A note on endomorphisms of hypercentral groups

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Abstract: Let H be a subnormal subgroup of a hypercentral group G. We prove that endomorphisms of G are uniquely determined by their restrictions to H if and only if $\mathrm{Hom}\big(G/H^G, G\big) = 0$, and draw some consequences from this fact.

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In [1] we defined 'bases' of groups with respect to sets of endomorphisms as follows. Let G be a group and let Γ be a set of endomorphisms of G. A subset X of G is a *Γ-basis* if and only if, for every $\alpha, \beta \in \Gamma$, we have $\alpha = \beta$ if $\alpha_{|X} = \beta_{|X}$, where $\alpha_{|X}$ and $\beta_{|X}$ denote restrictions to X. For instance, an $\mathrm{Inn}\, G$-basis is simply a subset X of G such that $C_G(X) = Z(G)$. Of course X is a Γ-basis if and only if $\langle X \rangle$ is, hence one can always reduce the study of Γ-bases to subgroups rather than to arbitrary subsets of G. In [1] we were mainly interested in studying the property of being an Aut-basis as an embedding property for subgroups. (We write 'End-basis', 'Aut-basis' or 'Inn-basis' of a group G to mean $\mathrm{End}\, G$-, $\mathrm{Aut}\, G$- or $\mathrm{Inn}\, G$-basis respectively.) In particular, we discussed the consequences for the structure of a group of the property of having a subgroup satisfying some group-theoretic condition as an Aut-basis.

Here we aim at a different direction: to give some more explicit description of End-bases and Aut-bases in certain groups. A subgroup H of a group G is an End-basis of G if and only if the restriction map $res : \mathrm{End}\, G \to \mathrm{Hom}(H, G)$ is injective. Thus a necessary condition for H to be an End-basis of G is the following: the only endomorphisms ε of G such that $H \le \ker \varepsilon$ (that is: such that $\varepsilon^{res} = 0$) is the zero endomorphism. We say that H is a *zero-basis* of G if this condition holds. As is clear, this is also equivalent to the condition $\mathrm{Hom}\big(G/H^G, G\big) = 0$, and H is a zero-basis of G if and only if H^G is. If G is abelian the restriction map res is a homomorphism, and $\ker res$ is the set of all $\varepsilon \in \mathrm{End}\, G$ such that $H \le \ker \varepsilon$. Hence H is an End-basis of G if and only if it is a zero-basis, a fact that provides an easy characterization of End-bases of abelian groups. This raises the problem of determining to which extent analogous characterizations of End-bases can be obtained for wider classes of groups. As observed, every End-basis is a zero-basis, but the converse is not true in general. However, we shall show that the conditions of being a zero-basis and that of being an End-basis are equivalent for nilpotent groups. More generally, we shall prove that every subnormal subgroup of a hypercentral group is an End-basis provided it is a zero-basis (see Theorem 1.3). Some counterexamples bar the way to the most obvious attempts to further improvements on this result. This theorem and related remarks on hypercentral groups are the subject of Section 1; some consequences for the explicit description of End-bases and Aut-bases will be given in Section 2.

1. Zero-bases and End-bases

Our first lemma is an easy remark about the property of having no non-zero homomorphism to a given group G.

Lemma 1.1. *Let K and G be two groups, of which at least one is hypercentral. Then* $\mathrm{Hom}(K,G)=0$ *if and only if* $\mathrm{Hom}(K_{\mathrm{ab}},G)=0$.

Proof — We have to prove the 'if' part of the statement. To this end, assume $\mathrm{Hom}(K_{\mathrm{ab}},G)=0$ and let $\varepsilon\in\mathrm{Hom}(K,G)$. Then K^ε is a subgroup of G and $\mathrm{Hom}(K^\varepsilon_{\mathrm{ab}},G)=0$, whence $\mathrm{Hom}(K^\varepsilon_{\mathrm{ab}},K^\varepsilon)=0$. But K^ε is hypercentral, hence $K^\varepsilon=1$ and $\varepsilon=0$. Thus $\mathrm{Hom}(K,G)=0$, as required. □

A consequence of this lemma is that a subgroup H of a hypercentral group G is a zero-basis if and only if HG' is a zero-basis.

