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Locally finite groups all of whose subgroups are boundedly finite over their cores

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Abstract. For n a positive integer, a group G is called core-n if H/H_G has order at most n for every subgroup H of G (where H_G is the normal core of H, the largest normal subgroup of G contained in H). It is proved that a locally finite core-n group G has an abelian subgroup whose index in G is bounded in terms of n.

1. Introduction

Given a positive integer n, a group G is called core-n if H/H_G has order at most n for every subgroup H of G. Here H_G denotes the normal core of H, the largest normal subgroup of G contained in H. Our main result is as follows.

Theorem 1. *Every locally finite core-n group G has an abelian subgroup whose index in G is bounded in terms of n.*

By the Mal'cev Local Theorem, it is sufficient to prove the theorem assuming the group G to be finite. A further argument [10] reduces the proof to the case where G is a finite p-group. In view of this reduction, it is natural to reformulate Theorem 1 for finite core-p^k p-groups, since the function bounding the index of an abelian subgroup then involves p and k naturally.

Theorem 2. *Let p be a prime and let G be a finite core-p^k p-group, where k is a positive integer. Then G has an abelian subgroup of index at most $p^{f(k)}$, where*

$$f(k) = k(k + (k/2 + 1)(k + 1)(2k + 1))((k + 1)(k^2 + k + 2) - 1) \qquad \text{if} \quad p \neq 2,$$

and

$$f(k) = (k + 1)(k + (k/2 + 1)(k + 1)(2k + 1))((k + 1)(k^2 + k + 2)) + 1 \qquad \text{if} \quad p = 2.$$

The first step in proving Theorem 2 is the following result, which is also interesting in its own right.

Theorem 3. *Let G be a finite core-p^k p-group, where k is some positive integer.*

(a) *If $p \neq 2$, then the nilpotency class of G is at most $(k + 1)(k^2 + k + 2)$.*

(b) *If $p = 2$, then G has a subgroup of index 2 whose nilpotency class is at most $(k + 1)(k^2 + k + 2) + 1$.*

The upper bounds obtained in Theorems 1 and 2 may well be far from the truth. It might be interesting, if very difficult, to find best possible bounds. We have succeeded in doing this for finite core-p p-groups for p odd, in which case the bound is p^2, and we have an 'almost' best possible bound for (arbitrary) core-2 2-groups; these results appear in [2]. Theorem 1 complements the result of [1], where it is proved that if all subgroups of a locally finite group are finite over their cores, then the group is abelian-by-finite (there is no function bounding the index of an abelian subgroup there, but also no restriction on the orders $|H/H_G|$ in the hypothesis). These results can also be viewed as duals of B. H. Neumanns theorem [7], stating: groups in which all subgroups are of finite index in their normal closures are finite-by-abelian. Some other results and discussion of further problems relating to groups all of whose subgroups are finite over their cores can be found in [1, 2, 3, 5, 6, 9, 10]. Our final remark is that (unlike in B. H. Neumanns theorem) one has to impose some finiteness condition on a core-finite group (even for core-p groups). Indeed, the Tarski p-groups, constructed by A. Yu. Ol'shanskii [8] for sufficiently large p, are core-p but not abelian-by-finite.

1. The nilpotency class of finite core-p^k p-groups

We begin by establishing some notation and making some elementary observations. We denote by $[A, {}_mB]$ the commutator subgroup

$$[\dots[A, \underbrace{B], \dots, B}_{m}].$$

Throughout this section, in which we prove Theorem 3, G will denote a finite core-p^k p-group, where $k \in \mathbb{N}$ and p is an arbitrary prime unless otherwise stated. Since $\langle g^{p^k} \rangle$ is normal in G for all $g \in G$, it follows easily that $[G^{p^k}, G'] = 1$ (and hence, incidentally, that G^{p^k} has class at most 2). Writing $B = G^{p^k} \cap G'$ and $A = B^{p^k}$, we see that B is abelian and that all subgroups of A are therefore G-invariant. Since $G/C_G(A)$ is abelian and of exponent at most p^k, it is clear that the following lemma ought to be of some use to us.

