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QUASI-POWER AUTOMORPHISMS OF INFINITE GROUPS

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§1. Introduction

A *power automorphism* of a group G is an automorphism fixing every subgroup of G. Power automorphisms have been studied by many authors, mainly by C.D.H. Cooper [2]. The set $\mathrm{PAut}\, G$ of all power automorphisms of a group G is a normal, abelian, residually finite subgroup of the full automorphism group $\mathrm{Aut}\, G$ of G.

The aim of this paper is the study of *quasi-power automorphisms* of infinite groups. We say that an automorphism of a group G is a quasi-power automorphism if it fixes all but finitely many subgroups of G. It is clear that the set of all quasi-power automorphisms of G is a normal subgroup $\mathrm{QAut}\, G$ of $\mathrm{Aut}\, G$ containing $\mathrm{PAut}\, G$ and that $\mathrm{QAut}\, G = \mathrm{Aut}\, G$ if G is finite.

It is easily verified that quasi-power automorphisms fix all infinite subgroups (see Lemma 2.2 below). Automorphisms fixing infinite subgroups of groups have been studied by M. Curzio, S. Franciosi and F. de Giovanni [3] under the name of I-automorphisms. They prove that, under certain solubility or finiteness conditions for the group G, the group $\mathrm{IAut}\, G$ of all I-automorphisms of G coincides with $\mathrm{PAut}\, G$, provided G is not a Černikov group. They also give some sufficient conditions on a non-Černikov group G to ensure the commutativity of $\mathrm{IAut}\, G$ and exhibit, by contrast, an infinite Černikov group G such that $\mathrm{IAut}\, G$ is not abelian.

Stronger results hold for quasi-power automorphisms. Indeed, if G is an infinite group, then $\mathrm{QAut}\, G$ is always abelian and residually finite, as happens for $\mathrm{PAut}\, G$. Furthermore, it turns out that the existence of quasi-power automorphisms which are not power automorphisms affects the structure of an infinite group strongly, even if no further condition on this group is imposed. Our main result illustrating this is the following Theorem A, which also gives information on the subgroups which are not fixed under the action of quasi-power automorphisms.

Theorem A. *Let G be an infinite group such that $\mathrm{QAut}\, G \neq \mathrm{PAut}\, G$. Then the subgroups of G which are not fixed under the action of $\mathrm{QAut}\, G$ generate a finite characteristic subgroup $\Theta(G)$ of G. Moreover:*

(*i*) *$G/\Theta(G)$ is a finite extension of a p-subgroup (p prime). In particular, G is periodic;*
(*ii*) *every subgroup of G which is not fixed under the action of $\mathrm{QAut}\, G$ has order divisible by p, and $\Theta(G)$ has order divisible by p^2;*
(*iii*) *every infinite subgroup of G which is locally finite is Prüfer-by-finite.*

Here, by a *Prüfer-by-finite* group we mean a finite extension of a Prüfer group.

Quasi-power automorphisms are connected with the study of autoprojectivities (i.e., automorphisms of the subgroup lattice) of groups. In fact the group $\bar{\mathrm{Q}}\mathrm{Aut}\,G = \mathrm{QAut}\,G/\mathrm{PAut}\,G$ is isomorphic with the group of all autoprojectivities induced by $\mathrm{Aut}\,G$ on the subgroup lattice $\ell(G)$ which act on $\ell(G)$ as finitary permutations. The finiteness of $\Theta(G)$ (Theorem A) exactly means that $\bar{\mathrm{Q}}\mathrm{Aut}\,G$ is always finite. Further information on $\bar{\mathrm{Q}}\mathrm{Aut}\,G$ is contained in

Theorem B. *Let G be an infinite group such that $\mathrm{QAut}\,G \neq \mathrm{PAut}\,G$. Then $\bar{\mathrm{Q}}\mathrm{Aut}\,G$ is a finite (abelian) p-group, where p is the prime such that $G/\Theta(G)$ is a finite extension of a p-group. Moreover, if P is a Sylow p-subgroup of G, then $\bar{\mathrm{Q}}\mathrm{Aut}\,G$ can be embedded in $\bar{\mathrm{Q}}\mathrm{Aut}\,P$. Conversely, every finite abelian p-group is isomorphic with $\bar{\mathrm{Q}}\mathrm{Aut}\,G$ for a suitable abelian p-group G.*

Another problem considered in this paper (§3) is whether quasi-power automorphisms of infinite groups are central, the motivation being the well-known theorem by Cooper (Theorem 2.2.1 in [2], see Newell [5] for a generalization) stating that power automorphisms are central. Recall that an automorphism of a group G is said to be *central* if it acts trivially on the factor group $G/Z(G)$ or, equivalently, if it centralizes every inner automorphism of G. Cooper's centrality theorem generalizes a previous result by E. Schenkman [7] stating that the intersection of the normalizers of the subgroups of a group G (called the *norm* of G) is contained in $Z_2(G)$, the second centre of G (i.e., inner power automorphisms are central). Cooper's theorem cannot be generalized to quasi-power automorphisms of infinite groups (see Example 3.4). However some sufficient conditions under which quasi-power automorphisms are central are given. These imply, for instance, a generalization of Schenkman's result.

The last section of the paper contains a description of $\mathrm{QAut}\,G$ for infinite groups in some classes of nilpotent groups (including abelian groups).

