}_{p,n}$. Now $Q = \bigcup_{n\in\mathbb{N}} A^*_n\langle y\rangle$ and each $\overline{W}_{p,n}$ can be identified with a subgroup of $\overline{W}_{p,\infty}$ in such a way that $\overline{W}_{p,\infty} = \bigcup_{n\in\mathbb{N}} \overline{W}_{p,n}$, so we only have to show that, for all n, any isomorphism $A^*_n\langle y\rangle \to \overline{W}_{p,n}$ can be extended to an isomorphism $A^*_{n+1}\langle y\rangle \to \overline{W}_{p,n+1}$. To this aim it is enough to observe the following: given $b, c \in A^*$ such that $A^*_n = \langle b\rangle^{\langle y\rangle}$ and $c^p = b$, we have $A^*_{n+1} = \langle c\rangle^{\langle y\rangle}$. To prove that, note that the mapping $u \mapsto u^p$ induces a $\langle y\rangle$-isomorphism from A^*_{n+1}/A^*_1 to A^*_n, hence $\langle c\rangle^y A^*_1 = A^*_{n+1}$ and so $\langle c\rangle^y = A^*_{n+1}$, because $A^*_1 \le (A^*_{n+1})^p$. Thus we prove that $Q \simeq \overline{W}_{p,\infty}$ if $G \in (\mathrm{CO})$.

Conversely, if $G/Z(G) \simeq \overline{W}_{p,\infty}$ then G has a subgroup A of index p such that $A/Z(G)$ is a divisible abelian p-group, therefore such is $A/Z(A)$. But this implies that A is abelian, hence $G \in (\mathrm{CO})$ by Theorem 3.5. □

Thus, if $G \in (\mathrm{CO})$ and $G/Z(G)$ is a locally finite p-group, also in the case when this latter quotient is infinite the abelian maximal subgroup of G has rank $p-1$ modulo $Z(G)$.

We can sum up the results in the last part of this section as follows:

Theorem 3.11. *Let G be a locally nilpotent group. Then $G \in (\mathrm{CO})$ if and only if either G is abelian or there are finitely many, pairwise distinct primes $p_1, p_2, \dots, p_k$ such that G has a central decomposition $G = G_1G_2\cdots G_k$, where, for each i, either $G_i/Z(G_i) \simeq \overline{W}_{p_i,\infty}$ or G_i has an abelian subgroup of index p_i and $G_i/Z(G_i) \simeq \overline{W}_{p_i,n}/N$, where n is a positive integer and N is a $\overline{W}_{p_i,n}$-invariant subgroup of $\overline{W}'_{p_i,n}$.*

A couple of final remarks are in order. Firstly, as happens when $G_i/Z(G_i)$ is infinite, also when $p_i = 2$ the requirement that G_i has an abelian subgroup of index p_i is redundant.

Corollary 3.12. *Let G be group such that $G/Z(G)$ is a nontrivial locally finite 2-group. Then $G \in (\mathrm{CO})$ if and only if $G/Z(G)$ is either dihedral or locally dihedral.*

Proof. The corollary follows from Theorem 3.11 and from the remark that if $G/Z(G)$ is dihedral or locally dihedral then it has a locally cyclic subgroup $A/Z(G)$ of index 2, and A must be abelian. □

The second remark is that a similar statement fails for all other primes:

Example 3.13. For every odd prime p and every integer $n > 1$ there exists a p-group G such that $G/Z(G) \simeq \overline{W}_{p,n}/Z(\overline{W}_{p,n})$ but no maximal subgroup of G is abelian, so that $G \notin (\mathrm{CO})$. Our examples are defined as follows. Let $q = p^{n-1}$ and let A be the group in the variety of nilpotent groups of class 2 and exponent p^n generated by elements $g_1, g_2, \dots, g_{p-2}, c$ subject to the extra defining relations:

$$[g_i, g_j] = c^{q\nu_{j-i}},\ \text{for } 1 \le i < j \le p-2; \qquad [g_i, c] = c^{q\mu},\ \text{for } 1 \le i \le p-2. \tag{$\mathfrak{R}$}$$

for some integers $\nu_1, \nu_2, \dots, \nu_{p-3}, \mu$, not all divisible by p. For a suitable choice of these integers an automorphism y of A is defined by the assignments:

$$g_1 \mapsto g_2 \mapsto \cdots \mapsto g_{p-2} \mapsto \Big(\big(\textstyle\prod_{i=1}^{p-2} g_i^i\big)c^{-1}\Big)^\tau \qquad \text{and} \qquad c \mapsto \big(\textstyle\prod_{i=1}^{p-2} g_i^{i-p-i\tau}\big)c^\tau,$$

