J. Algebra, 424 (2016), pp. 399–417. DOI: 10.1016/j.jalgebra.2015.10.007

Groups with countably many subgroups

GIOVANNI CUTOLO AND HOWARD SMITH

ABSTRACT. We describe soluble groups in which the set of all subgroups is countable and show that locally (soluble-by-finite) groups with this property are soluble-by-finite. Further, we construct a nilpotent group with uncountably many subgroups in which the set of all abelian subgroups is countable.

1. INTRODUCTION

It is a trivial observation that a group is finite precisely when it has only finitely many subgroups. As a next step we consider the question: when is the set of all subgroups of a group G countable? The abelian groups with this property have been characterised by Rychkov and Fomin in [13], see Lemma 2.3 below. We shall mostly be concerned with the analogous problem for soluble groups.

We shall say that a group G has the property CMS (or $G \in$ (CMS)) if the set $\mathcal{L}(G)$ of all subgroups of G is countable. Quite plainly all group with CMS are countable, and all groups with the maximal condition on subgroups also satisfy CMS.

The class (CMS) is closed under taking subgroups and quotients, but not under extensions, nor even direct products. For example, if p is a prime, the Prüfer p-group $\mathcal{C}_{p^\infty}$ has CMS, but its direct square $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ has $2^{\aleph_0}$ subgroups. Groups with CMS satisfy a very strong rank-finiteness condition. Indeed, a direct product of infinitely many nontrivial groups certainly has at least $2^{\aleph_0}$ subgroups, therefore groups in (CMS) cannot have any such section. We shall prove that soluble groups without subnormal sections of the types excluded by these remarks satisfy CMS. Thus we shall see in Theorem 2.7 that *a soluble-by-finite group satisfies* CMS *if and only if it is minimax and has no (subnormal) sections of type $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$, for any prime p*. We shall also see that *all locally (soluble-by-finite) groups with* CMS *are soluble-by-finite* (Theorem 2.12). A side feature of both theorems is that they also show that locally (soluble-by-finite) groups with uncountably many subgroups have at least $2^{\aleph_0}$ of them.

Let p be a prime. In the description of a soluble CMS-group G a role is played by the maximal number of factors isomorphic to $\mathcal{C}_{p^\infty}$ in a finite series of G. We shall denote this number by $\mathrm{r}_{p^\infty}(G)$. For instance the result by Rychkov and Fomin cited above could be rephrased by saying that the abelian groups with CMS are precisely those abelian minimax groups G such that $\mathrm{r}_{p^\infty}(G) \leq 1$ for all primes p. It turns out that $\mathrm{r}_{p^\infty}(G) \leq 2$ for all soluble groups G with CMS and all primes p; and, unlike the abelian case, $\mathrm{r}_{p^\infty}(G) = 2$ may occur, even if G is nilpotent.

On a different thread, we show—in Theorem 3.1—that *there exists a nilpotent group with uncountably many subgroups, of which just countably many are abelian.*

In some contrast with these results, nothing can be said on soluble groups in which the set of subgroups has a given uncountable cardinality. Indeed, it is not difficult to prove that if κ is an uncountable cardinal, then every abelian and hence every soluble group of cardinality κ has as many subnormal subgroups as subsets, i.e., 2^κ—it is proved in [3] that if the group is supposed to be nilpotent-by-finitely generated abelian then 'subnormal' may even be replaced by 'normal' in this statement. A consequence of these remarks and of Theorem 2.7 is that if G is an infinite soluble-by-finite group then $|\mathcal{L}(G)|$ is either $\aleph_0$ or $2^{|G|}$ (see Corollary 2.8); moreover, G has $|\mathcal{L}(G)|$ subnormal subgroups. Regarding the latter (hardly surprising) remark, note that there exist countable locally soluble groups with uncountably many subgroups but only countably many subnormal subgroups. Examples of this kind are the groups constructed in [17], (2.3), in which all subnormal subgroups have finite index.