Notation and terminology used throughout the paper are mostly standard (see for instance [6]). The groups $\mathrm{PAut}\,G$, $\mathrm{QAut}\,G$, $\bar{\mathrm{Q}}\mathrm{Aut}\,G$ have been defined above. Slightly extending the terminology used by Cooper, if p is a prime, G is a p-group and π is a p-adic unit, we will call the map $x \longmapsto x^\pi$ the universal power automorphism of exponent π of G, whenever it is an automorphism of G. Here x^π has to be interpreted as x^n, where n is an integer congruent to π modulo the order of x. Finally $\mathcal{C}_n$ and $\mathcal{C}_{p^\infty}$ will denote a cyclic group of order n and a Prüfer p-group, respectively.

§2. First results and proof of Theorem A

It is obvious that the power automorphisms of a group G are precisely those automorphisms fixing all cyclic subgroups of G. It is easily seen that a similar statement holds for quasi-power automorphisms.

Proposition 2.1. *Let G be a group. Then $\mathrm{QAut}\,G$ consists of all automorphisms fixing all but finitely many cyclic subgroups of G.*

Proof. Let α be an automorphism of G fixing all but finitely many cyclic subgroups of G. Then the set $S = \{\, x \in G \mid \langle x\rangle^\alpha \neq \langle x\rangle \,\}$ is finite. For every subgroup H of G such that $H^\alpha \neq H$, it holds $H = \langle H \cap S\rangle$, otherwise $H = \langle H \smallsetminus S\rangle$ would be fixed by α. It follows that $\alpha \in \mathrm{QAut}\,G$. □

Since every infinite group is generated by the complement of each finite subset, an argument similar to that used in the proof of the previous proposition proves the following lemma, which allows us to apply the results of [3] referred to in §1.

Lemma 2.2. *A quasi-power automorphism of a group G fixes every infinite subgroup of G.*

It is shown in [3] that if G is a non-Černikov (locally radical)-by-finite group, then every I-automorphism of G is a power automorphism; hence $\mathrm{QAut}\,G = \mathrm{PAut}\,G$ in this case.

Proposition 2.3. *Let G be an infinite group. Then $\mathrm{QAut}\,G$ is abelian.*

Proof. Let α, $\beta \in \mathrm{QAut}\, G$ and let $\Gamma = \langle\alpha,\beta\rangle$. There exists a subset X of G such that $G \setminus X$ is finite and $\langle x\rangle^{\Gamma} = \langle x\rangle$ for all $x \in X$. Clearly $\Gamma/C_{\Gamma}(x)$ is abelian for all $x \in X$. Since $\langle X\rangle = G$, then $\bigcap_{x\in X} C_{\Gamma}(x) = 1$, so that Γ is abelian and so is $\mathrm{QAut}\, G$. □

Our next aim is the proof of Theorem A. We shall divide it into several steps.

Proof of Theorem A

Assume G is an infinite group such that $\mathrm{QAut}\, G \neq \mathrm{PAut}\, G$. Recall that $\Theta(G)$ is the (characteristic) subgroup of G generated by all (cyclic) subgroups of G which are not fixed under the action of $\mathrm{QAut}\, G$.

Step 1 — $\mathrm{QAut}\, G$ *is residually finite.*

The orbit $Orb(H)$ of a subgroup H of G under the action of $\mathrm{QAut}\, G$ is the orbit of H under the action of $\bar{\mathrm{Q}}\mathrm{Aut}\, G$. As remarked in §1, this can be identified with an abelian group of finitary permutations of $\ell(G)$, so that $Orb(H)$ is finite. It follows that for every cyclic subgroup H of G, the normalizer of H in $\mathrm{QAut}\, G$ has finite index in $\mathrm{QAut}\, G$, hence $|\,\mathrm{QAut}\, G : C_{QAut\, G}(H)|$ is finite. Therefore $\mathrm{QAut}\, G$ is residually finite.

From now on, let H be a subgroup of G which is not fixed by some quasi-power automorphism θ of G. By Lemma 2.2, H is finite.

Step 2 — $[G, \mathrm{QAut}\, G]Z(G)/Z(G)$ *is finite.*

Since $\mathrm{QAut}\, G \triangleleft \mathrm{Aut}\, G$, we have $[\mathrm{Inn}\, G, \mathrm{QAut}\, G] \leq \mathrm{QAut}\, G$. By Proposition 2.3 and Step 1, we deduce that $[G, \mathrm{QAut}\, G]Z(G)/Z(G)$ is abelian and residually finite. Hence $H[G, \mathrm{QAut}\, G]$ is a nilpotent-by-finite group on which θ acts as a quasi-power automorphism which is not a power automorphism. Then $H[G, \mathrm{QAut}\, G]$ is a Černikov group by the above-quoted result of [3]. It follows that $[G, \mathrm{QAut}\, G]Z(G)/Z(G)$ is finite.

Step 3 — H *has finitely many conjugates in* G*.*

Assume false. Let $C/Z(G)$ be the centralizer of θ in $G/Z(G)$. By Step 2, C has finite index in G. Then $|C : N_C(H)|$ is infinite. Hence there exists $x \in C$ such that $(H^x)^{\theta} = H^x$. Since $[x, \theta] \in Z(G)$, it follows that $(H^{\theta})^x = (H^x)^{\theta} = H^x$ and so $H^{\theta} = H$, a contradiction.

Step 4 — There exists a (unique) prime number p dividing the order of every subgroup of G which is not fixed under the action of $\mathrm{QAut}\, G$ *and such that G/F is a finite extension of a p-group for a suitable finite subgroup F of $\Theta(G)$ normal in G.*

Let F be the normal closure in G of the subgroup generated by the subgroups of G which are not fixed by θ. By Step 3, F is finite. It is clear that H has a cyclic subgroup H_1 of order a power of a prime p such that $H_1^{\theta} \neq H_1$. Let K be a subgroup of $N = N_G(H_1)$ without elements of order p. If K were not contained in F, then θ would fix H_1K and so H_1, the unique Sylow p-subgroup of H_1K. Hence $K \leq F \cap N$. We have proved that NF/F is a p-group. Since N has finite index in G (again by Step 3), it follows easily that every p'-subgroup of G is fixed by any quasi-power automorphism of G. Therefore Step 4 is proved.