where $\tau = 1 + p + p^2 + \cdots + p^{n-1}$. In fact, for any choice of $\nu_1, \nu_2, \dots, \nu_{p-3}, \mu$ the relations in $(\mathfrak{R})$ are preserved by these assignments with the possible exception of the $p-3$ relations $[g_i, g_{p-2}] = c^{q\nu_{p-2-i}}$ where $1 \le i < p-2$. The condition that any of these is preserved is expressed by a linear equation modulo p; thus we have a system of $p-3$ equation on the $p-2$ indeterminates $\nu_1, \nu_2, \dots, \nu_{p-3}, \mu$; each nontrivial solution of this linear system over the integers mod p gives rise to a group A in which the automorphism y is well defined. For such an A let $G = A \rtimes \langle y\rangle$; we have $1 \ne A' = \langle c^q\rangle \le Z(G)$. Now, let $W = W_{p,n} = B \rtimes \langle x\rangle$, where $B = \langle b\rangle^{\langle x\rangle}$ is the base group. Let bars to denote images modulo $Z_2(W)$ in W and modulo A' in G, and let $b_i = \bar{b}^{\bar{x}^{i-1}}$ for all $i \in \mathbb{N}$. It can be checked that the assignments $\bar{g}_i \mapsto \bar{b}_i$ for all $i \in \{1, 2, \dots, p-2\}$ and $\bar{c} \mapsto \prod_{i=1}^{p-1} \bar{b}_i^i$ define an isomorphism $\alpha\colon \bar{A} \to \bar{B}$ such that $(\bar{a}^{\bar{y}})^\alpha = \overline{(a^\alpha)^x}$ for all $a \in A$; thus y acts on A/A' in the same way as x acts on $B/Z_2(W)$; it follows that y induces a splitting automorphism of order p on A/A'. Since $[A', y] = 1$ this implies that y has order p. As a further consequence, α may be extended to an isomorphism $\bar{G} \to \bar{W}$. Yet another one is that $Z(G) = A'$, for now we know that $\bar{G}$ has maximal class and $A' < \Omega_1(A) = \langle g_1^q, g_2^q, \dots, g_{p-2}^q\rangle \times A' \lhd G$, hence $Z(G) \le \Omega_1(A)$ and the conclusion follows. Therefore

$G/Z(G) \simeq W/Z_2(W) \simeq \overline{W}_{p,n}/Z(\overline{W}_{p,n})$. If G had an abelian subgroup of index p then G/A' would be nilpotent of class 2 at most, which is clearly false. So G has all properties that we required.

4. Properties of soluble (CO)-groups

In this section we will consider soluble (CO)-groups. We do not aim at a classification of such groups. Rather, we shall prove some general properties, one of which will be useful for the upcoming discussion on supersoluble (CO)-groups.

Theorem 4.1. *Let G be a locally soluble (CO)-group. Then G is locally (abelian-by-finite), hence locally polycyclic, and G' is nilpotent of class at most 2. Thus G is soluble of derived length at most 3.*

If $F = \operatorname{Fit} G$ then G/F is cyclic. If G is not abelian-by-finite then F is abelian and G/F is infinite, and F is maximal among the locally nilpotent subgroups of G, hence it is the Hirsch-Plotkin radical of G.

Proof. We may assume that G is soluble, for if the result is true in this case then all finitely generated subgroups of G have derived length at most 3, hence G is soluble itself. So, let G be soluble. By Theorem 3.5, F is metabelian and abelian-by-finite. On the other hand $C_G(F) \leq F$, so $\operatorname{Out}_G(F) = G/F$, hence G/F is cyclic. Therefore G has soluble length 3 at most, that is, G'' is abelian. Hence $[G', G''] = 1$ by Lemma 2.4 and G' is nilpotent of class 2 at most. There is nothing more to prove in the case when G/F is finite, and we may assume that G/F is infinite. Let $X = C_G(Z(F))$. By Corollaries 3.6 and 3.7, F is nilpotent and $F/Z(F)$ is finite, so $Y := C_X(F/Z(F))$ has finite index in X. Now, $Y/Z(F)$ embeds in the stability group of the series $1 \leq Z(F) \leq F$, which is periodic. Hence $X/Z(F)$ is periodic. As G/F is infinite cyclic $X = F$. Now let A be a maximal normal abelian subgroup of F. Since $Z(F) \leq A$ we have $C_G(A) \leq C_G(Z(F)) = F$, hence $A = C_G(A)$. Let $N = N_G(A)$, then $F \leq N$ and $|G : N|$ is finite, because $F/Z(F)$ is finite, hence N/F is infinite. But $N/A = \operatorname{Out}_G(A)$ is cyclic; it follows that $A = F$, that is: F is abelian. If $F \leq S \leq G$ and S is locally nilpotent then $Z(S) \lhd G$, hence $Z(S) \leq F$. But $S/Z(S)$ is periodic (see Theorem 3.5) and it follows that $S = F$. Hence F is a maximal locally nilpotent subgroup of G.