Lemma 1.1. *Let p be a prime and let $\langle a\rangle$ be a cyclic group of order p^t, and suppose that Γ is a p-subgroup of* Aut $\langle a\rangle$ *having exponent at most p^k.*

(i) *If p is odd, then $\langle a\rangle$ has a Γ-central series of length at most $k+1$.*
(ii) *If $p=2$, then $\langle a\rangle$ has a Γ_1-central series of length at most $k+2$, where Γ_1 is a subgroup of index at most 2 in Γ.*

Proof. For $t=1$ the result is immediate, so we shall assume that $t\geq 2$. Let α denote the automorphism given by $a^\alpha=a^{1+p}$. If p is odd, then α generates the Sylow p-subgroup of Aut $\langle a\rangle$ and has order p^{t-1}, so that $\Gamma=\langle\alpha^{p^\lambda}\rangle$ for some λ such that $\lambda+\kappa\geq t-1$. Since $(1+p)^{p^\lambda}\equiv 1 \pmod{p^{\lambda+1}}$, we see that Γ acts trivially on each of the factors $\langle a^{p^{i(\lambda+1)}}\rangle/\langle a^{p^{(i+1)(\lambda+1)}}\rangle$, $i\geq 0$. Since $(k+1)(\lambda+1)\geq t$, part (i) of the lemma now follows. For $p=2$, we note that $\langle\alpha\rangle$ has index at most 2 in Aut $\langle a\rangle$ and, setting $\Gamma_1=\Gamma\cap\langle\alpha\rangle$, we have that $\Gamma_1=\langle\alpha^{2^\lambda}\rangle$ for some λ satisfying $\lambda+k\geq t-2$. Part (ii) is then proved as above. □

Our next lemma will enable us to deal with the factor B/A and will also be of use in bounding the class of G/G^{p^k} .

Lemma 1.2. *If E is a normal abelian section of G of exponent p^l $(l\geq 1)$, then E has a G-central series of length at most $l(k+1)$.*

Proof. Clearly, we may assume that $l=1$, and we may as well assume that E is a (normal) subgroup of G. Then $E=(E\cap Z(G))\times F$ for some F and, by the core-p^k property, F has order at most p^k (since every nontrivial normal subgroup of G intersects $Z(G)$ nontrivially). It follows easily that $[E,{}_{k+1}G]=1$, as required. □

Lemma 1.3. *G/G^{p^k} has nilpotency class at most $k+k^2(k+1)$.*

Proof. We may assume that $G^{p^k}=1$. Let U be a maximal normal abelian subgroup of G and let $C=C_G(U/U^p)$. Then C stabilizes the series

$$U\geq U^p\geq U^{p^2}\geq\cdots\geq U^{p^k}=1$$

and, since $U=C_G(U)$, we deduce that C/U has class at most $k-1$ (see [4, Theorem 1.C.1]) and hence that C has derived length at most k. Applying Lemma 1.2 to the factors of the derived series of C we see that $C\leq Z_{k^2(k+1)}(G)$. Further, G/C is isomorphic to a group of automorphisms of $\bar U=U/U^p$ and, by Lemma 1.2, $[\bar U,{}_{k+1}(G/C)]=1$, which gives G/C of class at most k. The result follows. □

Now we complete the proof of Theorem 3. With the notation as previously established, all we need to do is to provide the bounds obtained in our lemmas. For p odd, we use the facts that $[G'G^{p^k},A]=1$ and every subgroup of A is G-invariant, then apply Lemma 1.1 to show that $[A,{}_{k+1}G]=1$ (we remind the reader that $A=(G^{p^k}\cap G')^{p^k}$). By Lemma 1.2, $[B,{}_{k(k+1)}G]\leq A$, while $[G^{p^k},G]\leq G^{p^k}\cap G'=B$. Finally, we apply Lemma 1.3 and deduce that G has class at most

$$(k+1)+k(k+1)+1+(k+k^2(k+1))=(k+1)(k^2+k+2),$$

thus proving Theorem 3(a).

For $p=2$, we again have that G/A has class at most $(k+1)(k^2+k+1)$. Write $\Gamma=G/C_G(A)$, and let G_1 be the pre-image of the subgroup Γ_1 of index at most 2 in Γ which centralizes a series of length at most $k+2$ in A—the existence of Γ_1 is, of course, guaranteed by Lemma 1.1, as $[G'G^{2^k},A]=1$. Clearly, this subgroup Γ_1 satisfies our requirements, and the proof of Theorem 3 is complete.

2. An abelian subgroup of bounded index

Here we prove Theorems 1 and 2. For given m, the property that a group contains an abelian subgroup of index at most m can be written as a universal formula of predicate calculus. Hence, by the Mal'cev Local Theorem, it suffices to prove Theorem 1 for finite groups (see, for example, [4, Proposition 1.K.2]).

Next we show that Theorem 1 follows from Theorem 2. Let G be a group satisfying the hypothesis of Theorem 1. By the well-known result of Dedekind and Baer, if every subgroup of a group is normal, then the group has an abelian subgroup of index at most 2. If p is a prime greater than n, then every p-subgroup of G is normal in G, and so the Sylow p-subgroup of G is abelian. Suppose that P is a Sylow p-subgroup of G for some prime $p\leq n$. Then P is core-p^k for some k such that $p^k\leq n$ and so, by Theorem 2, P has a G-invariant abelian subgroup of index bounded in terms of p^k. Since G is the product of its Sylow p-subgroups (over all p), the result now follows easily.