Step 5 — Every infinite locally finite subgroup K of G is Prüfer-by-finite.

We may clearly assume that K contains the FC-centre of G. Then, by Step 3, θ acts on K as a quasi-power automorphism which is not a power automorphism. By Step 4, K is (locally nilpotent)-by-finite. Applying as above a result from [3], we get that K is a Černikov group. Let R be its finite residual. By Step 3, $H \triangleleft HR$. Assume, by contradiction that R is not a Prüfer group. Then, since H is finite, HR/H is not a Prüfer group, so that there exist two infinite subgroups A and B of HR such that $A \cap B = H$. By Lemma 2.2, θ fixes both A and B and so $H^{\theta} = H$, a contradiction.

By the above (Steps 3 and 5), to prove the finiteness of $\Theta(G)$, we may assume, at the expense of replacing G with its FC-centre, that G is a finite, central extension of a Prüfer p-group. Thus, before completing the proof of Theorem A, we examine in some details quasi-power automorphisms of nilpotent Prüfer-by-finite p-groups.

Proposition 2.4. *Let the nilpotent p-group G be extension of a Prüfer group A by a finite group Q and let $\alpha \in \mathrm{Aut}\, G$. Then:*

(*i*) *If* $\alpha \in \mathrm{PAut}\, G$, *then* α *is a universal power automorphism.*

(*ii*) $\alpha \in \mathrm{QAut}\, G$ *if and only if* α *induces on both* A *and* Q *a universal power automorphism with the same* p*-adic unit as exponent.*

(*iii*) *If* $\alpha \in \mathrm{QAut}\, G$ *and* H *is a subgroup of* G *not fixed by* α, *then* $|H| < |Q|^2$.

Proof. Let π be the p-adic unit such that $x^\alpha = x^\pi$ for each $x \in A$.

(*i*) For any $x \in G$, α induces a power automorphism on the abelian group $\langle A,x\rangle$. Therefore $x^\alpha = x^\pi$. Hence α is the universal power automorphism with exponent π.

(*ii*) Let $\alpha \in \mathrm{QAut}\, G$. By Lemma 2.2, the subgroup F generated by the subgroups H of G such that $H^\alpha \neq H$ is finite. Let B be any proper subgroup of A properly containing $A \cap F$. Then α induces a power automorphism on G/B. Thus, by (*i*), α induces the universal power automorphism of exponent π on G/B and hence on Q, as we wanted to show.

Conversely, assume that α induces on both A and Q the universal power automorphism of exponent a given p-adic unit π. Since $x^{-\pi}x^\alpha = y^{-\pi}y^\alpha$ whenever $x, y \in G$ with $x \equiv y \pmod A$, then $E = \langle x^{-\pi}x^\alpha \,|\, x \in G\rangle$ is a finite subgroup of A. Let now H be a subgroup of G such that $H \neq H^\alpha$. Then $E \not\leq H$, since α induces on G/E the universal power automorphism of exponent π. Then $H \cap A < E$, so that $|H| < |E|\,|Q|$. It follows that $\alpha \in \mathrm{QAut}\, G$.

(*iii*) With the notation of (*ii*), it will be sufficient to prove that $|E| \leq |Q|$. Let $n = |Q|$ and $x \in G$. Since $x^{-\pi}x^\alpha \in A \leq Z(G)$, then $(x^{-\pi}x^\alpha)^n = (x^{-\pi})^n(x^\alpha)^n = (x^n)^{-\pi}(x^n)^\alpha = 1$, because $x^n \in A$. Hence E is contained in $A[n]$, which has order n. □

We can now complete the proof of Theorem A.

Step 6 — $\Theta(G)$ *is finite.*

We may assume that G has a central subgroup $A \simeq \mathcal{C}_{p^\infty}$ of finite index. Let P be a Sylow p-subgroup of G and let $p^n = |P/A|$ and $m = |G : P|$. We shall prove that every $\alpha \in \mathrm{QAut}\, G$ fixes every (finite) subgroup H of G of order not dividing mp^{2n}.

Let S be a Sylow p-subgroup of H. Then S is contained in a conjugate of P and $S^\alpha = S$ by Proposition 2.4 (*iii*). By Step 4, α fixes every p'-subgroup of H, hence $H^\alpha = H$. Since G' is finite there are only finitely many elements of G of order at most mp^{2n}, so that $\Theta(G)$ is finite.

Step 7 — Every Sylow p*-subgroup* P *of* G *is fixed by every quasi-power automorphism* θ *of* G.

If this is not the case, then P is finite and $N = N_G(P)$ has finite index in G. Hence $N^\theta = N$. Since P is characteristic in N, we get $P^\theta = P$, as we wanted.