It remains to show that G is locally (abelian-by-finite). To this end we may assume that G is finitely generated, still retaining the assumption that G/F is infinite, and prove that it is abelian-by-finite. If G is not polycyclic, than it has a quotient Q which is just non-polycyclic (see, e.g., [7], 7.4.1). The FC-centre of Q is easily seen to be trivial, and this implies that $C_Q(N) = 1$ for all normal subgroups N of finite index in Q. Hence $Q/N = \operatorname{Out}_Q(N)$ is cyclic and $Q' \leq N$ for all such N. But Q is residually finite, because G is metabelian and by a well-known theorem by Jategaonkar (see [7], 7.2.1). Therefore Q is abelian, a contradiction. This argument shows that G must be polycyclic. Then F is finitely generated. Let $x \in G$ be such that $G = F\langle x\rangle$. There exists $A \leq F$ such that F/A is finite, $A \lhd G$ and A is free abelian. Fix a prime p; then $|F/A^p|$ is finite and there exists a positive integer n such that $[F, x^n] \leq A^p$. For all $k \in \mathbb{N}$ the group $F\langle x^n\rangle/A^{p^k}$ is nilpotent, because x^n centralises all factors $A^{p^i}/A^{p^{i+1}}$. Hence Lemma 3.2 shows that x^{np} centralises A/A^{p^k}. Therefore $[A, x^{np}] \leq \bigcap_{k\in\mathbb{N}} A^{p^k} = 1$. Now, x^{np} stabilises the series $1 < A \leq F$; since F/A is finite and A is torsion-free we have $[F, x^{np}] = 1$. This is a contradiction, as $C_{\langle x\rangle}(F) = 1$. The proof is complete. □

For any group X let $FC(X)$ denote the FC-centre of X.

Corollary 4.2. *Let $G \in$ (CO). Then:*

(i) *if S is a soluble subnormal subgroup of G then $FC(S) \leq FC(G)$;*
(ii) $\operatorname{Fit} G \leq FC(G)$.

Proof. To prove (i) we may assume $S \lhd G$. Let $a \in FC(S)$, let T be a right transversal of $C_S(a)$ in S (so T is finite), and $C = C_G(S)$. We have $G = CS\langle x\rangle$ for some element x. If $H = \langle a, x, T\rangle$ then $G = C_G(a)H$, and $a \in FC(G)$ if and only if $a \in FC(H)$. Thus, at the expense of replacing G with H we may assume that G is soluble and finitely generated. By Theorem 4.1, G has a normal abelian subgroup A of finite index. There exists a positive integer λ such that $x^\lambda \in A$ and $[x^\lambda, S] \leq S \cap A$. The stability group of the series $1 \leq S \cap A \leq S$ is finite, so that $x^\mu \in C$ for some positive integer μ. Therefore $|G : CS|$ is finite, and it follows that $a \in FC(G)$. Thus (i) is proved. Since $\operatorname{Fit} G$ is centre-by-finite, by Corollaries 3.6 and 3.7, part (ii) is a special case of (i). □

Note that soluble non-metabelian (CO)-groups can be easily constructed, the smallest example being the following. Let $G = Q \rtimes \langle x\rangle$, where Q is the quaternion group of order 8, x has order 3 and $[Q, x] \neq 1$, hence $[Q, x] = Q$. Then $G' = Q$ is not abelian and $G \in$ (CO). This follows from the fact that, for every subgroup H of G such that $|G : H|$ is not prime, H and $\operatorname{Aut} H$ are cyclic.

Next we show that soluble (CO)-groups which are not abelian-by-finite do exist. According to Theorem 4.1 such groups are bound to be metabelian and not finitely generated.

Example 4.3. Let A be an infinite periodic abelian group, all whose primary components have prime order. Fix any automorphism α of A of infinite order and form the semidirect product $G = A \rtimes \langle x\rangle$, where x (has infinite order and) acts on A like α. Then $G \in$ (CO) and, clearly, $A = \operatorname{Fit} G$.