The following is a slight generalization of Lemma 1.11 of [1]. We include a proof for the convenience of the reader.

Lemma 1.2. *Let H be a subgroup of the hypercentre of the group K, and let G be a hypercentral group. If* $\mathrm{Hom}(K/H^K,G)=0$ *then* $\mathrm{Hom}\big(H^K/H^{(H^K)},G\big)=0$.

Proof — Let $N:=H^K$, and suppose $\mathrm{Hom}(N/H^N,G)\neq 0$. Then $\mathrm{Hom}(N/H^NN',G)\neq 0$ by Lemma 1.1. Thus there exists a proper subgroup S of N such that $HN'\leq S$ and N/S embeds in G. Let $V=S_K$ and $C/V=(N/V)\cap Z(K/V)$. Obviously $C\cap S=V$, thus C/V embeds in N/S and so in G. Now $S\ntrianglelefteq K$ as $N=H^K$, hence $C<N$. As N lies in the hypercentre of K, there exists $d\in N\smallsetminus C$ such that dC is central in K/C. Let D/V be the centralizer of dV in K/V. Then $N\leq D$ because N/V is abelian, and K/D is isomorphic to $[d,K]V/V$, a non-trivial subgroup of C/V. Since the latter embeds in G, this gives rise to a non-zero homomorphism from K/N to G. Thus $\mathrm{Hom}(K/N,G)\neq 0$, a contradiction. □

Now we are able to prove the main result of this section.

Theorem 1.3. *Let H be a subnormal subgroup and a zero-basis of the hypercentral group G. Then H is an End-basis of G, and $C_G(H^\eta)=C_G(G^\eta)$ for all $\eta\in\mathrm{End}\,G$.*

Proof — We will argue by induction on the subnormal defect d of H in G. Of course we may assume $d>0$. Let $\eta\in\mathrm{End}\,G$. Suppose that $C:=C_G(H^\eta)$ does not centralize G^η, and let α be the minimal ordinal such that $[Z_\alpha(G)\cap C,G^\eta]\neq 1$. Then α is not a limit ordinal. Pick $x\in Z_\alpha(G)\cap C\smallsetminus C_G(G^\eta)$, and let $L=H^{G,d-1}$. Since L^η normalizes H^η and hence C we have $[x,L^\eta]\leq C\cap Z_{\alpha-1}(G)$, so $[x,L^\eta,G^\eta]=1$. Therefore $\varphi: g\in L\mapsto[g^\eta,x]\in G$ is a homomorphism. Clearly $H\leq\ker\varphi$. Lemma 1.2 yields $\mathrm{Hom}(L/H,G)=0$, thus $\varphi=0$, which amounts to saying $[L^\eta,x]=1$. By the induction hypothesis $C_G(L^\eta)=C_G(G^\eta)$, hence $x\in C_G(G^\eta)$. This contradicts our choice of x. So $C_G(H^\eta)=C_G(G^\eta)$.

Now let ε and η be endomorphisms of G such that $\varepsilon_{|H}=\eta_{|H}$. For all $g\in L$ and for all $h\in H$ we have

$$(h^\eta)^{g^\varepsilon}=(h^\varepsilon)^{g^\varepsilon}=(h^g)^\varepsilon=(h^g)^\eta=(h^\eta)^{g^\eta},$$

so $g^\varepsilon g^{-\eta}$ centralizes H^η and hence L^η by the above. It follows that the mapping $\delta: g\in L\mapsto g^\varepsilon g^{-\eta}\in G$ is a homomorphism. As $H\leq\ker\delta$ and $\mathrm{Hom}(L/H,G)=0$ we have $\delta=0$ and so $\varepsilon_{|L}=\eta_{|L}$. By the induction hypothesis L is an End-basis of G, hence $\varepsilon=\eta$. This proves that H is an End-basis of G.□

Corollary 1.4. *Let G be a nilpotent group. Then the zero-bases of G are precisely the End-bases of G.*

Corollary 1.5. *Let H be a nilpotent subnormal subgroup and a zero-basis of the hypercentral group G. Then G is nilpotent, of the same class as H.*