Now we prove Theorem 2. Applying Theorem 3 and an easy induction argument, we are left to prove the following proposition on p-groups of nilpotency class 2.

Proposition 2.1. *Let p be a prime and let G be a finite core-p^k p-group of nilpotency class 2. Then G has an abelian subgroup of index at most $p^{f(k)}$, where*

$$f(k) = k(k + (k/2 + 1)(k + 1)(2k + 1)) \qquad \text{if} \quad p \neq 2,$$

and

$$f(k) = (k + 1)(k + (k/2 + 1)(k + 1)(2k + 1)) \qquad \text{if} \quad p = 2.$$

Proof. We recall first some formulae that hold in any nilpotent group F of class 2; they will be used usually without reference:

$$[ab, c] = [a, c][b, c],$$
$$[a^m, b] = [a, b]^m = [a, b^m], \qquad m \in \mathbb{N}.$$

In particular, the exponents of $F/Z(F$ and of F' are the same. Another consequence is that $[h, F]$ and $F/C_F(h)$ are isomorphic groups for every $h \in F$. □

Lemma 2.1. *Let H be a finite core-p^k p-group of nilpotency class 2. Then $H/Z(H)$ (and H') is of exponent at most p^k if $p \neq 2$, and at most 2^{k+1} if $p = 2$.*

Proof. We consider the case $p \neq 2$ first. We have to prove that every commutator is of order at most p^k. Thus, without loss of generality, we may assume that $H = \langle a, b\rangle$, and that $Z(H)$ is cyclic. By induction, supposing the opposite, we may assume that $[a, b]$ has order p^{k+1}.

Now, as $[a, b]^{p^{k+1}} = 1$, both $a^{p^{k+1}}$ and $b^{p^{k+1}}$ are central, so without loss of generality, since $Z(H)$ is cyclic, we have $a^{p^{k+1}} = b^{\lambda p^{k+1}}$ for some $\lambda \in \mathbb{N}$. Thus

$$(ab^{-\lambda})^{p^{k+1}} = a^{p^{k+1}} b^{-\lambda p^{k+1}} [b, a]^{-\lambda p^{k+1}(p^{k+1}-1)/2} = 1,$$

since p is odd and $[b, a]^{p^{k+1}} = 1$. So, with $a_1 := ab^{-\lambda}$, we have $G = \langle a_1, b\rangle$, where $a_1^{p^{k+1}} = 1$. By the core-p^k property, $\langle a_1^{p^k}\rangle$ is a normal subgroup of H, so $a_1^{p^k} \in Z(H)$, since it is of order at most p. Thus $[a_1, b]^{p^k} = 1$, a contradiction, since $[a_1, b] = [a, b]$.

Let now $p = 2$. Similarly, we have to prove that every commutator is of order at most 2^{k+1}, and we may assume $H = \langle a, b\rangle$, $Z(H)$ is cyclic and $[a, b]$ has order 2^{k+2}. Again, $a^{2^{k+2}} = b^{\lambda 2^{k+2}}$, so

$$(ab^{-\lambda})^{2^{k+2}} = [b, a]^{-\lambda 2^{k+1}(2^{k+2}-1)}.$$

Hence, as above, without loss of generality, we may assume $\lambda = 1$ (else $a_1 := ab^{-\lambda}$ has order $\leq 2^{k+2}$, so $a_1^{2^{k+1}}$ is central and $[a_1, b] = [a, b]$ has order $\leq 2^{k+1}$, a contradiction). So, replacing $ab^{-\lambda}$ by a, we have

$$a^{2^{k+2}} = [b, a]^{2^{k+1}}$$

and a has order *precisely* 2^{k+3}. But $\langle a^{2^k}\rangle$ is a normal subgroup of G, so $[a^{2^k}, b] = a^{\varepsilon 2^k}$ for some ε. But $[a, b]^{2^k}$ has order 4, so $a^{\varepsilon 2^k}$ has order 4, so $\varepsilon = 2\delta$, where δ is odd, whence $[a, b]^{2^k} = a^{\delta 2^{k+1}}$. But then $[a^{\delta 2^{k+1}}, b] = 1$, so $[a, b]^{\delta 2^{k+1}} = 1$ and hence $[a, b]^{2^{k+1}} = 1$, a contradiction that completes the proof. □

We fix the notation $k_0 = k$ if p is odd, and $k_0 = k+1$ if $p = 2$,so that p^{k_0} is an upper bound for the exponent of $G/Z(G)$ and of G', by Lemma 2.1.

