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Groups with finite outer automizers

GIOVANNI CUTOLO

ABSTRACT. We describe generalised soluble or nilpotent groups G in which, for all $H \leq G$, the group of outer automorphisms induced on H by elements of G via conjugation (that is, the factor $N_G(H)/HC_G(H)$) is finite. It turns out that such groups are abelian-by-finite; the nilpotent ones are centre-by-finite. Also groups in which the same condition is imposed on abelian subgroups only are considered.

1. INTRODUCTION

Roughly speaking, this article is concerned with groups of automorphisms that subgroups of a group induce on other subgroups by conjugation. If G is a group and $H \leq G$, the *automizer* of H in G is the group $\operatorname{Aut}_G H := N_G(H)/C_G(H)$, obviously embeddable in $\operatorname{Aut} H$. Several articles in the literature have considered groups in which restrictions are imposed on the automizers of (some or all) subgroups. Mostly, the interest has been focused on groups with 'small' ([4, 5, 9]; also see [15]) or with 'large' ([3, 7, 10, 13]) automizers, where the automizer of H in G is called small when $N_G(H) = HC_G(H)$, that is, when only inner automorphisms are induced on H by $N_G(H)$, and 'large' when every automorphism of H is induced by some element of $N_G(H)$, or, in a weaker sense, when $N_G(H)/C_G(H) \simeq \operatorname{Aut} H$.

By discussing the embedding of a subgroup H in a group G, or more specifically in $N_G(H)$, it can be useful to disregard inner automorphisms of H and, rather than on $\operatorname{Aut}_G H$, focus on the group of outer automorphisms induced by $N_G(H)$ on H, that is, the image of the coupling of the natural extension $H \hookrightarrow N_G(H) \twoheadrightarrow N_G(H)/H$. Up to the obvious isomorphism, this group is $\operatorname{Out}_G H := N_G(H)/HC_G(H)$, which we call the *outer automizer* of H in G. This definition was introduced in [6].

Some important group theoretic properties can be described in terms of automizers or outer automizers. The most striking example is probably that of p-nilpotency. Both Burnside's and Frobenius' well known p-nilpotency criteria can be expressed in terms of automizers—for instance the latter can be stated by saying that, if G is a finite group and p is a prime, G is p-nilpotent if and only if $\operatorname{Aut}_G H$ (or, equivalently, $\operatorname{Out}_G H$) is a p-group for every p-subgroup H of G. Also, if P is a Sylow p-subgroup of G then the condition $\operatorname{Out}_P G = 1$ is equivalent to the requirement that $N_G(P)$ is p-nilpotent, and this latter property often has major consequences for the structure of G: sometimes it implies the p-nilpotency of G, as in Burnside's criterion or in its extension proved in [1]; if it holds for all Sylow subgroups of G then G is nilpotent, see [2]. More relevant for our purposes is the following result by Zassenhaus ([15], Theorem 7): if G is a finite group in which all abelian subgroups have trivial (outer) automizer, then G is abelian. Note that this theorem is not hard to prove in the case when G is a p-group, and once this has been done the full result follows from Burnside's criterion. It is also worth remarking that Zassenhaus' result cannot be extended to arbitrary infinite groups. Indeed, if F is a noncyclic free group, then all abelian subgroups of F are cyclic and have trivial automizer in F

Various group classes can be defined by imposing restrictions on outer automizers of subgroups. Indeed, every (group-theoretical) class $\mathfrak{X}$ gives rise to the class of those groups G such that $\operatorname{Out}_G H \in \mathfrak{X}$ for all $H \leq G$; of course more extended classes can be defined by imposing this restriction on specific subgroups of G only. Classes defined by the 'small automizers' property can be obtained in this context; indeed, if $\mathfrak{X}$ is the class of trivial groups the condition $\operatorname{Out}_G H \in \mathfrak{X}$ (or $\operatorname{Out}_G H = 1$) means exactly that H has 'small automizer' in G.

The object of [6] was the class (CO), defined as outlined in the previous paragraph when $\mathfrak{X}$ the class of cyclic groups. Thus, (CO) is the class of those groups in which every subgroup of has cyclic outer automizer. Here we consider the analogous class (FO), obtained by taking for $\mathfrak{X}$ the class of finite groups. So, we say that a group G is an *FO-group*, or that $G \in$ (FO), if an only if $\operatorname{Out}_G H$ is finite for all $H \leq G$. The more restricted class (BFO) is defined by saying that a group G is a *BFO-group* if and only if there is a finite upper bound for $|\operatorname{Out}_G H|$, when H ranges over the subgroups of G. Obvious examples of FO-groups are the centre-by-finite groups (which actually are BFO-groups), but also the quasifinite groups or, more generally, all groups in which all subgroups of infinite index are finite or cyclic, or, even more generally, have finitely many outer automorphisms. Less obvious examples are the abelian-by-finite, centre-by-(finite rank) groups. That these groups are in the class (FO), and even in (BFO), is shown by the first of our main results, which characterises abelian-by-finite FO-groups.