Proof — Theorem 1.3 shows in particular that H is an Aut-basis of G. Then G is nilpotent and has the same class as H by [1], Theorems 1.12 and 1.8. □

Every subgroup H of a group G such that $H^G = G$ is a zero-basis, obviously. This makes very easy to construct examples of zero-bases that are not End-bases in hypercentral (non-nilpotent) groups. For instance, let $D = C \rtimes \langle a \rangle$ be the locally dihedral 2-group: C is a Prüfer 2-group and a has order 2 and acts like the inversion map on C. Then $\langle a \rangle^D = D$, so $\langle a \rangle$ is a zero-basis of D. By considering the identity automorphism and the inner automorphism of D determined by a, one sees that $\langle a \rangle$ is not an Inn-basis of D (also see [1], Lemma 1.7), in particular $\langle a \rangle$ is not an End-basis. It is worth remarking that this example can be generalized to arbitrary primes. Indeed, let p be any prime and let A be the (external) direct product of $p-1$ copies of a Prüfer p-group. Let α be the automorphism of A acting like the companion matrix of the polynomial $1 + x + \cdots + x^{p-1}$, i.e., mapping every element $(a_1, a_2, \dots, a_{p-1})$ of A to $(a_{p-1}^{-1}, a_{p-1}^{-1}a_1, a_{p-1}^{-1}a_2, \dots, a_{p-1}^{-1}a_{p-2})$. Then α has order p and $[A, \alpha] = A$, so that $G = A \rtimes \langle \alpha \rangle$ is a hypercentral p-group in which the subgroup $\langle \alpha \rangle$ is a zero-basis (since $\langle \alpha \rangle^G = G$) but not an Inn-basis.

These examples show that the hypothesis that the zero-basis H is subnormal cannot be dismissed in Theorem 1.3, even if we only intend to prove that H is an Aut- (or at least an Inn-) basis, rather than an End-basis. The next example shows that Theorem 1.3 cannot be extended to arbitrary locally nilpotent groups.

Example 1.6. There exists a locally nilpotent metabelian group G having an abelian normal subgroup N such that N is a zero-basis but not an End-basis (not even an Inn-basis) of G.

Our construction starts with the standard wreath product $W = \mathcal{C}_p \wr \mathcal{C}_{p^\infty}$ of a cyclic group of prime order p by a Prüfer p-group. Let B be the base group of W, and let $\varphi : \mathcal{C}_{p^\infty} \to \operatorname{Aut} B$ be the homomorphism describing the conjugation action of $\mathcal{C}_{p^\infty}$ on B. Now let A be any abelian group such that there exists an epimorphism $\pi : A \twoheadrightarrow \mathcal{C}_{p^\infty}$ but A has no subgroups isomorphic to $\mathcal{C}_{p^\infty}$. Possible choices for A are the rational group $\mathbb{Q}$ and a direct product of cyclic p-groups of unbounded orders. Let $G = B \rtimes A$, where the conjugation action of A on B is described by $\pi\varphi$. Then $Z(G) = \ker \pi$ and $G/Z(G) \simeq W$, so G is locally nilpotent. Finally let $N = BZ(G)$. Then $N \lhd G$ and $G/N \simeq \mathcal{C}_{p^\infty}$. Since G has no subgroups isomorphic to $\mathcal{C}_{p^\infty}$ we have $\operatorname{Hom}(G/N, G) = 0$, hence N is a zero-basis of G. However, N is abelian, so N is not an Inn-basis of G (by [1], Lemma 1.7 again). □

Remark 1.7. The second part of the argument in the proof for Theorem 1.3 may be straightforwardly adapted to prove the following: *let N be a normal subgroup, an Inn-basis and a zero-basis of the group G. Then N is an Aut-basis of G.* For, under these hypotheses, N^η still is an Inn-basis of G, that is, $C_G(N^\eta) = Z(G)$ for all $\eta \in \operatorname{Aut} G$.

