GROUPS COVERED BY CONJUGATES OF PROPER SUBGROUPS

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Abstract. We consider possibilties for pairs (G, H), where G is a group, H a subgroup, and G is the union of conjugates of H. For instance, if G is locally finite and H finite, then $G = H$; and the same holds without the hypothesis of local finiteness if H is isomorphic with the alternating group $\mathbb{A}_5$.

1. Preliminaries and statement of results

We say that a group G lies in the class $\mathfrak{X}$ if G is not covered by conjugates of any proper subgroup. In other words, $G \in \mathfrak{X}$ if and only if $G = 1$ or in any transitive action of G on a set at least one element of G has no fixed points. This class was investigated in [11, 12], where it was shown that $\mathfrak{X}$ is closed under extensions and (restricted) direct products (though not cartesian products), as well as containing all hypercentral (hence all soluble) groups; it is well-known that all finite groups lie in $\mathfrak{X}$. This class $\mathfrak{X}$ is wide, but it does not contain, for example, all locally nilpotent periodic groups; no infinite transitive group G of finitary permutations is in $\mathfrak{X}$ since G is the union of the point stabilizers. Further, since a direct product is in $\mathfrak{X}$ if and only if each of the direct factors is, we see that every group is a direct factor of a group outside $\mathfrak{X}$.

Even if G is a group that is the union of conjugates of a (proper) subgroup H, there will be connections between the structures of G and H. It is obvious, for example, that G is periodic or of some finite exponent if and only if H has the same property, and G is perfect if H is (also see Lemma 6 in the next section). Less obvious but still easy is the next result, which is fundamental for our investigation.

Lemma 1. *Suppose that $H \leq G = \bigcup_{g \in G} H^g$. Then the mapping $N \mapsto H \cap N$ from the lattice of normal subgroups of G to the lattice of normal subgroups of H is injective and hence strictly increasing.*

This follows from the fact that if $N \lhd G$ then the hypothesis yields that $N = \bigcup_{g \in G}(H^g \cap N) = \bigcup_{g \in G}(H \cap N)^g$.

In particular, G will satisfy Min-n or Max-n, or will have finitely many normal subgroups only, if H has the corresponding property. Another immediate consequence of the previous remark is the following, which we state for ease of further reference.

Lemma 2. *Suppose that $H < G = \bigcup_{g \in G} H^g$ and that $N \lhd G$. Then $G/N = \bigcup_{g \in G/N}(HN/N)^g$ and either $HN/N < G/N$ or $H \cap N < N = \bigcup_{g \in N}(H \cap N)^g$.*

Our general theme can be summed up like this.

Problem. Suppose that the group G is the union of conjugates of a subgroup H. What conditions on H and G allow us to deduce that $G = H$?

Here we have in mind structural conditions, not conditions about where H sits in the lattice of subgroups of G. For example, G is certainly H if H is subnormal in G, as $G = H^G$ (but H may be ascendant and proper, see below).

There are striking, difficult examples where G is a Tarski monster and H is cyclic, but even more general examples are provided in the literature. For instance, given any countable group H containing an element of 'big enough' (possibly infinite) order but none of order 2, there exists

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a 2-generator simple group G such that $H < G$ and $G = \bigcup_{g \in G} H^g$ (see [3], Theorem 17). In a similar vein, we have:

Theorem 1. *Let G be a group of finitary permutations on an infinite set Ω. Suppose that every G-orbit is infinite, and that for every cardinal κ the union of all G-orbits of cardinality less than κ has itself cardinality less than κ. Then G is the union of the conjugates of an abelian subgroup H.*

This holds, in particular, if G is transitive or, more generally, if every G-orbit has cardinality $|\Omega|$. Note that if G is transitive then it is also the union of the conjugates of a point stabilizer S but S is not abelian; more than that, S cannot belong to any variety $\mathfrak{V}$ that is not the variety of all groups. Indeed, if $S \in \mathfrak{V}$ then it is clear that $G \in \mathfrak{V}$, but the variety generated by G is the class of all groups, as was proved by P.M. Neumann [6]. Therefore, in Theorem 1, G acts transitively on an infinite set with H as a point stabilizer and each element fixing at least one point, but this action of G is essentially different from the natural action of G on Ω, even if the latter is transitive.