The only remaining fact to be proved is that p^2 divides the order of $\Theta(G)$, where p is defined as in Step 4 above. Assume false. Then every p-subgroup H of G which is not fixed under the action of $\mathrm{QAut}\, G$ is a Sylow p-subgroup of $\Theta(G)$ and so has the form $H = P \cap \Theta(G)$, where P is a Sylow p-subgroup of G. This is a contradiction by Step 7. Theorem A is now completely proved. □

By Theorem A there are two possible kinds of groups G for which $\mathrm{QAut}\, G$ and $\mathrm{PAut}\, G$ do not coincide, namely Prüfer-by-finite groups and periodic non-(locally finite) groups. Indeed both possibilities occur. A class of groups of the first type has already been considered in Proposition 2.4, from which we deduce the following

Corollary 2.5. *Let the nilpotent* p*-group* G *be extension of a Prüfer group* A *by a finite group* Q. *Then* $Q_{ab} = Q/Q'$ *can be embedded in* $\bar{\mathrm{Q}}\mathrm{Aut}\, G = \mathrm{QAut}\, G/\mathrm{PAut}\, G$. *In particular,* $\mathrm{QAut}\, G = \mathrm{PAut}\, G$ *if and only if* $Q = 1$.

Proof. Let $\Gamma = C_{Aut\, G}(A) \cap C_{Aut\, G}(Q)$. Then $\Gamma \simeq \mathrm{Hom}(Q, A) \simeq Q_{ab}$. By Proposition 2.4, $\Gamma \leq \mathrm{QAut}\, G$ and $\Gamma \cap \mathrm{PAut}\, G = 1$. Hence $Q_{ab} \rightarrowtail \bar{\mathrm{Q}}\mathrm{Aut}\, G$. The second part of the statement is now clear, as $\mathrm{Aut}\, \mathcal{C}_{p^\infty} = \mathrm{PAut}\, \mathcal{C}_{p^\infty}$. □

We give now an example to show that Prüfer-by-finite groups G such that $\mathrm{QAut}\, G \neq \mathrm{PAut}\, G$ are not necessarily nilpotent.

Example 2.6. Let $A \simeq \mathcal{C}_{2^\infty}$, $\langle b\rangle \simeq \mathcal{C}_2$ and $\langle x\rangle \simeq \mathcal{C}_4$. Let G be the semidirect product of $(A \times \langle b\rangle)$ by $\langle x\rangle$ with x^2 amalgamated with the element a of order 2 of A and x acting as the inversion automorphism on $A \times \langle b\rangle$. Consider the automorphism α of G which acts trivially on $A\langle x\rangle$ and maps b to ab. We will prove that $\alpha \in \operatorname{QAut} G$. Let $H = \langle h\rangle$ be a cyclic subgroup of G. If $h \notin A\langle x\rangle \cup A\langle b\rangle$, then $h = cbx$ for a suitable $c \in A$. Then $a = x^2 = h^2$ and $h^\alpha = cbax = ah = h^3$, so that $H^\alpha = H$. If $h \in A\langle x\rangle$ then $h^\alpha = h$. Thus all cyclic subgroups of G which are not fixed by α are contained in $A\langle b\rangle$. By Proposition 2.4, α induces on $A\langle b\rangle$ a quasi-power automorphism. Hence $\alpha \in \operatorname{QAut} G$ by Proposition 2.1. $\square$

An example of a non-(locally finite) group G for which $\operatorname{QAut} G \neq \operatorname{PAut} G$ is the following.

Example 2.7. Ashmanov and Ol'shanskii ([1], Remark 4, p.72) have constructed, for a sufficiently large prime number p, an infinite group T with the following structure:
– T has a unique subgroup $\langle c\rangle$ of order p, namely its centre;
– every nontrivial proper subgroup of T different from $\langle c\rangle$ is cyclic of order p^2 (and contains $\langle c\rangle$).

Consider the group $G = T \times \langle a\rangle$, where a has order p, and the automorphism α of G defined by $a^\alpha = ac$ and $x^\alpha = x$ for all $x \in T$. Then α is a quasi-power automorphism of G. In fact, let $H = \langle xa^r\rangle$ ($x \in T$ and r an integer) be a cyclic subgroup of G which is not fixed by α. Since $[G, \alpha] = \langle c\rangle$, then $c \notin H$ and $H \cap T = 1$. Thus H has order p and $1 = (xa^r)^p = x^p$, so that $x \in \langle c\rangle$. Hence $H \leq \langle c,a\rangle$. It follows that $\alpha \in \operatorname{QAut} G \smallsetminus \operatorname{PAut} G$. $\square$

§3. Centrality of quasi-power automorphisms

In this section we consider the problem of determining to what extent Cooper's centrality theorem for power automorphisms can be generalized to quasi-power automorphisms of infinite groups. We have already remarked that quasi-power automorphisms are "near" to being central in the sense that $[G, \operatorname{QAut} G]Z(G)/Z(G)$ is finite for every group G (Step 2 of the proof of Theorem A). Another obvious observation is that, for every infinite group G, one has $[G, \operatorname{QAut} G, \operatorname{QAut} G] \leq Z(G)$, as $\operatorname{QAut} G$ is abelian and normal in $\operatorname{Aut} G$. We state now a property of quasi-power automorphisms which allows us to make use of a characterization of central automorphisms from [4].