To show that $G \in$ (CO), let $H \leq G$. Every subgroup of A is the product of some of the primary components of A, hence $H \cap A = A_\pi$ and $H = A_\pi\langle bx^\lambda\rangle$ for some set π of primes, $\lambda \in \mathbb{Z}$ and $b \in A_{\pi'}$ (following standard notation, $A_{\pi'}$ is the complement to A_π in A). Let $N = N_G(H)$. Then $N = A_\pi N_{A_{\pi'}\langle x\rangle}(H)$. If $a \in A_{\pi'}$ and $t \in \mathbb{Z}$ are such that $g := ax^t \in N$ then $k := [g, bx^\lambda] \in H \cap G' \leq H \cap A = A_\pi$. But $k = [a, x^\lambda]^{x^t}[x^t, b]^{x^\lambda} \in A_{\pi'}$, since $A_{\pi'} \lhd G$. It follows that $k = 1$. Therefore $N_{A_{\pi'}\langle x\rangle}(H) = C := C_{A_{\pi'}\langle x\rangle}(bx^\lambda)$ and $N = A_\pi C$. Now let $\xi\colon N \to \operatorname{Aut} H$ be the map describing the conjugation action of N on H, and let $I = \operatorname{Inn} H$. Then $\operatorname{Out}_G(H) \simeq N^\xi I/I = C^\xi I/I$, because $C \leq N \leq HC$. On the other hand, if $\rho\colon \operatorname{Aut} H \to \operatorname{Aut} A_\pi$ is the restriction map then $C^\xi \cap \ker\rho = 1$, hence $C^\xi \simeq C^{\xi\rho}$. But $C^{\xi\rho} \simeq C/C_C(A_\pi)$ is cyclic. Therefore $\operatorname{Out}_G(H)$ is cyclic. This proves that $G \in (\mathrm{CO})$, as required.

The rest of this section is devoted to polycyclic-by-finite groups. We shall show that such groups are in (CO) if and only if all of their finite quotients are in (CO). The following notation comes handy: if G is a polycyclic-by-finite group we let $\mathcal{F}(G)$ denote the set of all normal subgroups of finite index in G. It is well-known that $H = \bigcap\{HK \mid K \in \mathcal{F}(G)\}$ for all $H \leq G$. A further result by Segal ([11], Theorem B) is that for every positive integer n two n-tuples of elements of G are conjugate if and only if they are conjugate modulo every $K \in \mathcal{F}(G)$. It easily follows (see [12], Theorem 4) that an automorphism α of G is inner if and only if it induces an inner automorphism on G/K for every characteristic subgroup K of finite index in G. We shall use the following equivalent form of this result.

Corollary 4.4. *Let G be a polycyclic-by-finite group and $H \leq G$. For all $K \in \mathcal{F}(G)$ let $C(K)/K = C_{G/K}(HK/K)$. Then $HC_G(H) = \bigcap\{HC(K) \mid K \in \mathcal{F}(G)\}$.*

Proof. Let J be the intersection of the subgroups $HC(K)$ with K ranging over $\mathcal{F}(G)$; it is clear that J normalises H, as $H = \bigcap\{HK \mid K \in \mathcal{F}(G)\}$. Let $g \in J$. If N is a characteristic subgroup of finite index in H then $N = H \cap NK$ for some $K \in \mathcal{F}(G)$. Now $g \in HC(K)$, hence g induces an inner automorphism on $H/H \cap K$, hence on H/N too. By Segal's result cited in the previous paragraph, it follows that g induces an inner automorphism on H, that is, $g \in HC_G(H)$. Therefore $J \leq HC_G(H)$. The reverse inclusion is obvious, thus the proof is complete. □

Proposition 4.5. *Let G be a polycyclic-by-finite group. Then $G \in (\mathrm{CO})$ if and only if all finite quotients of G are in (CO).*

Proof. The necessity of the condition is clear. Suppose that all finite quotients of G are in (CO). Let $H \leq G$ and $N = N_G(H)$. For all $K \in \mathcal{F}(G)$ the quotient $N/N \cap HC(K) \simeq NC(K)/HC(K)$ is cyclic, as $G/K \in (\mathrm{CO})$. By using Corollary 4.4 it follows that $\operatorname{Out}_G(H) = N/HC_G(H)$ is abelian. Every finite subgroup $\bar F = F/HC_G(H)$ of $\operatorname{Out}_G(H)$ is cyclic because, by Corollary 4.4 again, there exists $K \in \mathcal{F}(G)$ such that $HC_G(H) = F \cap HC(K)$ and hence $\bar F \simeq C(K)F/HC(K)$. Finally, $\operatorname{Out}_G(H)$ is finitely generated abelian; if it is not cyclic then it has a subgroup $H_1/HC_G(H)$ of finite index such that N/H_1 is not cyclic. But $C_G(H_1) \leq C_G(H) \leq H_1$ and $N \leq N_G(H_1)$, hence N/H_1 is a finite subgroup of $\operatorname{Out}_G(H_1)$. By the previous part of the proof N/H_1 is cyclic. This contradiction shows that $\operatorname{Out}_G(H)$ must be cyclic, hence $G \in (\mathrm{CO})$, as required. □