We proceed with the proof of Proposition 2.1. We may assume the rank of $G/Z(G)$ to be greater than k, since otherwise the index of $Z(G)$ is at most $p^{k_0 k}$. Hence the rank of $G/\Phi(G)$ is also at least $k + 1$.

Choose the largest possible N and a set of elements $\{x_i \mid i = 1, \ldots, N\}$ satisfying the following conditions:

(1) the x_i are linearly independent modulo the Frattini subgroup $\Phi(G)$;
(2) $[x_i, x_j] = 1$ for all $i, j = 1, \ldots, N$;
(3) the rank of $[x_i, G]$ (which is equal to the rank of $G/C_G(x_i)$) is not greater than $(k/2 + 1)(k + 1)$.

To show that sets satisfying (1)–(3) do exist, take $k + 1$ elements $b_1, \ldots, b_{k+1}$ linearly independent modulo $\Phi(G)$, and generate a subgroup $B = \langle b_1, \ldots, b_{k+1}\rangle$. Note that B has rank at most $(k/2 + 1)(k + 1)$ (that is, each of its subgroups can be generated by $(k/2+1)(k+1)$ elements). Since $|B : B_G| \leq p^k$, we have $B_G \nleq \Phi(G)$. Take $x_1 \in B_G \setminus \Phi(G)$. Then $\{x_1\}$ clearly satisfies (1) and (2). The rank of $[x_1, G]$ is at most $(k/2 + 1)(k + 1)$ since $[x_1, G] \leq B_G \leq B$.

Let X be an abelian normal subgroup containing the (maximal) set $\{x_i \mid i = 1, \ldots, N\}$ and $Z(G)$ (for example, take $\langle\{x_i \mid i = 1, \ldots, N\}, Z(G)\rangle$). We shall prove that X is a desired abelian subgroup of index at most $p^{k_0(k+(k/2+1)(k+1)(2k+1))}$ in G. Since the exponent of $G/Z(G)$, and hence of G/X, is at most p^{k_0}, we need only prove that the rank of the abelian group G/X is at most $k + (k/2 + 1)(k + 1)(2k + 1)$. The latter rank obviously coincides with that of $G/X\Phi(G)$, since $X \geq Z(G) \geq G'$.

Lemma 2.2. *The rank of* $\Big(\bigcap_{i=1}^{N} C_G(x_i)\Big)\Phi(G)/X\Phi(G))$ *is at most k.*

Proof. Otherwise, we could pick $k+1$ elements $b_1, \ldots, b_{k+1}$ in $\bigcap_{i=1}^N C_G(x_i)$ which are linearly independent modulo $X\Phi(G)$. Again, for $B = \langle b_1, \ldots, b_{k+1}\rangle$, there is $x_{N+1} \in B_G \smallsetminus X\Phi(G)$. Then the set $\{i = 1, \ldots, N\} \cup \{x_{N+1}\}$ would also satisfy (1)–(3), contrary to the maximality of N. We have (1) by the choice of the b_i linearly independent modulo $X\Phi(G)$; we have (2), since $x_{N+1} \in B \leq \bigcap_{i=1}^N C_G(x_i)$; and (3) holds for x_{N+1} by the same argument as for x_1 above. □

Corollary. *It suffices to show that the rank of* $G/\bigcap_{i=1}^{N} C_G(x_i)$ *is at most* $(k/2+1)(k+1)(2k+1)$.

Proof. Note that the ranks of $G/\bigcap_{i=1}^N C_G(x_i)$ and $G/\big(\bigcap_{i=1}^N C_G(x_i)\big)\Phi(G)$ are the same. In the series

$$G \geq \Big(\bigcap_{i=1}^{N} C_G(x_i)\Big)\Phi(G) \geq X\Phi(G),$$

the rank of the second factor is at most k, by Lemma 2.2. If the Corollary holds true, then the rank of the first factor is at most $(k/2+1)(k+1)(2k+1)$, and hence the rank of $G/X\Phi(G)$ is at most $k+(k/2+1)(k+1)(2k+1)$, as required. □

Now choose the smallest M for which there is a subset $\{g_j \mid j = 1, \ldots, M\} \subseteq \{x_i \mid i = 1, \ldots, N\}$ such that

$$\bigcap_{j=1}^{M} C_G(g_j) = \bigcap_{i=1}^{N} C_G(x_i),$$

and fix the corresponding subset $\{g_j \mid j = 1, \ldots, M\}$.