Lemma 3.1. *Let θ be a quasi-power automorphism of the infinite group G. Then $[g, g^\theta] = 1$ for every $g \in G$.*

Proof. Assume false. Then $[g, g^\theta] \neq 1$ for some $g \in G$, so that $C = C_G(g)$ is not fixed by θ. Hence C is finite and, by Theorem A, $g \in C \leq F$, the FC-centre of G. Therefore $|G : C|$ is finite and so is G, a contradiction. $\square$

Lemma 3.2. *Let G be an infinite group. Then $[G, \operatorname{QAut} G]$ is abelian and, if $\theta \in \operatorname{QAut} G$, then:*
(i) *$[g, \theta, h, h] = 1$ and $[g, \theta, h]^{-1} = [h, \theta, g]$ for all g, $h \in G$;*
(ii) *$\langle [g,\theta]\rangle \lhd G$ for all $g \in G$ such that θ normalizes every conjugate of $\langle g\rangle$ in G. This holds in particular for all $g \in G \smallsetminus \Theta(G)$;*
(iii) *θ is a central automorphism of G if and only if $\langle [g,\theta]\rangle \lhd G$ for all $g \in G$.*

Proof. Let g, $h \in G$ and θ, $\psi \in \operatorname{QAut} G$. Since $[G, \operatorname{QAut} G, \operatorname{QAut} G] \leq Z(G)$, $[h, \psi, \theta, g] = 1$, which implies $[\,[g, \theta]\,, [h, \psi]\,] = 1$ by [4], Lemma 1 (ii). Hence $[G, \operatorname{QAut} G]$ is abelian. Furthermore Lemma 3.1 and $\operatorname{QAut} G \lhd \operatorname{Aut} G$ imply that $[g, \theta, h]$ centralizes h. Again by Lemma 1 (ii) of [4], it holds $[g, \theta, h]^{-1} = [h^{-1}, \theta, g^{-1}]$. By Lemma 3.1 and since $[G, \operatorname{QAut} G]$ is abelian, $[h^{-1}, \theta, g^{-1}] = [h, \theta, g]$ so that (i) is proved. Statements (ii) and (iii) follow immediately from [4], Lemma 3 and Theorem. $\square$

Our main result about centrality of quasi-power automorphisms is the following.

Theorem 3.3. *Let G be an infinite group. Then:*
(i) *A quasi-power automorphism θ of G fixing infinitely many elements of G is central.*
(ii) *Every quasi-power inner automorphism of G is central.*

(*iii*) $[G, \mathrm{QAut}\, G] \le Z_2(G)$. *In particular* $[G', \mathrm{QAut}\, G] \le Z(G)$ *and* $\mathrm{QAut}\, G$ *acts trivially on* $\gamma_3(G)$.
(*iv*) *If G is not a finite, central extension of a Prüfer group, then every quasi-power automorphism of G is central.*

Proof. (*i*) Let $g \in G$. Since $C = C_G(\theta)$ is infinite, there exists $c \in C$ such that $cg \notin \Theta(G)$. By Lemma 3.2 (*ii*), we get $\langle [g,\theta] \rangle = \langle [cg,\theta] \rangle \lhd G$. Hence θ is central by Lemma 3.2 (*iii*).

(*ii*) If $Z(G)$ is infinite, statement (*ii*) follows from (*i*). Assume then $Z(G)$ is finite. In this case $[G, \mathrm{QAut}\, G]$ is finite by Step 2 in the proof of Theorem A and so statement (*i*) implies that every quasi-power automorphism of G is central.

(*iii*) follows from (*ii*), as every element of $[G, \mathrm{QAut}\, G]$ induces on G a quasi-power automorphism.

(*iv*) Suppose that G has a noncentral quasi-power automorphism θ. Then $\gamma_3(G) \le C_G(\theta)$ is finite by (*i*). Hence G is finite-by-nilpotent and so it is Prüfer-by-finite by Theorem A (*iii*). □

In particular, centreless infinite groups have no nontrivial quasi-power automorphism.

Statement (*ii*) of Theorem 3.3 generalizes (for infinite groups) the analogous theorem by Schenkman on the norm quoted in §1. On the other hand, quasi-power automorphisms of infinite groups may well be non-central, as the following example shows.

Example 3.4. Let p be a prime and $n > 1$ an integer ($n > 3$ if $p = 2$). Let then F be the free group on two generators x, y in the variety of the nilpotent groups of class at most 2 and exponent dividing p^n. Then $F' = \langle [x,y] \rangle$ has order p^n if $p \ne 2$ and has order 2^{n-1} if $p = 2$. Consider the automorphism α of F defined by $x^\alpha = x^t$ and $y^\alpha = y^t$, where $t = 1 + p^{n-1}$ if p is odd and $t = 1 + 2^{n-2}$ if $p = 2$. Then α is not central, as $[x, \alpha, y] = [x, y]^{t-1} \ne 1$. Let $C = (F')^p$. Since F/C has derived subgroup of order p, then α induces on F/C the universal power automorphism of exponent t. Moreover $([x, y]^p)^\alpha = [x^\alpha, y^\alpha]^p = [x, y]^{pt^2} = [x, y]^p$, so that α centralizes C.

Let now G be the direct product of F by $A \simeq \mathcal{C}_{p^\infty}$ with C amalgamated with the corresponding subgroup of A. Then $(ga)^\theta = g^\alpha a^t$ (for all $g \in F$, $a \in A$) defines an automorphism θ of G which induces on both A and G/A the universal power automorphism of exponent t. By Proposition 2.4, $\theta \in \mathrm{QAut}\, G$ but θ is not central, as α is not.

It is worth remarking (see Proposition 3.7 below) that, if $p = 2$, the above construction gives automorphisms acting trivially on G'. It is also possible to define θ in such a way that $[G', \theta] \ne 1$. The construction is similar to the above, starting with $n > 4$ and $t = 1 + 2^{n-3}$. □

We fix now our attention on the case not settled in Theorem 3.3 (*iv*).