5. Supersoluble groups

The aim of this final section is a description of finite supersoluble (CO)-groups. Thanks to Proposition 4.5 this will also yield a characterization of arbitrary supersoluble (CO)-groups. If G is such a group, since $G/\operatorname{Fit} G$ is finite Lemma 2.7, together with the results in Section 3, shows that G is abelian-by-(finite cyclic). However, unlike nilpotent (CO)-groups, finite supersoluble (CO)-groups may fail to have an abelian normal subgroup of square-free index (an example being provided by the holomorph of a group of order 5), and this makes their stucture less transparent. Thus, rather than in terms of the action of the group on a big abelian normal subgroup, as we did in Section 3, our description will be worded in terms of a suitable Sylow basis. In the proof we shall make frequent use of the following (certainly well known) elementary lemma, whose proof we omit.

Lemma 5.1. *Let $G = AB$ be a group, where A and B are periodic subgroups and $\pi(A) \cap \pi(B) = \varnothing$. If $N \lhd G$ then $N = (N \cap A)(N \cap B)$.*

Let G be a finite supersoluble group, let $p_1, p_2, \ldots, p_n$ be the primes dividing $|G|$ and assume $p_1 > p_2 > \cdots > p_n$. For each $i \in I := \{1, 2, \ldots, n\}$ choose a Sylow p_i-subgroup P_i of G in such a way that $P_1, P_2, \ldots, P_n$ form a Sylow basis of G. Every subgroup of G has a conjugate X which is factorised with respect to the factorization $G = P_1P_2\cdots P_n$, that is, such that $X = \prod_{i\in I}(X \cap P_i)$. Let $H \leq G$. In order to study $\operatorname{Out}_G(H)$ we may replace H with a conjugate of its and assume that $N_G(H)$ is factorised with respect to $G = P_1P_2\cdots P_n$. Then, by Lemma 5.1, also H and $C_G(H)$ are factorised, and

$$\operatorname{Out}_G(H) = \frac{\prod_{i\in I} N_{P_i}(H)}{\prod_{i\in I} P_i^* C_{P_i}(H)}\ ,$$

where $P_i^* := H \cap P_i$ for all $i \in I$. Thus, $\operatorname{Out}_G(H)$ is cyclic if and only if each of the groups $X_i := N_{P_i}(H)/P_i^*C_{P_i}(H)$ is cyclic and the normalizers $N_{P_i}(H)$ centralise each other modulo $HC_G(H)$. If $i, j \in I$ and $i < j$ we have $[P_i, P_j] \leq P_i$ and $HC_G(H) \cap P_i = P_i^*C_{P_i}(H)$, hence the latter condition is equivalent to requiring that $[N_{P_i}(H), N_{P_j}(H)] \leq P_i^*C_{P_i}(H)$ for all such pairs i, j.

Interest in these conditions motivates the next three lemmas, leading to our classification theorem.

Lemma 5.2. *Let G be a finite supersoluble (CO)-group and p a prime, and let P be a nonabelian normal Sylow p-subgroup of G. If Q is a p'-Hall subgroup of G then*

(i) $P \cap G' \leq Z_{c-1}(P)$, *where c is the nilpotency class of P;*
(ii) *if P is two-generator then $G = P \times Q$;*
(iii) $[P, Q] \leq Z(P)$.

Proof. Let $Z := Z_{c-1}(P)$. Then $|P/Z| = p^2$, by Corollary 3.8. In order to prove (i) we may factor out $Z_{c-2}(P)$ and assume that $Z = Z(P)$. Since G is supersoluble it has a normal subgroup A such that $Z < A < P$. By Maschke's theorem G also has a normal subgroup $B \neq A$ such that $Z < B < P$. Now A and B are abelian and $P = AB$, hence Lemma 2.4 yields $[P, G'] = 1$, so $P \cap G' \leq Z(P) = Z$. Thus (i) is proved. As a consequence, $[P, Q] \leq P \cap G' \leq Z$ by (i), i.e., Q acts trivially on P/Z. Now, if P is two-generator then Z is the Frattini subgroup of P, hence $[P, Q] = 1$, which proves (ii). Finally, if $c = 2$ then $Z(P) = Z$, hence $[P, Q] \leq Z$ gives (iii); if $c > 2$ then (iii) follows from (ii) and the fact that $P/Z(P)$ is nonabelian and two-generator by Corollary 3.7. □