Another observation on zero-bases of arbitrary groups is that if, for any group G, we call $\operatorname{End}_c G$ the ring of those endomorphisms of G whose image is contained in $Z(G)$ then all zero-bases clearly are End_c-bases. (Indeed the restriction map $\operatorname{End}_c G \to \operatorname{Hom}\big(H, Z(G)\big)$ is a homomorphism for every $H \leq G$.) This easily implies that every zero-basis of a group G is an $\operatorname{Aut}_c G$-basis, where $\operatorname{Aut}_c G$ is the group of the central automorphisms of G.

The remaining part of this section is about some inheritance questions related to zero- and End-bases. One of the difficulties in studying 'bases', like End- or Aut-bases, of groups lies in the fact that the property of being a 'basis' of some sort in not generally preserved under taking subgroups or epimorphic images. Easy counterexamples showing this are implicitly suggested by the results in the next section. However, passing to some subgroups or some factors may preserve the properties which we are interested in. Before showing that, we record a further remark on the property considered in Lemma 1.1 and Lemma 1.2.

Lemma 1.8. *Let K and G be hypercentral groups. If* $\mathrm{Hom}(K_{\mathrm{ab}}, G) = 0$ *then* $\mathrm{Hom}(\gamma_i(K), G) = 0$ *for all $i \in \mathbb{N}$.*

Proof — Suppose $\mathrm{Hom}(\gamma_i(K), G) \neq 0$ for some $i \in \mathbb{N}$. It follows from Lemma 1.1 that $\gamma_i(K)$ has a nontrivial abelian quotient $A := \gamma_i(K)\big/N$ isomorphic to a subgroup of G. There is no loss of generality in assuming $N_K = 1$. Then $\gamma_i(K)$ is abelian. Let $a \in Z_2(K) \cap \gamma_i(K)$. Then $[K, a] \simeq K/C_K(a)$, hence $[K, a]$ is an epimorphic image of K_{ab} and so $\mathrm{Hom}\big([K, a], G\big) = 0$. For every $x \in K$, the quotient $\gamma_i(K)\big/N^x$ is isomorphic to A and embeds in G, hence $[K, a] \leq N^x$. Thus $[K, a] \leq N_K = 1$, so $a \in Z(K)$. As K is hypercentral we get $\gamma_i(K) \leq Z(K)$. Therefore K is nilpotent and $\gamma_i(K)$ is an epimorphic image of the tensor product of i copies of K_{ab}. Now, since $\mathrm{Hom}(K_{\mathrm{ab}}, A) = 0$ it easily follows $\mathrm{Hom}\big(\gamma_i(K), A\big) = 0$, a contradiction. □

Proposition 1.9. *Let H be a subgroup and a zero-basis of the hypercentral group G. For every $i \in \mathbb{N}$ we have:*
(i) *$HZ_i(G)\big/Z_i(G)$ is a zero-basis of $G\big/Z_i(G)$;*
(ii) *H is a zero-basis of $H^G\gamma_i(G)$.*

Proof — To prove (*i*) we may assume $i = 1$. Let $Z = Z(G)$ and suppose that there exists a non-zero endomorphism ε of G/Z such that $HZ/Z \leq \ker\varepsilon$. Let $V/Z := \mathrm{im}\,\varepsilon$, so that $V > Z$. We can pick an element $x \in G \smallsetminus C_G(V)$ belonging to the least term of the upper central series of G not centralizing V. Then $[x, V] \neq 1 = [x, V, V]$, hence the mapping $\eta : gZ \in V/Z \mapsto [g, x] \in G$ is a homomorphism. Now $g \in G \mapsto (gZ)^{\varepsilon\eta} \in G$ is a non-zero endomorphism of G whose kernel contains H. This contradiction proves (*i*).

(*ii*) We have $\mathrm{Hom}(G/H^G, G) = 0$, hence Lemma 1.2 and Lemma 1.8 yield $\mathrm{Hom}(H^G/S, G) = 0$ and $\mathrm{Hom}(L/H^G, G) = 0$, where $S = H^{(H^G)}$ and $L = H^G\gamma_i(G)$. Let $T = H^L$. Then $S \leq T \lhd H^G$, so $\mathrm{Hom}(H^G/T, G) = 0$. It follows $\mathrm{Hom}(L/T, G) = 0$, which proves (*ii*). □