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Theorem A. *Suppose that the group G has an abelian normal subgroup A of finite index n. Then $G \in$ (FO) if and only if $G/Z(G)$ has finite torsion-free rank and, for all primes p dividing n, if P/A is a Sylow p-subgroup of G/A then $A/C_A(P)$ has finite p-rank. If this is the case, then $G \in$ (BFO).*

(In accordance with standard terminology, a group has finite torsion-free rank if and only if it has a finite series whose factors are either periodic or cyclic.) The relevance of Theorem A comes from the fact that in a rather extended class of groups, containing for instance the soluble-by-finite groups, all FO-groups are abelian-by-finite. By a *generalised radical* group we mean a group with an ascending normal series whose infinite factors are locally nilpotent.

Theorem B. *Every generalised radical FO-group is abelian-by-finite, and therefore a BFO-group.*

So, Theorem A actually describes all FO-groups in classes like those of soluble-by-finite, hyperabelian and FC-hypercentral groups. In the special case of locally nilpotent groups this description takes a simpler form:

Theorem C. *Let G be a locally nilpotent group. Then $G \in$ (FO) if and only if $G/Z(G)$ is a Chernikov group. If this happens, then G is hypercentral and satisfies BFO.*

It is a consequence of Theorems B and C that locally supersoluble FO-groups are abelian-by-finite (Corollary 4.2) but we leave open the question of whether all locally soluble FO-groups are abelian-by-finite (that is, soluble) or at least in (BFO). Note that, despite the fact that most of the FO-groups discussed so far are abelian-by-finite and hence in (BFO), the classes (FO) and (BFO) do not coincide. Indeed, let $(G_i)_{i\in\mathbb{N}}$ be a sequence of finite groups of odd orders with the property that for every $n \in \mathbb{N}$ there exist $i \in \mathbb{N}$ and $H \leq G_i$ such that $|\operatorname{Out}_{G_i} H| > n$ (for instance, one can choose suitable metacyclic groups as the G_i). Theorem 35.1 in [11] shows that there exists a simple group G with the properties that each of the groups G_i is embedded in G and every proper noncyclic subgroup of G is contained in a subgroup of G isomorphic to one of the G_i. It is clear that G is in (FO) but not in (BFO).

We also consider groups satisfying a weaker property than FO, and define the class (FOA) as follows. We say that G is an *FOA-group* if and only if $\operatorname{Out}_G H$ is finite for all *abelian* subgroups H of G. Of course, $\operatorname{Out}_G H = \operatorname{Aut}_G H$ in this case, but we leave a reference to outer automorphisms in the notation for (FOA), for the sake of uniformity. It is not hard to construct examples of FOA-groups which are not in (FO). Indeed, as already remarked, if F is a noncyclic free group, then $\operatorname{Out}_F H = 1$ for all abelian $H \leq F$, thus $F \in$ (FOA), in a very strong sense. Since (FO) is quotient-closed (see Lemma 2.1), certainly $F \notin$ (FO). We shall prove that properties FO and FOA are equivalent for Baer groups and for hypercentral groups (Propositions 3.8 and 3.9), but we have not decided whether the same is true for all locally nilpotent groups. For a Baer FO-group G, the condition in Theorem C implies that G is nilpotent and $G/Z(G)$ is finite; thus *the Baer FOA-groups are just the centre-by-finite nilpotent groups.* Finally, we shall see that soluble (and even hyperabelian) FOA-groups are bound to be abelian-by-finite (Corollary 3.10), but not necessarily in (FO).

2. Abelian-by-finite groups

Our first lemma establishes obvious closure properties for the classes (FO), (BFO) and (FOA). The proof is essentially the same as in [6], Lemma 2.1; we reproduce it here for the convenience of the reader.