Lemma 5.3. *Let $G = P \rtimes Q$ be a periodic* (CO)*-group, where Q is locally nilpotent q-group for some prime q and P is a q'-group. Then either $[Q, P] = 1$ or $Q_P = C_Q(P)$ is abelian.*

Proof. By Lemma 3.2, Q has a maximal subgroup A which is abelian. Assume that Q_P is not abelian. Then $Q_P \nleq A$ and $K := A \cap Q_P \nleq Z(Q)$, hence $C_Q(K) = A$. Let $H = PK$. As $H \lhd G$, Lemma 5.1 shows that $C_G(H) = C_P(H)C_Q(H)$. Now $C_Q(H) = C_Q(K) \cap C_Q(P) = A \cap Q_P = K$, hence $C_G(H) \leq H$. Therefore $\mathrm{Out}_G(H) = G/H \simeq Q/K$, hence Q/K is cyclic. Since $Q_P \nleq A$ this shows that $A < Q_P$, that is $Q_P = Q$, or, equivalently, $[Q, P] = 1$. Thus the proof is complete. □

Lemma 5.4. *Let $G = R \rtimes (P \rtimes Q)$ be a finite group, where $|P|$ and $|RQ|$ are coprime. Let $H \leq G$ and assume that $H = R^*P^*Q^*$, where $R^* = R \cap H$, $P^* = P \cap H$ and $Q^* = Q \cap H$.*

(i) *Suppose that $RP \in$* (CO) *and $[P, Q] \leq Z(RP)$. Then $[N_P(H), N_Q(H)] \leq P^*C_{[P,Q]}(H)$ and $N_P(H)/P^*C_P(H)$ is cyclic.*
(ii) *Let $R = 1$. Suppose that Q is a q-group for some prime q and $Q \in$* (CO)*. If Q_P is abelian and Q/Q_P is cyclic then $N_Q(Q^*)/Q^*C_Q(H)$ is cyclic.*

Proof. In the hypothesis of (i), let $N = N_P(H)$ and $M = N_Q(H)$. By standard results on coprime actions, and since $[P, Q] \leq Z(P)$, we have $N = [N, Q^*] \times E$, where $E = C_N(Q^*)$. Note that $[N, Q^*] \leq P \cap H = P^*$, because Q normalises P and N normalises H. Moreover M normalises $[N, Q^*]$, hence $[N, M] \leq P^*[E, M]$. On the other hand, by looking at the action of M on N we get $N = [N, M] \times C_N(M)$, hence $E = C_{[N,M]}(Q^*)C_N(M)$ and $[E, M] = [C_{[N,M]}(Q^*), M] \leq C_{[N,M]}(Q^*)$. But $[N, M] \leq Z(RP)$, hence $C_{[N,M]}(Q^*) \leq C_{[P,Q]}(H)$; therefore $[N, M] \leq P^*[E, M] \leq P^*C_{[P,Q]}(H)$, as required. Now consider $J := N/P^*C_P(H)$. Since $N = [N, Q^*] \times E$ and $[N, Q^*] \leq P^*$, so that $P^* = [N, Q^*](E \cap P^*)$, we have $J \simeq J_1 := E/(E \cap P^*)C_P(H)$. Next, $C_P(H) = C_E(P^*R^*) = C_E([N, Q^*](E \cap P^*)R^*) = C_E((E \cap P^*)R^*)$, because $[N, Q^*] \leq Z(RP)$. Now, E normalises R^* and $E \cap R^* = 1$, hence $J_1 \simeq J_2 := ER^*/(E \cap P^*)C_E((E \cap P^*)R^*)R^*$. Finally, $RP \in$ (CO), hence $\mathrm{Out}_{ER^*}((E \cap P^*)R^*)$ is cyclic. By Lemma 5.1, $C_{ER^*}((E \cap P^*)R^*) = C_E((E \cap P^*)R^*)C_{R^*}((E \cap P^*)R^*)$, therefore $J_2 = \mathrm{Out}_{ER^*}((E \cap P^*)R^*)$ and J_2 is cyclic. Thus the proof of (i) is complete.