Leaving subnormality aside, Corollary 2.8 raises a question: what (non-trivial) restrictions are there (if any) on the possible cardinalities of the sets of all subgroups of infinite groups if the (generalised) solubility hypothesis is dropped? The following example is of interest in this regard. The construction in Theorem 35.2 of [9] provides a simple group G of cardinality $\aleph_1$ with exactly $\aleph_1$ subgroups; moreover, all proper subgroups of G are countable and, for any sufficiently large prime p, it may be arranged for G to be a p-group. It is also worth remarking that intermediate steps in this construction provide examples of infinite simple, two-generated groups in (CMS) not satisfying the maximal condition: in the notation of [9] all the groups $G^{\nu+1}$ where $\omega \leq \nu < \omega_1$ (or just $0 < \nu < \omega_1$ if the construction is started from, say, a Prüfer p-group) have this property. Finally we mention that a similar problem, counting conjugacy classes of subgroups, has been considered by S. Shelah [14] for uncountable cardinals (we thank Simon Thomas for drawing our attention to this paper).

2010 *Mathematics Subject Classification.* 20E07,20F16,20F19,20F99.

Key words and phrases. soluble groups; subgroups; countable sets; abelian subgroups.

2. Proofs

As remarked in the introduction, the class (CMS) is not extension-closed. Our first results provide sufficient conditions for an extension of CMS-groups to satisfy CMS.

Lemma 2.1. *Let the group G have a normal subgroup N with* CMS *such that $G/N \in \mathrm{Max}$. Then G has* CMS.

Proof. Given $H \leq G$, let $L = N \cap H$. Then L is one of the countably many subgroups of N. Also, $L \lhd H$ and $H/L \simeq HN/N$ is finitely generated. Therefore $H = L\langle F\rangle$ for some finite subset F of G. Since G is countable, there are countably many possible choices for F. It follows that G has CMS. □

Lemma 2.2. *Let the group G have a normal subgroup N such that both N and G/N have* CMS. *Then $G \notin$ (CMS) if and only if G has a section V/U, such that $U \leq N$ and $(V \cap N)/U$ has uncountably many complements in V/U.*

Proof. Assume $G \notin$ (CMS). Then uncountably many subgroups of G have the same image under the mapping $H \mapsto (H \cap N, HN/N)$ from $\mathcal{L}(G)$ to the countable set $\mathcal{L}(N) \times \mathcal{L}(G/N)$. In other words, there exist subgroups $U, W \leq G$ such that $U \leq N \leq W$ and the set $\mathcal{K}$ of all $H \leq W$ such that $H \cap N = U$ and $HN = W$ is uncountable. Now $U \lhd H$ for all $H \in \mathcal{K}$, hence $U \lhd V := \langle H \mid H \in \mathcal{K}\rangle$. For all $H \in \mathcal{K}$ we have $V/U = (V \cap N/U) \rtimes (H/U)$, thus the result follows. □

There is a well-known correspondence between complements in semidirect products and derivations (see, for instance, [7], p. 196). So, in the notation of the previous lemma, if $G \notin$ (CMS) there are a section M $(= V \cap N/U)$ of N and a group $\bar{V}$ isomorphic to a subgroup of G/N (namely $V/V \cap N$) acting on M via conjugation in such a way that $\mathrm{Der}(\bar{V}, M)$ is uncountable. Of course, in the special case when $\bar{V}$ acts trivially on M (that is, when M is a central factor in V) then $\mathrm{Der}(\bar{V}, M) = \mathrm{Hom}(\bar{V}, M)$.

We are already in a position to give a proof of the result by Rychkov and Fomin describing the abelian groups with CMS; we include it for the reader's convenience.

Lemma 2.3 ([13])**.** *Let G be an abelian group. Then the following conditions are equivalent:*

(i) *$G \in$ (CMS);*
(ii) *$|\mathcal{L}(G)| < 2^{\aleph_0}$;*
(iii) *G is minimax and, for every prime p, it has no section isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$;*
(iv) *G is minimax and, for every prime p, $\mathrm{r}_{p^\infty}(G) \leq 1$;*
(v) *G has a finitely generated subgroup H such that G/H is the direct product of finitely many, pairwise nonisomorphic Prüfer groups.*

Proof. Plainly, (i) implies (ii). Now assume (ii). For any prime p, G has no section isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$, because the latter group has $2^{\aleph_0}$ subgroups. For the same reason, as remarked in the introduction, no section of G is the direct product of infinitely many nontrivial groups, which amounts to saying that G is minimax. This shows that (ii) implies (iii). Next, by the structure of abelian minimax groups, (iii), (iv) and (v) are clearly equivalent. Finally, assume (v). Then $\mathcal{L}(G/H)$ is lattice-isomorphic to the direct product of the subgroup lattices of the primary components of G/H, hence it is countable. If X is a subgroup of G/H and M is a section of H, then $\mathrm{Hom}(X, M)$ is finite. Therefore, the remarks following Lemma 2.2 show that $G \in$ (CMS). The proof is complete. □

We shall see that conditions (i–iii) in Lemma 2.3 are still equivalent for arbitrary soluble groups, while (iv) and (v) are generally stronger than the property CMS.