Lemma 2.1. *The classes* (FO) *and* (BFO) *are subgroup- and quotient-closed; the class* (FOA) *is subgroup-closed.*

Proof. Let G be a group and $U \leq G$. For all $H \leq U$ we have $HC_U(H) = HC_G(H) \cap U$, hence $\operatorname{Out}_U H = (N_G(H) \cap U)/(HC_G(H) \cap U)$ is isomorphic to a subgroup of $\operatorname{Out}_G H$. It follows that U belongs to (FO), (BFO) or (FOA) if G belongs to the same class. Now suppose that $N \lhd G$, and let $H/N \leq G/N$. Then $C_G(H)N/N \leq C_{G/N}(H/N)$ and $N_{G/N}(H/N) = N_G(H)/N$. Therefore $\operatorname{Out}_{G/N} H/N$ is an epimorphic image of $\operatorname{Out}_G H$. It follows that $G/N \in$ (FO) if $G \in$ (FO), and $G/N \in$ (BFO) if $G \in$ (BFO). The proof is complete. □

It should be noticed that (FOA) is not quotient-closed; this follows from the fact that all free groups are in (FOA), as we observed in the introduction. Nonetheless, a weak form of closure under quotients will be established in Corollary 3.4.

As anticipated in the introduction, all FO-groups in the classes that we will be concerned with turn out to be abelian-by-finite. Thus, our first aim will be discussing sufficient conditions for an abelian-by-finite group to be an FO-group. This will lead to the proof of the 'if' part of Theorem A.

So, let the group G have a normal abelian subgroup A of finite index, say $n = |G/A|$. Let $H \leq G$ and $N = N_G(H)$. Then N acts on H by conjugation, call $\xi\colon N \to \operatorname{Aut} H$ the homomorphism describing this action. Further, let $\nu\colon \operatorname{Aut} H \twoheadrightarrow \operatorname{Out} H$ be the natural epimorphism. Then $\operatorname{Out}_G H \simeq N^{\xi\nu}$. As can be expected, the structure of $\operatorname{Out}_G H$ is related to a 1-dimensional cohomology group (for standard facts about cohomology and group derivations the reader is referred to, e.g., [8]). Indeed, let $A_0 = H \cap A$, $\bar{H} = H/A_0$ and $B = N_A(H)$, so that $A_0C_A(H) \le B \le N$ and $B = C_A(\bar{H})$ has index at most n in N. Plainly, $\mathrm{Out}_G H$ is finite if and only if $B^{\xi\nu}$ is finite, and $|\mathrm{Out}_G H| \le n|B^{\xi\nu}|$. Now, B^{ξ} stabilises the series $1 \le A_0 \le H$; this yields a natural monomorphism $\mu\colon B^{\xi} \rightarrowtail \mathrm{Der}(\bar{H}, A_0)$. Moreover, $A_0^{\xi\mu}$ is the group of inner derivations $\mathrm{IDer}(\bar{H}, A_0)$, hence B^{ξ}/A_0^{ξ} embeds in $H^1(\bar{H}, A_0) \simeq \mathrm{Der}(\bar{H}, A_0)/\mathrm{IDer}(\bar{H}, A_0)$. On the other hand $A_0^{\xi} \le \ker\nu$, therefore $B^{\xi\nu}$ is an epimorphic image of B^{ξ}/A_0^{ξ}. It follows that $B^{\xi\nu}$ is isomorphic to a section of $H^1(\bar{H}, A_0)$. This shows that $\mathrm{Out}_G H$ *is finite provided* $H^1(\bar{H}, A_0)$ *is finite.*

Of course $H^1(\bar{H}, A_0)$ has finite exponent, dividing $|\bar{H}|$ and hence n. We can almost immediately deduce that *if $G/Z(G)$ has finite torsion-free rank and finite p-rank for all primes p dividing n, then* $G \in (\mathrm{BFO})$. As a matter of fact, a stronger result holds:

Proposition 2.2. *Suppose that the group G has an abelian normal subgroup A of finite index n. Assume that $G/Z(G)$ has finite torsion-free rank and that, for all primes p dividing n, if P/A is a Sylow p-subgroup of G/A then $A/C_A(P)$ has finite p-rank. Then* $G \in (\mathrm{BFO})$.

Proof. For the purpose of this proof, if $\kappa(H)$ is a cardinal defined in terms of a subgroup H of G we shall say that $\kappa(H)$ is uniformly bounded if it is finite and bounded above by an integer independent of the choice of H. For instance, what we have to prove is that $|\mathrm{Out}_G H| = |N_G(H) : HC_G(H)|$ is uniformly bounded while H varies over the subgroups of G. One can observe that all uniformly bounded integers appearing in this proof can actually be bounded above in terms only of n and of the ranks considered in the statement.