Lemma 3.5. *Let G be a finite, central extension of a Prüfer p-group. Then* $G/C_G([G, \mathrm{QAut}\, G])$ *is a finite, abelian p-group.*

Proof. Note first that $G/C_G([G, \mathrm{QAut}\, G])$ is finite and abelian by Theorem 3.3 (*iii*). By Lemma 3.2 (*i*), the statement is equivalent to saying that $[g, \theta] \in Z(G)$ for all $\theta \in \mathrm{QAut}\, G$ and for all elements g of G of order a power of a prime $q \ne p$. Let g be a counterexample. There exists an element $h \in G$ of prime-power order such that $c = [g, \theta, h] \ne 1$. By Lemma 3.2 (*i*), the order of c divides both the order of g and that of h, hence h is a q-element. If Q is a Sylow q-subgroup of G containing g, there exists $x \in G$ such that $h^x \in Q$. Since θ acts on Q as a power automorphism (see Theorem A (*ii*)), then $[g, \theta, h^x] = 1$. By Theorem 3.3 (*iii*) $G' \le C_G([G, \mathrm{QAut}\, G]) \le C_G([g, \theta])$, so that the latter is normal in G and h centralizes $[g, \theta]$, a contradiction. □

Theorem 3.6. *Let G be a finite, central extension of a Prüfer p-group A. If the Sylow p-subgroups of G/A are abelian, then every quasi-power automorphism of G is central.*

Proof. Assume false. By Lemma 3.5 and Lemma 3.2 there exist p-elements g and h of G and $\theta \in \mathrm{QAut}\, G$ such that $c = [g, \theta, h] \ne 1$. Arguing as in the previous lemma, we may assume that g and h belong to the same Sylow p-subgroup P of G. By hypothesis $h^{g^{-1}} = ha$ for a suitable $a \in A$. Moreover, by Proposition 2.4, there exists a p-adic unit π such that θ^{-1} induces on both A and P/A the universal power automorphism of exponent π. Let $h^{\theta^{-1}} = h^\pi b \quad (b \in A)$. Then

$$(h^{\theta^{-1}})^{g^{-1}} = (h^\pi b)^{g^{-1}} = (h^{g^{-1}})^\pi b = h^\pi a^\pi b = (h^{g^{-1}})^{\theta^{-1}}$$

which amounts to saying $[g, \theta, h] = 1$, a contradiction. $\square$

As a final observation we note:

Proposition 3.7. *Let p be an odd prime, let G be a finite, central extension of a Prüfer p-group and let $\theta \in \operatorname{QAut} G$. Then θ is a central automorphism of G if and only if it acts trivially on G'.*

Proof. By Lemma 3.5, it is enough to prove that $[g, h, \theta] = [g, \theta, h^2]$ holds for all $g,\ h \in G$. Since $G' \leq C_G([G, \operatorname{QAut} G])$, by Lemma 3.1 the element $gh = hg[g, h]$ centralizes $[g^{-1}h^{-1}, \theta]$. Hence, applying Lemma 3.2, we get:

$$\begin{aligned}
[g, h, \theta] &= [g^{-1}h^{-1}, \theta]\,[gh, \theta] = [g^{-1}, \theta]^{h^{-1}}[g, \theta]^{h[h,\theta]} \\
&= [g^{-1}, \theta]^{h^{-1}}[g, \theta]^{h} = \left([g, \theta]^{-1}\,[g, \theta]^{h^2}\right)^{h^{-1}} \\
&= [g, \theta, h^2]^{h^{-1}} = [g, \theta, h^2].
\end{aligned}$$

$\square$

Example 3.4 shows that Proposition 3.7 does not hold if $p = 2$.

§4. Quasi-power automorphisms of Prüfer-by-finite groups and proof of Theorem B

A theorem by Cooper ([2], Theorem 2.3.1) states that the group of power automorphisms of a finite group G can be embedded in the direct product of the power automorphism groups of the Sylow subgroups of G. A result of the same type holds for quasi-power automorphisms of Prüfer-by-finite groups.

Lemma 4.1. *Let G be a finite extension of a Prüfer p-group. Let P be a Sylow p-subgroup of G, let $q_1, q_2, \ldots, q_n$ be the primes in $\pi(G)$ different from p and let Q_i be a Sylow q_i-subgroup of G for each $i \leq n$. Then $\operatorname{QAut} G$ can be embedded in $D = \operatorname{QAut} P \times \operatorname*{Dr}_{i=1}^{n} \operatorname{PAut} Q_i$.*

Proof. Every $\theta \in \operatorname{QAut} G$ defines by restriction a power automorphism θ_i on each Q_i (Theorem A (ii)) and a quasi-power automorphism θ_0 on P (Lemma 2.2). Consider the homomorphism $\varphi : \theta \in \operatorname{QAut} G \longmapsto (\theta_0, \theta_1, \ldots, \theta_n) \in D$. Let $\theta \in \ker\varphi$. Since G is clearly generated by P and the Q_i's, then $\theta = 1$ and φ is a monomorphism. $\square$

If G is nilpotent the monomorphism θ of the proof above is actually an isomorphism. Hence, modulo the description of power automorphisms of finite groups of prime-power order, the study of $\operatorname{QAut} G$ may be reduced to the case that G is a p-group. Thus quasi-power automorphisms of infinite nilpotent groups are described by Proposition 2.4 above. In some cases a more explicit description can be given.