If the subgroup H in the last proposition is also subnormal, we may substitute 'End-basis' for 'zero-basis' in the statement, because of Theorem 1.3. As regards part (*i*) of the proposition, if G is not hypercentral and $Z = Z(G)$, then HZ/Z is not necessarily a zero-basis of G/Z (a counterexample is the group in Example 1.6) but it is at least an End_c-basis in G/Z, as follows by an argument similar to that in the proof above. On the other side, terms of the lower central series behave worse with respect to taking quotients. Indeed, let $G = AH$ be a central product, where A is isomorphic to the additive rational group, H is a finitely generated group and $H' = Z(H) = A \cap H \neq 1$. Then G is nilpotent of class 2 and H is an End-basis of G, but $H/G' = H/H'$ is not an End-basis (or, equivalently, a zero-basis) of G/G'.

2. Description of End- and Aut-bases.

For periodic locally nilpotent groups the description of bases can be reduced to the case of p-groups. Indeed, it is immediate to check the following lemma.

Lemma 2.1. *Let $G = \mathrm{Dr}_{i\in I}\, G_i$ be a direct product of periodic pairwise coprime groups G_i (i.e., $\pi(G_i) \cap \pi(G_j) = \varnothing$ if $i \neq j$). Let $H \leq G$ and let $H_i = H \cap G_i$ for all $i \in I$. Then $H = \mathrm{Dr}_{i\in I}\, H_i$ is a zero- (resp. End-, Aut-, Inn-) basis of G if and only if H_i is a zero- (resp. End-, Aut-, Inn-) basis of G_i for all $i \in I$.*

A special case of the following theorem provides a characterization of End-bases in periodic nilpotent groups.

Theorem 2.2. *Let p be a prime and let G be a hypercentral p-group. If H is a proper subnormal subgroup of G then:*
(i) *H is an End-basis of G if and only if G/H^G is divisible and G is reduced;*
(ii) *if H is an Aut-basis of G then either G/H^G is divisible and $Z(H^G)$ is reduced, or $|G| = 2$ and $H = 1$. In particular, $G' \leq H^G$.*

Proof — If $H = 1$ then H is an Aut-basis of G if and only if $\operatorname{Aut} G = 1$, that is if and only if $|G| = 2$ (we have $G \neq 1$ since $H < G$), while H cannot be an End-basis as $\operatorname{End} G \neq 0$. This is in accordance with our statement, so we may assume $H \neq 1$.

Theorem 1.3 ensures that H is an End-basis of G if and only if $\operatorname{Hom}(G/H^G, G) = 0$. Since G/H^G and G are nontrivial hypercentral p-groups, this is equivalent to the property that G/H^G is divisible and G is reduced. Thus (i) is proved. If H is an Aut-basis of G then $\operatorname{Hom}\big(G/H^G, Z(H^G)\big) = 0$, because $Z(H^G) \leq Z(G)$ and so $\operatorname{Hom}\big(G/H^G, Z(G)\big)$ is isomorphic to the group of the automorphisms of G acting trivially on H^G and on G/H^G. Thus also (ii) follows (every hypercentral periodic divisible groups is abelian, see [4], part 2, p. 125). □

Theorem 2.3. *Let p be a prime and let G be a nilpotent p-group. If $H \leq G$ then H is an Aut-basis of G if and only if either H is an End-basis of G or $|G| = 2$ and $H = 1$.*

Proof — To prove our statement we shall assume that H is an Aut-basis but not an End-basis of G and show that H is trivial. Suppose $H \neq 1$. It is a consequence of Theorem 1.3 that H^G is not an End-basis of G, so we may replace H with H^G, that is, we may assume $H \lhd G$. Theorem 2.2 shows that G has a subgroup $P \simeq \mathcal{C}_{p^\infty}$ but $Z(H)$ is reduced. As G is nilpotent, $P \leq Z(G)$, hence $P \nleq H$ and $PH/H \simeq \mathcal{C}_{p^\infty}$. Now $G' \leq H$, because G/H is divisible, therefore PH/H has a complement K/H in G/H. Also, $P \cap K = P \cap H$ is finite, of order p^n, say. Then G has an automorphism α defined by $[K, \alpha] = 1$ and $x^\alpha = x^{p^n+1}$ for all $x \in P$. This contradicts the hypothesis that H is an Aut-basis. Therefore $H = 1$, and so $|G| = 2$. Conversely, the stated condition clearly implies that H is an Aut-basis of G, hence the result is proved. □

Theorem 2.2 shows that a hypercentral p-group G has a proper subnormal subgroup as an End-basis if and only if G is reduced and G_{ab} has a nontrivial divisible quotient. This latter condition is equivalent to G_{ab} being of infinite exponent. For nilpotent groups this can be stated as follows.