It is easy to see that the hypothesis of the proposition is inherited by all subgroups of G, with n possibly replaced by a smaller integer and the p-ranks possibly increased by $r_0(G/Z(G))$ at most. Fix $H \le G$. If $H \le G_1 \le G$ then $HC_{G_1}(H) = HC_G(H) \cap G_1$, and it follows that $|\mathrm{Out}_G H| \le |\mathrm{Out}_{G_1} H|\,|N_G(H) : N_{G_1}(H)| \le |\mathrm{Out}_{G_1} H|\,|G : G_1|$. Therefore, provided we know that $|G : G_1|$, or at least $|N_G(H) : N_{G_1}(H)|$, is uniformly bounded, we may safely substitute G_1 for G in our argument. Thus, after replacing G with $N_G(H)$ first and then with AH, we may assume $H \lhd G = AH$. Then $[A, H] \le A_0 := A \cap H$. We may further assume that $n = |G/A|$ is minimal for G being a counterexample; it follows that there exists $t \in \mathbb{N}$ such that $|\mathrm{Out}_X K| \le t$ for all X such that $A \le X < G$ and all $K \le X$. Let K be such that $A_0 \le K < H$. Then $K \lhd AK < G$ and $A/(A \cap KC_{AK}(K)) \simeq \mathrm{Out}_{AK} K$ has order at most t. Also, $A_0C_A(K) = A \cap KC_A(K)$, and hence

$$\left|\frac{A \cap KC_{AK}(K)}{A_0C_A(K)}\right| \le \left|\frac{KC_{AK}(K)}{KC_A(K)}\right| \le \left|\frac{C_{AK}(K)}{C_A(K)}\right| \le \left|\frac{AK}{A}\right| < n,$$

so $|A/A_0C_A(K)| < tn$. Now let $A^* = \bigcap\{A_0C_A(K) \mid A_0 \le K < H\}$. Since the number of the subgroups between A_0 and H can be bounded in terms of n only, $|G/A^*H| = |A/A^*|$ is uniformly bounded, so we may replace G with A^*H. Thus we may assume $A^* = A$, that is, $A = A_0C_A(K)$ for all K such that $A_0 \le K < H$.

From the paragraphs preceding this proof, we know that $\bar{A} := A/A_0C_A(H)$ is embedded in $H^1(H/A_0, A_0)$, hence it has finite exponent dividing n. Also, $\mathrm{Out}_G H \simeq A/(A \cap HC_G(H))$ is isomorphic to a quotient of $\bar{A}$. Therefore, to complete the proof, all we have to do is show that the p-rank $r_p(\bar{A})$ is uniformly bounded for every prime p dividing n. Let p be such a prime, P/A_0 a Sylow p-subgroup of H/A_0, and $C = C_A(P)$. Then PA/A is a Sylow p-subgroup of G/A and $C = C_A(PA)$, hence $r_p(A/C)$ is finite by hypothesis. Also, $r_p(A/C_A(P^H)) = r_p(A/\bigcap\{C_A(P^x) \mid x \in H\})$ is uniformly bounded, since P has at most n conjugates in G. If $P^H = H$, then $\bar{A} = A/A_0C_A(P^H)$, hence we may disregard this case. So, assume $P^H < H$. By using either the Schur-Zassenhaus Theorem (if $P \lhd H$) or the Frattini Argument (if $P \ntriangleleft H$) we can choose K such that $A_0 \le K < H = P^HK$. Then $C_A(H) = C_A(P^H) \cap C_A(K)$; thus $r_p(C_A(K)/C_A(H)) \le r_p(A/C_A(P^H))$, so $r_p(C_A(K)/C_A(H))$ is uniformly bounded. On the other hand, $A = A_0C_A(K)$ by the reductions in the previous paragraph, hence $\bar{A}$ is an epimorphic image of $C_A(K)/C_A(H)$. Therefore $r_p(\bar{A})$ is uniformly bounded. Now the proof is complete. □

As suggested in the first part of the proof, a (uniform) upper bound for $|\mathrm{Out}_G H|$ can be expressed in terms of the index n and the (finitely many) ranks mentioned in the statement. Note that all these ranks are ranks of quotients of $A/A \cap Z(G)$. Perhaps it is also worth noting that the hypothesis that the torsion-free rank of G be finite is only used in the proof to ensure that the assumptions on G are inherited by all of its subgroups. However, this hypothesis cannot be disposed of, as Theorem A, once proved, will make clear.

3. Locally nilpotent groups

In this section we aim at proving Theorem C, describing locally nilpotent FO-groups. We shall also show that property FOA is sufficient to ensure FO for certain locally nilpotent groups. It is with FOA-groups that we start. A first, useful property of FOA-groups is that the condition on abelian subgroups required by the definition also holds for centre-by-finite subgroups.