The rank of $G/C_G(g_j)$ is at most $(k/2+1)(k+1)$ for every j, by condition (3), being actually the rank of one of the $G/C_G(x_i)$. By the Corollary above, it suffices to prove that $M \leq 2k+1$. For then the rank of

$$G/\bigcap_{i=1}^{N} C_G(x_i) = G/\bigcap_{j=1}^{M} C_G(g_j)$$

is at most $2k+1$ times $(k/2+1)(k+1)$, as required.

By the minimality of M, we have $C_G(g_j) \not\geq \bigcap_{l\neq j} C_G(g_l)$ for every $j = 1, \ldots, M$. So we choose $h_j \in \bigcap_{l\neq j} C_G(g_l) \smallsetminus C_G(g_j)$, for every $j = 1, \ldots, M$. Then $[g_l, h_l] \neq 1$ and $[g_l, h_j] = 1$ for all $j \neq l$, $j, l = 1, \ldots, M$.

We shall need the following elementary lemma.

Lemma 2.3. *Suppose that A is a finite abelian p-group of rank m and that B is a subgroup of A, and let r be the rank of B. Then there is a subgroup A_1 of A such that $A_1 \cap B = 1$ and the rank of A_1A^p/A^p is $m-r$.*

Proof. Taking $\Omega_1(B)$ instead of B, we may assume B to be of exponent p. Applying then induction on r, we are left with the case where $B = \langle b\rangle$ is cyclic of order p. Write A as the direct product of cyclics $\langle a_i\rangle$, and let $b = a_{i_0}^{\alpha} w$, where $a_{i_0}^{\alpha} \neq 1$ and w is a group word in the a_j, $j \neq i_0$. Then we can take $A_1 = \prod_{j\neq i_0} \langle a_j\rangle$. □

We introduce the subgroup $K = \langle [g_k, h_k] \mid k = 1, \ldots, M\rangle$. Let r be the rank of K. We shall prove the required inequality $M \leq 2k+1$ in two steps, in the following lemmas.

Lemma 2.4. *We have $M \leq k + r$.*

Proof. Suppose that $M \geq k+r+1$. Consider the abelian group

$$A = \langle g_k \mid 1, \ldots, M\rangle.$$

The rank of A is M, since the g_j are linearly independent, even modulo $\Phi(G)$ which contains A_p.

Set $D = A \cap K$, the rank of D being at most r. By Lemma 2.3, there is a subgroup H of A that intersects D (and hence K) trivially, such that the rank of HA^p/A^p is $\geq M-r \geq k+1$. Then $H_G \not\leq A^p$, since $|H : H_G| \leq p^k$.

Pick an element $u \in H_G \smallsetminus A^p$. Since the g_j are linearly independent modulo $\Phi(G)$, there are s and $\alpha \not\equiv 0 \pmod p$ such that $u = g_s^{\alpha} \cdot w$, where $w \in \langle g_i \mid i \neq s\rangle$. Now $[u, h_s] = [g_s, h_s]^{\alpha}$ is a nontrivial element of K that does not belong to H and hence does not belong to the normal subgroup H_G containing u, a contradiction. □

Lemma 2.5. *We have $r \leq k+1$.*

Proof. Suppose that $r \geq k+2$. In order to get a contradiction, we actually reduce the situation to that in the proof of Lemma 2.4 with $r = 1$, for some section of G. (Note that the core-p^k property is inherited by both subgroups and homomorphic images.)

First we factor out K^p. The rank of the image of K in G/K^p remains the same, and the images of the g_k remain linearly independent modulo the Frattini subgroup, since $K^p \leq \Phi(G)$.

Now we choose a subset $\{g_{j_s} \mid s = 1, \dots, r\}$ such that the commutators $[g_{j_s}, h_{j_s}]$, $s = 1, \dots, r$, are linearly independent, that is, generate a subgroup of rank r. Then we glue these commutators to one cyclic subgroup, by factoring out the subgroup

$$\Delta := \big\langle [g_{j_s}, h_{j_s}] \cdot [g_{j_t}, h_{j_t}]^{-1} \mid s \neq t,\ s, t = 1, \dots, r \big\rangle.$$

Again, the images of the g_{j_s} in $G/K^p\Delta$ remain linearly independent modulo the Frattini subgroup. Now the argument from the proof of Lemma 2.4 can be applied to the image of the subgroup $\langle g_{j_s}, h_{j_t} \mid s, t = 1, \dots, r \rangle$ in $G/K^p\Delta$, to arrive at a contradiction. □

By the remarks above, the proof of Proposition 2.1, and hence those of Theorems 1 and 2, is now complete.

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