Corollary 2.4. *Let p be a prime and let G be a nilpotent p-group. Then G has a proper subgroup as an End-basis if and only if G is reduced and has infinite exponent.*

Proof — If G has a proper subgroup as an End-basis then it is reduced and of infinite exponent by Theorem 2.2. Conversely, assume that G is reduced and has infinite exponent. Then G_{ab} has infinite exponent. Let N/G' be a basic subgroup of G_{ab}. By Theorem 2.2, to prove that G has a proper (normal) subgroup as an End-basis it suffices to check that G_{ab} has a nontrivial divisible quotient. Of course this is true if $N < G$. If $N = G$, then G_{ab} is a direct product of cyclic subgroups of unbounded orders, hence it has a quotient isomorphic to $\mathcal{C}_{p^\infty}$. □

By the remark preceding last corollary a periodic hypercentral group all whose primary components have finite rank (that is: are Černikov groups) has no proper subnormal subgroup as an End-basis. As a matter of fact, here the subnormality hypothesis can be dropped, as is shown by the following—slightly more general—result.

Proposition 2.5. *Let p be a prime and let G be a hypercentral p-group. Assume that G has a divisible abelian subgroup of finite index A. If $|G| \neq 2$ then G has no proper subgroup as an Aut-basis.*

Proof — By a theorem of Zaicev ([5], Theorem 1; see [3], p. 218) if Γ is a finite group of automorphisms of A and B is a divisible Γ-invariant subgroup of A then there exists a Γ-invariant divisible subgroup C of A such that $A = BC$ and $B \cap C$ has finite exponent dividing $|\Gamma|$.

Let the subgroup H be an Aut-basis of G. As $|G| \neq 2$ Theorem 2.2 (ii) shows that G/H^G is divisible, hence $G = HA$. Let B be the maximal divisible subgroup of $H \cap A$. Let $p^n = \big|H/H \cap A\big|$, and let $S = A[p^n]$, the nth socle of A. It easily follows from Zaicev's theorem that A has a set $\{C_i \mid i \in I\}$ of H-invariant divisible subgroups of finite rank such that

$$A/S = \big(BS/S\big) \times \operatorname*{Dr}_{i\in I}\big(C_iS/S\big).$$

For every $i \in I$ the subgroup $H \cap C_i$ is finite. Let $p^{t_i} = \max\{\exp(H \cap C_i), p^n\}$. Then it is clear that one can define an automorphism α of G by setting $h^\alpha = h$ for all $h \in H$ and $g^\alpha = g^{1+p^{t_i}}$ for all $i \in I$ and $g \in C_i$. Since $\alpha \neq 1$ and H was supposed to be an Aut-basis of G this is a contradiction. □

Our next example shows that part *(ii)* of Theorem 2.2 cannot be improved to a characterization as in Theorem 2.3.

Example 2.6. For every prime number p there exists a hypercentral p-group having a normal subgroup which is an Aut-basis but not an End-basis.