Lemma 3.1. *Let H be a centre-by-finite subgroup of the FOA-group G. Then $\mathrm{Aut}_G H = N_G(H)/C_G(H)$ is finite.*

Proof. Let $N = N_G(H)$ and $M = C_N(H/Z(H))$. Then N/M is finite. For all $x \in H$ the subgroup $A_x := \langle x\rangle Z(H)$ is abelian and normalised by M; hence $M/C_M(A_x)$ is finite by property FOA. Since $H = \langle A_x \mid x \in S\rangle$ for a finite set S, we deduce that $M/C_M(H)$ is finite. It follows that $N/C_G(H)$ is finite, as required. $\square$

Anther obvious remark is the following lemma:

Lemma 3.2. *Let G be a hypercyclic FOA-group. Then every maximal normal abelian subgroup of G has finite index in G. As a special case, every hypercentral FOA-group is abelian-by-finite.*

Proof. If A is a maximal normal abelian subgroup of G then $A = C_G(A)$. Thus $G/A = \operatorname{Out}_G A$ is finite. Since hypercentral groups are hypercyclic, the result follows. $\square$

We are already in a position to characterise nilpotent FOA-groups and show that they actually are in (FO).

Lemma 3.3. *Every nilpotent FOA-group is centre-by-finite.*

Proof. Let G be a nilpotent FOA-group of minimal nilpotency class subject to $G/Z(G)$ being infinite. Lemma 3.2 shows that G has an abelian normal subgroup A of finite index, hence $G = A\langle F\rangle$ for some finite set F. For all $x \in F$ the subgroup $\langle x\rangle G'$ has class less than that of G, hence it is centre-by-finite, so $G/C_G(\langle x\rangle G')$ is finite by Lemma 3.1. Now, $Z(G) \geq \bigcap\{C_A(x) \mid x \in F\}$; it follows that $G/Z(G)$ is finite. $\square$

Corollary 3.4. *Let $N \lhd G \in$ (FOA), and assume that $N \leq Z_n(G)$ for some $n \in \mathbb{N}$. Then $G/N \in$ (FOA).*

Proof. Let A/N be an abelian subgroup of G/N. Then A is nilpotent, hence $N_G(A)/C_G(A)$ is finite by Lemmas 3.3 and 3.1. The result follows. $\square$

Corollary 3.5. *Let G be a locally nilpotent FOA-group. Then G' is periodic.*

Proof. Every element x of G' is in X' for some finitely generated $X \leq G$. But such an X is nilpotent, hence $X/Z(X)$ is finite and so X' is finite. Therefore, x is periodic. $\square$

Lemma 3.3 can be strengthened as follows.

Proposition 3.6. *Let $G \in$ (FOA). Then $Z_n(G)/Z(G)$ is finite for all positive integers n.*

Proof. Let T be a transversal to $C_G(Z_n(G))$ in G. We know from Lemmas 3.3 and 3.1 that T is finite. The former lemma also shows that $\langle t\rangle Z_n(G)$ is centre-by-finite for all $t \in T$. It follows that $Z(G) = C_{Z(Z_n(G))}(T)$ has finite index in $Z_n(G)$. $\square$

To extend Lemma 3.3 in a different direction, we make use of the following observation.

Lemma 3.7. *Let G be a Baer group such that:*

(i) *for every subnormal subgroup H of G, $\operatorname{Fit} H$ is nilpotent;*
(ii) *$\operatorname{Fit} G \leq Z_n(G)$ for some $n \in \mathbb{N}$.*

Then G is nilpotent (of class n at most).

Proof. The following consequence of (i) is not hard to prove: for every subnormal subgroup H of G, $\operatorname{Fit} H \leq \operatorname{Fit} G$. It is also easy to show that $\bar{G} := G/Z_n(G)$ inherits condition (i) from G, and $\operatorname{Fit} \bar{G} = 1$ by (ii). Therefore, every subnormal subgroup of $\bar{G}$ has trivial Fitting subgroup. Since G is a Baer group, this means that $\bar{G} = 1$. Then $G = Z_n(G)$ and the proof is complete. $\square$

Proposition 3.8. *Let G be a Baer group. Then $G \in$ (FOA) if and only if $G/Z(G)$ is finite.*

Proof. Assume $G \in$ (FOA). If A is a normal abelian subgroup of G and $C = C_G(A)$ then G/C is finite. Moreover, $A\langle x\rangle$ is nilpotent for all $x \in G$, because $\langle x\rangle$ is subnormal in G, hence $A/C_A(x)$ is finite by Lemma 3.3. It follows that $A/A\cap Z(G)$ is finite. Now let N be a normal nilpotent subgroup of G. Then, by the same lemma, $N/Z(N)$ is finite, and since $Z(N)/Z(N)\cap Z(G)$ also is finite by the argument in the previous lines, $N/N\cap Z(G)$ is finite.