Theorem 4.2. *Let the nilpotent p-group G be extension of a Prüfer group A by a finite abelian group Q. Let $p^t = |G'|$. If $p \neq 2$ or G is abelian, then:*
(i) $\operatorname{PAut} G$ *is isomorphic with the group of the p-adic units π such that $\pi \equiv 1 \,(\operatorname{mod} p^t)$.*
(ii) $\operatorname{QAut} G = \operatorname{PAut} G \times \mathit{\Gamma}$, *where $\mathit{\Gamma} = C_{Aut\,G}(A) \cap C_{Aut\,G}(Q) \simeq Q$.*

Proof. (i) Let $p \neq 2$. By the identity $(xy)^n = x^n y^n [y, x]^{\frac{n(n-1)}{2}}$, which holds in G for any integer n, as G has class at most 2, it follows that the map $\alpha_\pi : x \mapsto x^\pi$ (π a p-adic unit) is a monomorphism in G if and only if $\pi \equiv 1 \,(\operatorname{mod} p^t)$. Clearly the same holds if G is abelian. Furthermore, if $\pi \equiv 1 \,(\operatorname{mod} p^t)$, then α_π is an automorphism of G, since $A^{\alpha_\pi} = A$ and G/A is finite. Finally $\operatorname{PAut} G = \{\,\alpha_\pi \,|\, \pi \equiv 1 (\operatorname{mod} p^t)\,\}$ by Proposition 2.4 (i).

(ii) Clearly $\mathit{\Gamma} \lhd \operatorname{Aut} G$, as A is characteristic in G. By Corollary 2.5 we have only to prove that $\operatorname{QAut} G = \mathit{\Gamma}\operatorname{PAut} G$. Let $\alpha \in \operatorname{QAut} G$. There exists a p-adic unit π such that $a^\alpha = a^\pi$ for all $a \in A$. By Proposition 3.6 α is central and so acts trivially on $G' \leq A$. Hence $\pi \equiv 1 \,(\operatorname{mod} p^t)$. Therefore $\alpha_\pi : x \mapsto x^\pi$ is a power automorphism of G and $\alpha_\pi^{-1}\alpha \in \mathit{\Gamma}$. Thus $\operatorname{QAut} G = \operatorname{PAut} G \times \mathit{\Gamma}$. $\square$

Since power automorphisms of abelian groups are very well understood, the above theorem totally describes quasi-power automorphisms of abelian groups.

It is worth remarking that the proof of Theorem 4.2 actually shows this more general result: *Let the nilpotent p-group G of class at most 2 be extension of a Prüfer group A by a finite group Q. Let p^t be the exponent of G'. If $p \neq 2$ or G is abelian, then* $\mathrm{PAut}\, G$ *is isomorphic with the group of the p-adic units π such that $\pi \equiv 1 \pmod{p^t}$ and the group of the central quasi-power automorphisms of G is* $\mathrm{PAut}\, G \times \varGamma$, *where* $\varGamma = C_{Aut\, G}(A) \cap C_{Aut\, G}(Q) \simeq Q_{ab}$.

A result weaker than Theorem 4.2 holds when $p = 2$.

Proposition 4.3. *Let the nilpotent 2-group G be an extension of a Prüfer group A by a finite abelian group Q. Let $2^t = |G'|$. If G is not abelian, then:*
(i) $\mathrm{PAut}\, G$ *is isomorphic with the group of the 2-adic units π such that $\pi \equiv 1 \pmod{2^{t+1}}$.*
(ii) *Let $\varGamma = C_{Aut\, G}(A) \cap C_{Aut\, G}(Q)$. Then* $\mathrm{PAut}\, G \times \varGamma$ *is a subgroup of index 2 of* $\mathrm{QAut}\, G$.

Proof. (i) can be proved like the analogous statement of Theorem 4.2. To prove (ii) consider the homomorphism φ of $\mathrm{QAut}\, G$ in the group of the 2-adic units, defined by $a^{\alpha^\varphi} = a^\alpha$ for all $a \in A$. As in the proof of Theorem 4.2 we see that $\mathrm{PAut}\, G \times \varGamma$ is the preimage under φ of the group of all 2-adic units $\pi \equiv 1 \pmod{2^{t+1}}$. It will be enough to show that the image of φ is the group of the 2-adic units $\pi \equiv 1 \pmod{2^t}$. A 2-adic unit π belongs to $im\, \varphi$ if and only if there exists an automorphism of G inducing on both A and Q the universal power automorphism of exponent π. By the Universal Coefficients Theorem, the cohomology class of $A \rightarrowtail G \twoheadrightarrow Q$ may be identified with a homomorphism $f : M(Q) \to A$, where $M(Q)$ is the Schur multiplicator of Q. Hence $\pi \in im\, \varphi$ if and only if $\pi^2 f = \pi f$. Since the image of f is G', the order of f is 2^t, so that $im\, \varphi$ is the set of the 2-adic units $\equiv 1 \pmod{2^t}$, as we wanted to show. □

In the case that the factor Q is not abelian, we can describe periodic quasi-power automorphisms.

Proposition 4.4. *Let the nilpotent p-group G be an extension of a Prüfer group A by a finite nonabelian group Q. Then* $\mathrm{tor}\, \mathrm{QAut}\, G = C_{Aut\, G}(A) \cap C_{Aut\, G}(Q) \simeq Q_{ab}$.

Proof. Consider the exact sequence

$$C_{Aut\, G}(A) \cap C_{Aut\, G}(Q) \rightarrowtail \mathrm{QAut}\, G \xrightarrow{\varphi} \mathcal{U}$$

where $\mathcal{U}$ is the group of p-adic units and φ is defined as in the proof of Proposition 4.3. If $1 \neq \pi \in im\, \varphi$, then, by Proposition 2.4, Q has a universal power automorphism of exponent π. This implies $-1 \neq \pi \equiv 1 \pmod{p}$ (by a direct argument or by a theorem by Huppert, see [2], Theorem 5.1.1). Therefore π has infinite order. Thus $im\, \varphi$ is torsion-free and the proposition is proved. □

Proof of Theorem B

The fact that $\bar{\mathrm{Q}}\mathrm{Aut}\, G$ is finite is a direct consequence of Theorem A. We prove now that $\bar{\mathrm{Q}}\mathrm{Aut}\, G \rightarrowtail \bar{\mathrm{Q}}\mathrm{Aut}\, P$, where P is a Sylow p-subgroup of G. By Step 7 in the proof of Theorem A, every quasi-power automorphism θ of G induces by restriction a quasi-power automorphism θ_0 on P. It will be enough to show that $\theta \in \mathrm{PAut}\, G$ if $\theta_0 \in \mathrm{PAut}\, P$. Assume then $\theta_0 \in \mathrm{PAut}\, P$. By Theorem A ($ii$), we have only to show that $H^\theta = H$ for all p-subgroups H of G. We distinguish two cases.