On the positive side, we have :

Theorem 2. *If G is a locally finite group and is the union of conjugates of a finite subgroup H, then $G = H$.*

As could be expected, our proof requires the classification theorem for finite simple groups. A consequence is:

Corollary 3. *If H is a finite group with all elements of order less than 5, then no group G other than H is the union of conjugates of H.*

This is because of Sanov's result (see [5], Theorem 5.25): G is locally finite if all its elements have order less than 5. The hypothesis of finiteness on H cannot be dropped in Theorem 2. Of the many examples to illustrate this, we note the example in [11] of an infinite transitive p-group P of finitary permutations. Obviously, P is the union of the point stabilizers, which are, incidentally, hypercentral and ascendant in P (which is itself a Fitting group). Also, P is the union of conjugates of an abelian subgroup, by Theorem 1.

The next result is a little harder to prove, but the proof is fairly self-contained. Note that there are no finiteness conditions imposed on G.

Theorem 3. *If G is a group that is the union of conjugates of a subgroup H isomorphic to the alternating group $\mathbb{A}_5$, then $G = H$.*

Quite possibly, Theorem 3 remains true whenever H is a finite non-abelian simple group, though this looks like a very difficult problem. The examples mentioned above show that G need not be H when H has odd order and large enough exponent; the remark after Corollary 3 shows that G must be H if H is $\mathbb{S}_4$ or $\mathbb{S}_3$, for example. So the precise picture is somewhat tangled.

Question 1. *Which finite groups H can be embedded in some group $G \neq H$ such that G is the union of conjugates of H?*

It seems that it is finiteness conditions on G and/or H that give the best chance of interesting results. For instance,

Theorem 4. *Let G be a group having a finite series whose factors are either locally supersoluble or in the class $\mathfrak{X}$. Suppose that H is a subgroup of G such that $G = \bigcup_{g \in G} H^g$. If H has finite abelian section rank then $H = G$.*

(According to standard terminology, a group has finite abelian section rank if it has no infinite elementary abelian p-group as a section, for every prime p.) This requires some results that are in fact preliminary, but we prefer to state and prove them in the final section. However, we highlight at this point a problem that seems very intractable. In a sense, it is dual to Theorem 4.

Question 2. *Suppose that G is a locally finite p-group that is the union of conjugates of an elementary abelian subgroup H. Is $G = H$?*

We have no idea of what happens here, even though G is of exponent p. Clearly, only the infinite case is of interest, and by Lemma 6 below we may assume that G is perfect. Such a G has no maximal subgroups; we point out that locally finite groups of prime exponent without maximals do exist (see [9]).

2. Proofs

The proof of Theorem 2 relies on the following lemma, which is certainly well-known and whose proof we reproduce for the convenience of the reader.

Lemma 4. *Every simple locally finite group G of finite exponent is finite.*

Proof. Since G has finite exponent it does not involve all finite groups, hence it is linear (see [2], Theorem 2.6). Then, as an immediate consequence of a theorem of Burnside ([10], Corollary 1.23), G has a unipotent normal subgroup of finite index, hence it is finite. □

Proof of Theorem 2. Supposing the result false, let a counterexample be chosen for which H has the least possible order. Then G is simple by Lemma 2, and therefore finite by Lemma 4. This is a contradiction, and the result follows. □

The next lemma will be used in the proof of Theorem 3. As a matter of fact, computer calculation shows that any two-generator group satisfying the hypothesis of the lemma is finite; we thank M.F. Newman for pointing this out, thus inspiring us to find the following elementary proof.

Lemma 5. *Let G be a group in which every nontrivial element has order 3 or 5. If G is not cyclic and contains an element of order 3 then it contains a noncyclic finite subgroup.*

Proof. Assume false. Then G has more than one subgroup of order 3, and hence we may assume that it is generated by two elements of order 3, say a and b. As is well known and easy to check, if both ab and ab^{-1} have order 3 then $[a,b,b]=[a,b,a]=1$ and G is a finite 3-group. Thus we may further assume that ab has order 5. Now $[ba,ab]=a^{-1}b^{-1}b^{-1}a^{-1}baab=(a^{-1}b)^3$, hence, if ab^{-1} has order 3 then $ba\in\langle ab\rangle$, so that a normalizes $\langle ab\rangle$, which leads to a contradiction. This shows that ab^{-1} has order 5; more than that, we may now assume that the product of any two elements of G of order 3 which do not generate the same subgroup has order 5. In particular, $c:=[a,b]$ has order 5. Then:

$$\begin{aligned} c^2 &= a^{-1}b(ba)^2ab^{-1}ab = a^{-1}b(ba)^{-3}ab^{-1}ab = a^{-1}b(a^{-1}b^{-1})^2a^{-1}(b^{-1}a)^2b \\ &= a^{-1}b(a^{-1}b^{-1})^2a^{-1}(a^{-1}b)^3b = a^{-1}b(a^{-1}b^{-1})^2aba^{-1}ba^{-1}b^{-1} \end{aligned}$$

and so $c^3=t^2$, where $t=a^{-1}b(a^{-1}b^{-1})^2ab$. It follows that $\langle t\rangle=\langle t^2\rangle=\langle c\rangle$ and so $t=c^{-1}$. Then $(a^{-1})^b=ta^{-1}=b^a(b^{-1})^{aba^{-1}}$. Now this element has order 3 and is the product of two elements of order 3; by the above assumption then $b^a=(b^{-1})^{aba^{-1}}$, so ba^{-1} acts via inversion on $\langle b^a\rangle$. This is a contradiction, because ba^{-1} has odd order. □

Proof of Theorem 3. Let G be a group covered by conjugates of its proper subgroup $H\simeq\mathbb{A}_5$. Then G must be infinite. Also, by looking at conjugacy in H we get that G has four or five conjugacy classes of elements: three classes consisting of the identity and all elements of orders 2 or 3, respectively, and one or two conjugacy classes of elements of order 5, according to whether an element of order 5 in H is conjugate in G to its square or not. Note that the subgroups of order 5 are conjugate anyway. We shall focus attention on centralizers of elements.

If x is a nontrivial element of G then x has prime order p, say, and $C:=C_G(x)$ has exponent p, because nontrivial elements of different orders cannot commute, otherwise their product would have composite order. If $p=2$ then C is abelian—let us show that the same also holds if $p\neq 2$. In this latter case there exists an element u of order 2 in G inverting x (that is, $x^u=x^{-1}$; such a u can be found in a conjugate of H containing x). Let y be any element of C. Then also uy inverts x, so it has even order, necessarily 2. Now both u and uy have order 2 and so u inverts y as well. Therefore u acts like the inverting automorphism on C and C is abelian. A consequence is that C is the centralizer of every nontrivial element of itself, regardless of the value of p. Therefore, if $N=N_G(C)$, then N/C acts fixed-point-freely on C. This action induces a transitive action on the set of all nontrivial cyclic subgroups of C, for any two such subgroups are conjugate in G, and if $g\in G$ is such that $x^g\in C$ then $C^g=C_G(x^g)=C$, hence $g\in N$.

Suppose that $p\neq 2$, and let vC be an involution in N/C. Then v has order 2, because its order is even, hence xx^v is centralized by v, so that $xx^v=1$ and $x^v=x^{-1}$. This shows that $vC=uC$, where u is as above, and N/C has only one involution, which therefore lies in the centre of N/C. If $w \in N \smallsetminus C\langle v\rangle$ then wC has odd order and vwC has composite order, which is a contradiction. Therefore $|N/C| = 2$; as the action of N on the nontrivial cyclic subgroups of C is transitive then $C = \langle x\rangle$. It follows that all nontrivial finite subgroups of odd order in G have prime order.

Now consider C and N in the case when $p = 2$. It follows from [4], Theorem 2.1, that C is infinite and therefore (because of transitivity) N/C is also infinite. Moreover, N/C has no involutions, because its action on C is fixed-point-free, and if F/C is a nontrivial finite subgroup of N/C then it has prime order, since F splits over C. Also, N/C has some element of order 3, since in H the centralizer of every involution is normalized by an element of order 3. Lemma 5 shows that this leads to a contradiction. □

In certain cases, the study of our Problem (see Section 1) can be reduced to the case when G is perfect, thanks to the following lemma.