If G is a Fitting group this shows that $G/Z(G)$ is a (periodic) FC-group, hence it is hypercentral by [12], Theorem 4.38, and therefore abelian-by-finite by Lemma 3.2. But abelian-by-finite Baer groups are nilpotent, hence G is nilpotent and so centre-by-finite (Lemma 3.3, once again) in this case.

Coming back to the general case, the previous paragraph shows that $F := \operatorname{Fit} G$ is nilpotent and, more generally, condition (i) of Lemma 3.7 is satisfied by G. By the first paragraph, $F/Z(G)$ is finite, hence $F = Z_n(G)$ for some $n \in \mathbb{N}$. Now the proof can be completed by invoking Lemmas 3.7 and 3.3. $\square$

The following consequence is worth noting. For every FOA-group G, if B is the Baer radical of G, then $B = \operatorname{Fit} G$ and $G/C_G(B)$ is finite (by Lemma 3.1).

Proposition 3.9. *Let G be a hypercentral group in (FOA). Then G is abelian-by-finite and $G/Z(G)$ is a Chernikov group. Therefore, $G \in$ (BFO).*

Proof. Lemma 3.2 shows that G has an abelian normal subgroup A of finite index such that $Z(G) \leq A$. Let T be a transversal to A in G. If $t \in T$ then $A/C_A(t) \simeq [A, t]$ is periodic, by Corollary 3.5. As $Z(G) = \bigcap\{C_A(t) \mid t \in T\}$, we conclude that $A^* := A/Z(G)$ is periodic, hence $G^* := G/Z(G)$ is periodic. Let S be the socle of A^*, and let $S_0 = S \cap Z(G^*)$. By Proposition 3.6, S_0 is finite. Now, S_0 has a complement K in S. Let N be the normal core of K in G^*. Then S/N is finite, because S/K and G/A are finite. But $N \cap Z(G^*) = 1$. Since G is hypercentral this yields $N = 1$. Therefore, S is finite, which amounts to saying that A^* satisfies the minimal condition. Hence G^* is a Chernikov group, as required. Finally, Proposition 2.2 yields that $G \in$ (BFO). □

Proof of Theorem C. Let G be a locally nilpotent FO-group. In view of Proposition 3.9, to prove that $G/Z(G)$ is Chernikov we only have to show that G is hypercentral. Assume false. Then we may factor out the hypercentre of G and assume that $Z(G) = 1$. Hence G has trivial FC-centre (by [12], Theorem 4.38) and so, by Proposition 3.8 and Lemma 3.1, trivial Baer radical. As a consequence, $C_G(G') = 1$; as $G \in$ (FO) this implies that G/G' is finite. Then $G = G'H$ for some finitely generated (hence nilpotent) subgroup H. Let c be the nilpotency class of H and $L = \gamma_{c+2}(G)$; then $G = LH$. It follows that L is the hypocentre of G; moreover G/L is finite. Since G is locally nilpotent L has no proper subgroup of finite index. Thus, at the expense of replacing G by L, we may assume that G has the same property. Now let $N \lhd G$ and $K = C_G(N)$. Then $K \cap N = Z(N) = 1$, because $\operatorname{Fit} G = 1$, hence property FO shows that G/KN is finite. But then $G = KN$, by our recent assumption. Hence $G = N \times K$. We have shown that the lattice of normal subgroups of G is complemented. Then G is a direct product of simple groups (see [14]), hence it is abelian, therefore trivial. Thus we have proved that G is hypercentral and hence $G/Z(G)$ is a Chernikov group.

Conversely, suppose that G is a (not necessarily locally nilpotent) group such that $G/Z(G)$ is Chernikov. Then G has a normal subgroup A of finite index such that $Z(G) \leq A$ and $A/Z(G)$ is abelian, divisible and periodic. This condition implies that A is abelian, therefore $G \in$ (FO) by Proposition 2.2. □

Abelian-by-finite, hypercentral groups of finite rank are certainly in (FO), by Proposition 2.2, but they are not necessarily nilpotent, an easy example being provided by the locally dihedral 2-group. This makes clear that there exist plenty of non-nilpotent locally nilpotent FO-groups.

In view of Propositions 3.8 and 3.9, one could ask whether all locally nilpotent groups in (FOA) are hypercentral (or, equivalently, abelian-by-finite) and so satisfy FO. This question is left open. Note that the results in this section readily allow us to reduce the problem to the case of (countable, locally finite) p-groups with trivial Baer radical.