– *G is Prüfer-by-finite.* Let $C = C_G([G, \theta])$. By Theorem 3.3 (iv) and Lemma 3.5, we have $PC = G$. It follows that the Sylow p-subgroups of G form a unique conjugacy class under the action of C. In particular there exists $c \in C$ such that $H^c \leq P$. Thus $(H^c)^\theta = H^c$. Since $[c, \theta] \in Z(G)$ (by Lemma 3.2 (i)) we get $H^\theta = H$.

– *G is not Prüfer-by-finite.* In this case θ is central by Theorem 3.3 (iv). Assume $H^\theta \neq H$. Since $\Theta(G)$ is finite it is easy to show that $P \cap \Theta(G)$ is a Sylow p-subgroup of $\Theta(G)$. As $H \leq \Theta(G)$, there exists $x \in \Theta(G)$ such that $H^x \leq P \cap \Theta(G) \leq P$. As in the previous case, from $[x, \theta] \in Z(G)$ it follows $H^\theta = H$.

Therefore $\bar{\mathrm{Q}}\mathrm{Aut}\, G \rightarrowtail \bar{\mathrm{Q}}\mathrm{Aut}\, P$. We have now to prove that $\bar{\mathrm{Q}}\mathrm{Aut}\, G$ is a p-group.

By the above, we may assume that G is a p-group. Let α be a quasi-power automorphism of G of prime-power order modulo $\mathrm{PAut}\, G$ and let H be a cyclic subgroup of G such that $H^\alpha \neq H$. It will be clearly enough to prove that α^{p^n} fixes H, for a suitable positive integer n.

Assume first G is soluble-by-finite. Then, by Theorem A (*iii*), G has a subgroup A isomorphic with $\mathcal{C}_{p^\infty}$ and $[A,H]=1$. Since α fixes the abelian group AH and $\bar{\mathrm{Q}}\mathrm{Aut}\,AH$ is a p-group by Theorem 4.2, the claim is proved in this case.

Let now G be not soluble-by-finite. Consider $C=C_G(H)$ and $D=C_C(C\cap\Theta(G)\,)$. Obviously $H\le D$ and $|G:D|$ is finite by Theorem A. Let now $E=D'H$. Since G is not soluble-by-finite, D' is infinite and not abelian. In particular $E^\alpha=E$. Moreover α acts trivially on D', as α is a central automorphism by Theorem 3.3 (*iv*). By the choice of D, it is clear that every subgroup of E which is not contained in $Z(E)$ is fixed by α. Further $Z(E)/Z(D')=HZ(D')/Z(D')$ is cyclic, so that α fixes every subgroup between $Z(D')$ and $Z(E)$. Thus α induces a power automorphism on $E/Z(D')=(D'/Z(D')\,)\times(Z(E)/Z(D')\,)$. Since α centralizes $D'/Z(D')\ne 1$, it must centralize the subgroup of order p of $Z(E)/Z(D')$ and so α^{p^t} centralizes $Z(E)/Z(D')$ for a suitable positive integer t. Hence α^{p^t} induces on E a periodic automorphism which acts trivially on $E/Z(D')$ and $Z(D')$. This can be identified with an element of $\mathrm{Hom}(E/Z(D'),Z(D')\,)$, whose torsion subgroup is a p-group. Therefore there exists a positive integer n such that $[E,\alpha^{p^n}]=1$. In particular α^{p^n} fixes H, as we wanted to show.

The last part of the statement follows from Theorem 4.2. □

A consequence of Theorem B and Corollary 5.1.2 of [2] is that *if G is a nonabelian p-group generated by elements of bounded order, then* $\mathrm{QAut}\,G$ *is an abelian p-group of finite exponent.*

Our last result is an observation about quasi-power automorphisms of a group G which is finite noncentral extension of a Prüfer subgroup A. It follows by Theorem 3.3 that $\mathrm{QAut}\,G$ centralizes A. In view of Lemma 4.1, we are mainly interested in the case that G is a p-group. Obviously this is possible only when $p=2$.

Proposition 4.5. *Let G be a finite noncentral extension of a Prüfer group A. Then,* $\mathrm{QAut}\,G$ *is finite. Moreover, if G is a 2-group, then also* $\mathrm{QAut}\,G$ *is a 2-group.*

Proof. It holds $G=AF$ for a suitable finite subgroup F of G fixed by $\mathrm{QAut}\,G$. Since $\mathrm{QAut}\,G$ acts trivially on A, then $\mathrm{QAut}\,G$ can be embedded in $\mathrm{Aut}\,F$ and so is finite.

If G is a 2-group, let $C=C_G(A)$. Then G/C has order 2 and C is clearly centralized by every power automorphism of G. Thus, if $z\in G\smallsetminus C$, then $\mathrm{PAut}\,G$ can be embedded in $\mathrm{Aut}\,\langle z\rangle$, which is a 2-group. The statement follows now by Theorem B. □

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