Lemma 6. *Let G be a group and H a subgroup such that $G = \bigcup_{g\in G} H^g$ and H is an extension of a perfect group by a soluble group of derived length n. Then G is an extension of a perfect normal subgroup P by a soluble group of derived length at most n, and $G = PH$. Moreover, if $H < G$ then $H \cap P < P = \bigcup_{g\in P}(H \cap P)^g$.*

Proof. Let N be a normal subgroup of G such that G/N is soluble. Since $G/N \in \mathfrak{X}$ then $HN = G$ and so G/N is isomorphic to a quotient of H, hence it has derived length at most n. Therefore $P := G^{(n)}$ is the soluble residual of G, hence it is perfect. Now $G/P \in \mathfrak{X}$, hence $G = PH$ and the last part of the statement follows. □

Proof of Theorem 4. Since every supersoluble group has nilpotent derived subgroup, it follows that locally supersoluble groups are (locally nilpotent)-by-abelian, and so by repeated application of Lemma 2 we may assume that G is locally nilpotent. Suppose first that H is periodic. Then G is also periodic. For every primary component G_p of G we have $G_p = \bigcup_{g\in G_p} H_p^g$; since H_p is a Černikov group we deduce from Lemma 1 that G_p has Min-n, hence it too is Černikov—here we are using the fact that primary periodic locally nilpotent groups whose abelian subgroups have finite rank are Černikov and hypercentral (see [8], vol. 2, p. 38, Corollary 1). Therefore G is hypercentral, hence $G \in \mathfrak{X}$ and $H = G$.

In the general case, let T be the torsion subgroup of G. Then $HT/T \simeq H/H \cap T$ is nilpotent (see [8], vol. 2, Theorem 6.36). Lemma 6 shows that G/T has a perfect normal subgroup P/T such that $G = PH$ and so $P = \bigcup_{g\in P}(H \cap P)^g$. At the expense of replacing G by P and H by $H \cap P$, we may assume that G/T is perfect. Now H has a finite subset F such that $H/\langle F\rangle^H$ is periodic. Let $N = T\langle F\rangle^G$. Then HN/N is periodic. By the previous case, this yields $HN = G$. As G/T is perfect it follows that $G = N$, hence G/T is the normal closure of a finite subset. Since, again, G/T is perfect then $G = T$, hence $H = G$ by the previous case. □

Two more easy consequences of Lemma 6 are the following. To prove the second we make use of the fact that with the given hypothesis the group G has Max-n by Lemma 1, and since every chief factor of G is abelian, G cannot have any nontrivial normal perfect subgroups.

Corollary 7. *Let G be a residually-$\mathfrak{X}$ group and let H be a soluble subgroup of G. If $G = \bigcup_{g\in G} H^g$ then $H = G$.*

Corollary 8. *Let G be a locally soluble group and let H be a soluble subgroup of G satisfying* Max-n. *If $G = \bigcup_{g\in G} H^g$ then $H = G$.*

The first corollary applies to residually finite groups; it is worth remarking that residually finite groups do not need to lie in $\mathfrak{X}$, for instance the free group of rank 2 is not in $\mathfrak{X}$ ([12]).

Also note that the second corollary applies when H is a polycylic subgroup of a locally soluble group. One could ask whether the same conclusion holds if we replace the Max-n hypothesis on H by Min-n, or Min. If this were true then the same statement would hold in the case that H is only supposed to be minimax. Indeed, reduction arguments similar to those employed for the previous proofs show that if there exists a locally soluble group G with a proper minimax subgroup H such that $G = \bigcup_{g\in G} H^g$ then there exists such an example in which G is perfect and periodic and H

is divisible abelian. We leave also this question open; for example, what if $H \simeq \mathcal{C}_{p^\infty} \times \mathcal{C}_{q^\infty}$ for distinct primes p and q?