Next we shall prove (ii). If $Q_P = Q$ the conclusion is clear since in this case $N_Q(Q^*)/Q^*C_Q(H) = \mathrm{Out}_Q(Q)^*$. In the other case, by Lemma 3.2, the (only) maximal subgroup A of Q containing Q_P is abelian. If $Q^* \leq A$ then $A \leq C_Q(Q^*)$, hence $C_Q(H) \geq Q_P$. As Q/Q_P is cyclic the result follows. Thus we may assume $Q^* \nleq A$, hence $Q = Q_PQ^*$. Let $N = N_Q(Q^*)$. Then $Q_P \cap Z(N) \leq C_Q(H)$. Moreover $|N/N'Z(N)| \leq q^2$, by Corollary 3.8, and $N' \leq Q^* \cap Q_P$, hence $Q^*C_Q(H) \geq Q^*(Q_P \cap Z(N)) = Q^*(Q_P \cap N'Z(N))$. Let $Y = Q_P \cap N$ and note that $N = Q^*Y$. If $|N/Q^*C_Q(H)| > q$ then, as a consequence of our previous remarks, $|Y/Q_P \cap N'Z(N)| = |Y/Y \cap N'Z(N)| > q$ and so $N = YN'Z(N)$. Therefore $N = YZ(N)$ is abelian, so that $N = C_Q(Q^*)$ and $C_Q(H) = Y$; hence $N/Q^*C_Q(H) = N/Q^*Y = 1$, a contradiction. This shows that $|N/Q^*C_Q(H)| \leq q$; thus also (ii) is proved. □

Theorem 5.5. *Let G be a finite supersoluble group. Then $G \in$* (CO) *if and only if G has a Sylow basis $(P_1, P_2, \dots, P_n)$, where each P_i is a Sylow p_i-subgroup and the primes $p_1, p_2, \dots, p_n$ are such that $p_1 > p_2 > \cdots > p_n$, and the following conditions hold for all $i, j \in \{1, 2, \dots, n\}$:*

(i) $P_i \in$ (CO)*;*
(ii) $P_i/C_{P_i}(P_1P_2 \cdots P_{i-1})$ *is cyclic;*
(iii) *if $i < j$ then $[P_i, P_j] \leq Z(P_i)$;*
(iv) *if $i < j$ either $[P_i, P_j] = 1$ or $(P_j)_{P_i}$ is abelian.*

Proof. Let $(P_1, P_2, \dots, P_n)$ be a Sylow basis of G, where, for every $i \in \{1, 2, \dots, n\}$, P_i is a p_i-subgroup and the primes $p_1, p_2, \dots, p_n$ satify $p_1 > p_2 > \cdots > p_n$. Suppose that $G \in$ (CO). If $i \in \{1, 2, \dots, n\}$ and $R := P_1P_2 \cdots P_{i-1} \lhd G$, every nontrivial automorphism induced by conjugation by P_i on R is obviously not inner, hence $P_i/C_{P_i}(R)$ is cyclic and (ii) holds. Part (i) is obvious, (iii) and (iv) follow from Lemmas 5.2 and 5.3.

Conversely, assume that (i–iv) hold. Let $H \leq G$; we have to show that $\mathrm{Out}_G(H)$ is cyclic. As we observed earlier, there is no loss of generality in assuming that $N_G(H)$ is factorised with respect to the factorisation $G = P_1P_2 \cdots P_n$ and, further, once this assumption has been made, it will be enough to show that for all $i, j \in \{1, 2, \dots, n\}$ the factor $X_i := N_{P_i}(H)/P_i^*C_{P_i}(H)$ is cyclic and, if $i < j$, $[N_{P_i}(H), N_{P_j}(H)] \leq P_i^*C_{P_i}(H)$; here $P_i^* = P_i \cap H$. We argue by induction on n. We consider X_n first. Let $P = P_1P_2\cdots P_{n-1}$ and $Q = P_n$. If $Q \lhd G$ then $G = P \times Q$ and it follows that $X_n = \mathrm{Out}_Q(Q \cap H)$ is cyclic. If $Q \ntriangleleft G$ then Q_P is abelian by Lemma 5.3 and Lemma 5.4 (ii) yields that X_n is cyclic, since $N_Q(H) \le N_Q(Q \cap H)$. Now let $i < n$, and set $R = P_1P_2\cdots P_{i-1}$, $P = P_i$ and $Q = P_{i+1}P_{i+2}\cdots P_n$. By induction hypothesis $RP \in$ (CO). Also, $[P,Q] \le Z(P)$ by (iii), while $[P,Q,R] \le [(PQ)',R] = 1$ because G is supersoluble and $|PQ|$ and $|R|$ are coprime. Thus $[P,Q] \le Z(RP)$. Now Lemma 5.4 (i) shows that X_i is cyclic. The same lemma also shows that $[N_P(H), N_{P_j}(H)] \le (P \cap H)C_P(H)$ for all $j \in \{i+1,\dots,n\}$. Thus we have proved that $G \in$ (CO). □

As regards condition (ii), note that $C_{P_i}(P_1P_2\cdots P_{i-1})$ is the p_i-component of the Fitting subgroup of G.