Finally, although we shall not pursue the topic of FOA-groups any further, we observe that a very easy consequence of Proposition 3.8 is that all hyperabelian FOA-groups are abelian-by-finite, hence soluble, but not necessarily FO-groups.

Corollary 3.10. *Let the FOA-group G have an ascending series with subnormal terms and abelian factors. Then G is abelian-by-finite.*

Proof. Let B be the Baer radical of G. It is known, and is easy to check, that $C_G(B) \leq B$, that is, $C_G(B) = Z(B)$. Then Proposition 3.8 and Lemma 3.1 show that $G/Z(B)$ is finite. □

An easy example of a group as in the corollary but not in (FO) is the following. Let A be an abelian group of infinite torsion-free rank and without elements of order 2. Let $G = A \rtimes \langle x \rangle$, where $x^2 = 1$ and $a^x = a^{-1}$ for all $a \in A$. The abelian subgroups of G not contained in A have order 2, hence $|\operatorname{Out}_G U| \leq 2$ for all abelian subgroups U of G. Therefore, G is a metabelian group which satisfies FOA (boundedly). However, $Z(G) = 1$, hence Theorem A (which we shall prove soon) shows that $G \notin$ (FO).

4. Proofs of Theorems A and B

To prove Theorem A we only have to show that Proposition 2.2 can be inverted. We make use of the following (certainly well-known) result.

Lemma 4.1. *Let A be an abelian torsion-free group of infinite rank, acted on by a finite group X. Then A has an X-invariant subgroup which is free abelian of infinite rank.*

Proof. Let $1 \neq a_1 \in A$ and $B_1 = \langle a_1 \rangle^X$. Plainly, B_1 is free abelian of finite rank. Choose A_1 as a subgroup of A which is maximal subject to $A_1 \cap B_1 = 1$. Then A/A_1B_1 is periodic and $r_0(A/A_1) = r_0(B_1)$ is finite. Let $K_1 := (A_1)_X$, the largest X-invariant subgroup of A_1, so $r_0(A/K_1)$ is finite and $r_0(K_1)$ is infinite. Now let $1 \neq a_2 \in K_1$ and $B_2 = \langle a_2 \rangle^X$. Then $B_1B_2 = B_1 \times B_2$ is free abelian of finite rank and X-invariant. As above, we can choose A_2 as a subgroup of A which is maximal subject to $A_1 \cap B_1B_2 = 1$ and, after letting $K_2 := (A_2)_X$, a nontrivial element of K_2 will generate, as an X-submodule, a subgroup B_3 such that $B_1 \times B_2 \times B_3$ is free abelian of finite rank. This procedure can be carried on to produce an infinite sequence of X-invariant subgroups of A whose direct product is free abelian and so satisfies the requirements in the statement. □

Proof of Theorem A. Sufficiency of the condition given in the statement of the theorem has already been proved as Proposition 2.2, now we have to prove that it is also necessary. So, assume that G is an FO-group with a normal abelian subgroup A of finite index n. Let p be a prime divisor of n, P/A a Sylow p-subgroup of G/A, $C = C_A(P)$, and S/C the p-component of the socle of A/C. If S/C is infinite, then a standard argument—similar to that for Lemma 4.1—shows that S/C, viewed as a (P/A)-module, contains a direct sum X/C of infinitely many nonzero cyclic submodules, each of order p^q at most, where $q = |P/A|$. Clearly, P acts nilpotently on each of these direct summands. It follows that $Z/C := C_{X/C}(P)$ is infinite. But $C = X \cap Z(P)$ and $Z = X \cap Z_2(P)$, hence this conclusion is excluded by Proposition 3.6. Therefore, S/C is finite, and this amounts to saying that the p-rank of A/C is finite, as required.