Proof of Theorem 1. Let $\kappa = |\Omega|$. Let A be the union of all G-orbits of cardinality less than κ and let $B = \Omega \smallsetminus A$, the union of all orbits of size κ. By hypothesis, $|A| < \kappa$, hence $B \neq \varnothing$ and $|G| = \kappa$. Every element g of G can be uniquely written as $g_1 g_2$, where g_1 stabilizes B (pointwise) and g_2 stabilizes A; we will also refer to g_1 and g_2 as the A- and B-components of g respectively. Consider the group of permutations induced by G on A. Arguing by induction on κ we may assume that this group is the union of conjugates of an abelian subgroup. Therefore there exists a complete set R of representatives of the conjugacy classes of elements of G such that $[g_1, h_1] = 1$ for every $g, h \in R$. We shall show, by a recursive argument, that R can be replaced by a similar set of representatives whose elements commute pairwise. Since $|G| = \kappa$ we can arrange the elements of R in a sequence $(x_\alpha)_{\alpha<\lambda}$ indexed by an ordinal $\lambda \leq \kappa$. Let S be the stabilizer of A in G. Then $|G/S| = |A| < \kappa$, unless $A = \varnothing$, and it follows that the S-orbit of every element of B has size κ. Let β be an ordinal less that λ, and suppose that for every ordinal $\alpha < \beta$ a conjugate $\tilde{x}_\alpha$ of x_α by an element of S has been chosen in such a way that these elements $\tilde{x}_\alpha$ commute pairwise. For every $g \in G$ let $s(g)$ be the support of the B-component g_2 of g. Since $s(x_\beta)$ is finite and $X := \bigcup_{\alpha<\beta} s(\tilde{x}_\alpha)$ has cardinality less than κ, by an extension of a theorem of B.H. Neumann ([1], Exercise 6(viii), page 56) there exists $g \in S$ such that $X \cap s(x_\beta)^g = \varnothing$. Let $\tilde{x}_\beta := x_\beta^g$. Then $\tilde{x}_\beta$ commutes with each of the already defined elements $\tilde{x}_\alpha$, for $\tilde{x}_\beta$ and $\tilde{x}_\alpha$ have the same A-components as x_β and x_α, while the corresponding B-components are disjoint.

This establishes our claim that there exists a complete set of pairwise commuting representatives of the conjugacy classes of elements of G. The subgroup generated by this set has the property required for H. This proves the theorem. □

By Lemma 6 the groups of Theorem 1 have perfect commutator subgroups; this fact is easily proved directly and is well-known for every group of finitary permutations with no finite orbit (see [7]). Still in the context of Lemma 6, note that all groups referred to in the next remark are perfect or at least have perfect commutator subgroups.

Remark. An argument somewhat like that in the proof of Theorem 1 shows that every proper normal subgroup of the full symmetric group on an infinite set Ω is covered by conjugates of an abelian subgroup: Ω has a permutation x with $|\Omega|$ orbits of each possible cardinality between 1 and $\aleph_0$ (in particular, $|\Omega|$ fixed points); if κ is an infinite cardinal not greater than $|\Omega|$ and G is the group of all permutations whose supports have cardinality less than κ then every element of G is conjugate (in G) to a permutation acting like x on some of the x-orbits and trivially on the other ones, and all these permutations commute, hence G has the required property; the same argument applies if G is the alternating group.

The same property does not hold for the full symmetric group, though $\operatorname{Sym}\Omega \notin \mathfrak{X}$ as long as Ω is infinite. Indeed, if $\operatorname{Sym}\Omega$ is covered by conjugates of a subgroup H then H contains a transposition and a permutation displacing all symbols and with no orbit of size 2, and two such permutations cannot commute.

The fact that $\operatorname{Sym}\Omega \notin \mathfrak{X}$ if Ω is infinite can be proved as follows: if Ω is uncountable then the obvious, transitive action of $\operatorname{Sym}\Omega$ on the set of all countably infinite subsets of Ω has the property that every element of $\operatorname{Sym}\Omega$ fixes at least one point, so $\operatorname{Sym}\Omega \notin \mathfrak{X}$. If Ω is countable a slightly more elaborate argument is needed. Let $\mathcal{L}$ be the set of all partitions of Ω consisting of two infinite sets. Define an equivalence relation $\sim$ on $\mathcal{L}$ as follows: for every $\{A, B\}, \{C, D\} \in \mathcal{L}$ let $\{A, B\} \sim \{C, D\}$ if and only if $\{A \smallsetminus F, B \smallsetminus F\} = \{C \smallsetminus F, D \smallsetminus F\}$ for some finite subset F of Ω. It is not hard to see that $\operatorname{Sym}\Omega$ acts transitively on $\mathcal{L}/{\sim}$ (in the obvious way) and every element of $\operatorname{Sym}\Omega$ fixes at least one element of $\mathcal{L}/{\sim}$, so $\operatorname{Sym}\Omega \notin \mathfrak{X}$ also in this case. (We remark that, if $|\Omega| = \aleph_0$, then every subgroup H whose conjugates cover $\operatorname{Sym}\Omega$ must contain $\operatorname{FSym}\Omega$.)

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