Easy examples of finite and possibly nonnilpotent, supersoluble (CO)-groups provided by Theorem 5.5 are, for instance, the groups with cyclic Sylow subgroups. Such groups are metacyclic; the wreath products considered in the final part of Section 3 (see Theorem 3.11) provide (nilpotent) examples of arbitrarily high rank. Somehow more complex (nonnilpotent) examples can be constructed as follows. For a fixed positive integer n, starting from p_n choose primes $p_1, p_2, \dots, p_n$ such that $p_i \equiv 1 \pmod{p_{i+1}p_{i+2}\cdots p_n}$ for all i. Still for all $i \in I := \{1,2,\cdots,n\}$, let $X_i = A_i \rtimes \langle x_i\rangle$ be a nonabelian, centrally indecomposable p_i-group in the class (CO), where A_i is abelian and x_i has order p_i—for instance, X_i may be the wreath product of a cyclic group of order p_i^λ, for some positive λ, by a group of order p_i, and A_i may be the base group of X_i. Next let $P_i = X_i \times C_i$, where $C_i = \mathrm{Dr}_{j=i+1}^n C_{ij}$ is elementary abelian of order p_i^{n-i}, all subgroups C_{ij} having order p_i. Now, if $i, j \in I$ and $i < j$ we can let P_j act on P_i as follows: $[A_jC_j, P_i] = 1 = [x_j, X_i] = [x_j, C_{ik}]$ for all $k \ne j$, and x_j induces an automorphism of order p_j on C_{ij}. It is not hard to check that repeated semidirect products defined by these actions yield a finite supersoluble group $G = P_1P_2\cdots P_n$ satisfying the conditions in Theorem 5.5, hence $G \in$ (CO). Also, $Z(G) = \langle Z(X_i) \mid i \in I\rangle$ lies in the Frattini subgroup of G, hence G cannot be centrally factorised as in Lemma 2.3. This example can be modified by letting each element x_i have arbitrary order divisible by p_i, or possibly infinite. All properties discussed for G are retained with the only exception that if the order of some x_i is finite and not a power of p_i then a nontrivial abelian direct factor appears—we use Proposition 4.5 to prove that this group is in (CO) in the infinite case. Also note that we can thus obtain supersoluble (CO)-groups of arbitrarily high torsion-free rank which are not centrally decomposable in the sense of Lemma 2.3; this is in contrast with what we remarked in the case of nilpotent groups.

The example of this group G also shows that in nonnilpotent supersoluble (CO)-groups the Fitting subgroup (which is in our case $P_1A_2C_2A_3C_3\cdots A_nC_n$) may be nonabelian. There are obvious examples sharing this property, like suitable direct products, however G makes a more significant example in that the (nonabelian) p_1-component P_1 of $\mathrm{Fit}(G)$ is not even a central factor in G.

Still, there are restrictions on the Fitting subgroup of a finite supersoluble (CO)-group, in the case when the former is not abelian, which we feel worth recording.

Proposition 5.6. *Let G be a finite supersoluble* (CO)*-group whose Fitting subgroup F is not abelian. Then*

(i) *if A is a maximal normal abelian subgroup of G then $\pi(F) = \pi(A)$, G/A is cyclic, $|F/A|$ is square-free and coprime with $|G/F|$;*

(ii) *if P is a nonabelian primary component of F then P is a Sylow subgroup in G. Moreover, P is a direct factor in G modulo $Z(P)$, and if P is two-generator then it is a direct factor in G.*

Proof. Of course $A \le F$ and $\pi(A) = \pi(F)$, and G/A is cyclic by Lemma 2.7 (i), Also, $A = C_G(A)$ is maximal abelian in F, hence $|F/A|$ is square-free by Theorem 3.5. Let $p \in \pi(G/F)$ and let x be a p-element in G of order p modulo F. Then $A_p\langle x\rangle$ is a p-group, so $[A_p, x^p] = 1$ by Lemma 3.2. As $x^p \in F$ we also have $[A_{p'}, x^p] = 1$, hence $x^p \in C_G(A) = A$. Since G/A is cyclic this argument shows that $p \notin \pi(F/A)$. Thus (i) is proved. Now let P be as in (ii) and let p be the prime diving $|P|$. As $P \nleq A$ we have $p \in \pi(F/A)$ and so $p \notin \pi(G/F)$, hence P is a Sylow p-subgroup of G. Then Lemma 5.2 applies and the proof is complete. □

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