Let A be the direct product of $p-1$ copies of the Prüfer group $\mathcal{C}_{p^\infty}$. As we recalled above, A has an automorphism α (of order p) such that $1+\alpha+\alpha^2+\cdots+\alpha^{p-1} = 0$. From the latter equality it follows that $C_A(\alpha)$ has exponent p. For every $n \in \mathbb{N}$ let H_n be the standard wreath product of A by a cyclic group $\langle a_n\rangle$ of order p^n. Let B_n be the base group of H_n. Let α_n be the automorphism of H_n that acts like α on A (identified with a direct factor of B_n in the standard way) and centralizes a_n. Finally, set $H := \mathrm{Dr}_{n\in\mathbb{N}} H_n$ and $G = H \rtimes \langle t\rangle$, where t has order p and acts on each factor H_n like α_n. Set also $B := \mathrm{Dr}_{n\in\mathbb{N}} B_n$ and let $N = \langle B, t, a_n a_{n+1}^{-p} \mid n \in \mathbb{N}\rangle$. Then $G' \leq B \leq N$, and $G/N \simeq \mathcal{C}_{p^\infty}$. We shall prove that N is an Aut-basis of G. Since G is not reduced N cannot be an End-basis. By Lemma 1.3 of [1] it is enough to show that $C_G(N)$ coincides with $Z(G)$ and $Z(N)$, and that it is reduced, so that $\mathrm{Hom}\big(G/N, Z(N)\big) = 0$. To this end, let $x \in C_G(N)$. Then $x = ht^u$ where u is a non-negative integer less than p and $h = (h_n)_{n\in\mathbb{N}} \in H$; here $h_n \in H_n$ for every $n \in \mathbb{N}$, and $h_n = 1$ for all but finitely many subscripts n. Let $i \in \mathbb{N}$ be such that $h_i = 1$. Then $1 = [B_i, x] = [B_i, t^u]$, hence $u = 0$, and $x = h \in H$. For every $n \in \mathbb{N}$ we have $1 = [x, a_n a_{n+1}^{-p}]$, hence $1 = [h_n, a_n]$ and $1 = [x, B_n] = [h_n, B_n]$, so h_n lies in $Z(H_n)$, which is the diagonal subgroup of B_n. Thus $x \in Z(H) \leq B \leq N$. This shows $C_G(N) = Z(N)$ and, since $G = NH$, also $C_G(N) = Z(G)$. Finally, $C_B(t)$ is contained in the socle of B, hence $Z(G)$ has exponent p and is therefore reduced. □

Note that the groups in Example 2.6 are not reduced but have a proper subgroup as an Aut-basis. We leave open the question whether a hypercentral p-groups with a proper subgroup as an End-basis must necessarily be reduced.

Finally, we have a look at End-bases and Aut-bases of non-periodic groups. Such bases seem to be much more difficult to describe in this case, even for abelian groups. Easy examples of End-bases of non-periodic locally nilpotent groups can be obtained by considering the primes involved in periodic sections of the groups.

Recall that if G is a locally nilpotent group and π is a set of primes then $I_{G,\pi}(H) := \{g \in G \mid g^n \in H$ for some π-number $n\}$ is a subgroup of G (the π-isolator of H in G).

Proposition 2.7. *Let G be a locally nilpotent group, and let π be a set of primes such that the π-component G_π of the torsion subgroup of G is trivial. If H is a subgroup of G such that $G = I_{G,\pi}(H)$ then H is an End-basis of G.*

Proof — Let $\varepsilon, \eta \in \mathrm{End}\, G$ and let K be the equalizer of ε and η in G, that is to say, the subgroup $\{g \in G \mid g^\varepsilon = g^\eta\}$. For each $g \in I_{G,\pi}(K)$ there exists a π-number n such that $g^n \in K$, hence $g^{\varepsilon n} = g^{\eta n}$. Since $G_\pi = 1$ the mapping $x \in G \mapsto x^n \in G$ is injective, so $g^\varepsilon = g^\eta$ and $g \in K$. Therefore $K = I_{G,\pi}(K)$. Assume now $\varepsilon_{|H} = \eta_{|H}$, that is $H \leq K$. Then $K = I_{G,\pi}(K) \geq I_{G,\pi}(H) = G$. Hence $K = G$, which amounts to saying $\varepsilon = \eta$. □

Thus every maximal independent subset of a torsion-free abelian group is an End-basis. The converse of this last result does not hold. Indeed, there exist many torsion-free abelian groups A such that $\mathrm{End}\, A$ only consists of the universal power endomorphisms $\varepsilon_n : x \mapsto x^n$ for $n \in \mathbb{Z}$. Groups with this property may have any arbitrary finite rank, or even be uncountable (see [2], vol. II, Theorem 89.2 and p. 133, Ex. 2). Clearly every nontrivial subgroup of such an A is an End-basis.

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