Now we have to prove that the torsion-free rank of $G/Z(G)$ is finite. Again, let p be a prime divisor of n and P/A be a Sylow p-subgroup of G/A. We can write P as AF, where F is a finitely generated subgroup. Let $D = A \cap F$, so $D \lhd P$, let $Z/D = (A/D) \cap Z(P/D)$ and $K/D = (A/D) \cap \bar{Z}(P/D)$. Then KF/D is a hypercentral FO-group and $Z/D = (A/D) \cap Z(KF/D)$. Hence K/Z is periodic, by Theorem C. Assume that A/Z has infinite torsion-free rank. Let K_*/K be the torsion subgroup of A/K. Lemma 4.1 provides a subgroup $B \leq A$ such that $K_* \leq B \lhd P$ and B/K_* is free abelian of infinite rank. Let $p^\lambda = |P/A|$ and $B_1 = B^{p^{\lambda+1}} K_*$. Then BF/B_1 is a nilpotent FO-group (because it has finite exponent; see [12], Lemma 6.34) and so B/Y is finite, where $Y/B_1 = (B/B_1) \cap Z(BF/B_1)$. Now fix a tranversal T to D in F. For every $a \in A$, we have $a^* := \prod_{t \in T} a^t \in A \cap Z(P) \leq Z \leq B_1$ but, if $a \in Y$, then $a^* \in a^{p^\lambda} B_1$. This shows that $Y^{p^\lambda} \leq B_1$. However, this is impossible, because B/B_1 is infinite homocyclic of exponent $p^{\lambda+1}$ and B/Y is finite. Hence $r_0(A/Z)$ is finite. Now, $A \cap Z(P) = \bigcap\{C_Z(t) \mid t \in T\}$ and $Z/C_Z(t) \simeq [Z, t] \leq D$ for all $t \in T$. Then, since D is finitely generated, $Z/A \cap Z(P)$ is finitely generated; it follows that $r_0(A/A \cap Z(P))$ is finite. Since $A \cap Z(G)$ is the intersection of all the subgroups $A \cap Z(Q)$ where Q/A ranges over the finite set of all Sylow subgroups of G/A, we have that $r_0(G/A \cap Z(G)) = r_0(A/A \cap Z(G))$ is finite. Thus $r_0(G/Z(G))$ is finite, as required, and the proof is complete. □

It is easy to find examples of abelian-by-finite FO-groups G in which, in the notation of Theorem A, $A/Z(G)$ has infinite p-rank for some (possibly all) primes p dividing $|G/A|$. One such example is as follows. Let A be an abelian group of exponent 6, whose 2-component A_2 is infinite. Write A_2 as $U \times V$, where $U \simeq V$, and fix an isomorphism $*\colon U \to V$. Then A has an automorphism x of order 6 acting like the inversion map on the 3-component A_3 of A and such that $u^x = u^*$ and $(u^*)^x = uu^*$ for all $u \in U$. Theorem A shows that the natural split extension $G = A \rtimes \langle x \rangle$ is an FO-group: here $\langle x^3 \rangle A/A$ and $\langle x^2 \rangle A/A$ are the Sylow 2-subgroup and the Sylow 3-subgroup of G/A, respectively, and their centralisers in A are A_2 and A_3. Of course $Z(G) = 1$, and we can make both the 2- and 3-rank of A infinite and arbitrarily large.

Proof of Theorem B. Consider the special case of radical groups first. So, let G be a radical FO-group, and let H be its Hirsch-Plotkin radical. Then $C_G(H) \leq H$, and hence $G/H = \operatorname{Out}_G H$ is finite. But H is abelian-by-finite by Theorem C, and hence G is abelian-by-finite, as required.

Now assume that G is a generalised radical FO-group. In view of the previous paragraph, in order to complete the proof, we only need to show that G is soluble-by-finite. Let S be the soluble radical of G. Then S is contained in the ω-th term of the upper Fitting series of G, hence it is soluble by the previous case. It follows that G/S has trivial Fitting subgroup. So, at the expense of substituting G/S for G we may assume $\operatorname{Fit} G = 1$. We shall prove that this implies that G is finite. For every finite normal subgroup N of G, if $C = C_G(N)$ we have $NC = N \times C$, because $\operatorname{Fit} G = 1$. Moreover, G/C is finite, and $\operatorname{Fit} C$ is trivial because it is nilpotent by Proposition 3.8. Hence, as G is generalised radical, C has a finite nontrivial G-invariant subgroup, unless $C = 1$, in which case G is finite. If G is infinite this makes it easy to construct an infinite direct product $D = \operatorname{Dr}_{i \in \mathbb{N}} F_i$ of finite, nontrivial normal subgroups of G. None of these factors F_i is abelian. Then the already-quoted theorem of Zassenhaus ([15], Theorem 7) shows that for all $i \in \mathbb{N}$ there exists $H_i \leq F_i$ such that $\operatorname{Out}_{F_i} H_i \neq 1$. If $H = \operatorname{Dr}_{i \in \mathbb{N}} H_i$, then $\operatorname{Out}_D H$ is infinite, and this is a contradiction. Therefore, G must be finite. Now the proof is complete. □

Corollary 4.2. *Let G be a locally supersoluble FO-group. Then G is abelian-by-finite and therefore hypercyclic.*

Proof. G' is locally nilpotent, hence soluble by Theorem C. Therefore G is soluble, and the corollary follows from Theorem B. □

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Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Via Cintia — Monte S. Angelo, I-80126 Napoli, Italy

E-mail address: cutolo@unina.it

URL: http://www.dma.unina.it/~cutolo/