Lemma 2.4. *Let G be a group and $N \leq Z(G)$. Then $G \in$ (CMS) if and only if G/N and all abelian sections of G satisfy* CMS.

Proof. Only the sufficiency needs a proof. By hypothesis, both N and G/N satisfy CMS. If $G \notin$ (CMS), then G has a section V/U as described in Lemma 2.2. Let K/U be a complement to $W := (V \cap N)/U$ in V/U. Then $V' \leq K$, because $N \leq Z(G)$; it follows that V/UV' is an abelian section of G in which $(N \cap V)V'/UV'$ has uncountably many complements, a contradiction. □

To prove Theorem 2.7 we shall make use of some information on derivations of nilpotent groups of finite rank. We state the relevant result in a slightly more general form than strictly needed. The lemma is most probably known, the proof is an adaptation of that of Theorem 10.3.6 in [7].

Lemma 2.5. *Let G be a nilpotent group of finite rank and M a $\mathbb{Z}G$-module which is artinian as an abelian group. If $H^0(G, M)$ is finite, then $H^n(G, M)$ is finite for all $n \in \mathbb{N}$.*

Proof. By [7], 10.3.7, saying that $H^0(G, M)$ $(\simeq M^G = C_M(G))$ is finite is equivalent to saying that $M_G = M/[M, G]$ is finite. Therefore, if N is a G-submodule of M then both N and $\overline{M} := M/N$ satisfy the hypothesis for M, as $N^G \leq M^G$ and $\overline{M}_G$ is an epimorphic image of M_G. The long exact cohomology sequence shows that, for all $n \in \mathbb{N}$, $H^n(G, M)$ is finite if both $H^n(G, N)$ and $H^n(G, \overline{M})$ are finite. Thus, to show that the result holds for M it is enough to prove it for N and M/N. Now, Lemma 4.4 of [11] shows that the result is certainly true for finite modules. Then, arguing by induction on the total rank of M as an abelian group, we may assume that M is $\mathbb{Z}$-divisible and has no proper infinite submodules.

Let X be a subgroup of G such that $H^n(X, M)$ is finite for all $n \in \mathbb{N}$. If $X \lhd Y \leq G$ then the Lyndon-Hochschild-Serre spectral sequence (see [7], p. 204), together with [11], Lemma 4.4 again, shows that $H^n(Y, M)$ is finite for all $n \in \mathbb{N}$. Arguing by induction on the subnormal defect of X in G we obtain the required result. Therefore, we may assume that no subgroup of G satisfies this property for X. Now, since M satisfies the minimal condition on subgroups, it is easily seen that $C_M(G) = C_M(X)$ for some finitely generated subgroup of G. Therefore we may substitute X for G and assume that G is finitely generated.

If G is periodic, it is finite and the result is obvious. We may now assume that there exists a non-periodic element $x \in Z(G)$. Let $N = C_M(x)$. Then N is a G-submodule of M, therefore either N is finite or $N = M$. In the former case $H^0(\langle x\rangle, M) \simeq N$ is finite, $H^1(\langle x\rangle, M) \simeq M/[M, x] = 0$ (see [12], exercise 11.3.2, p. 340) and $H^n(\langle x\rangle, M) = 0$ for all $n > 1$; this contradicts one of the assumptions in the previous paragraph. Otherwise, $N = M$ and M is a $\overline{G}$-module, where $\overline{G} = G/\langle x\rangle$. Also, for all $i \in \mathbb{N}$, $H^i(\langle x\rangle, M)$ is $\mathbb{Z}G$-isomorphic to M (if $i < 2$) or to 0 (if $i \geq 2$). Thus the hypotheses of the lemma are satisfied by $H^i(\langle x\rangle, M)$ in place of M and $\overline{G}$ in place of G. By induction on $\mathrm{r}_0(G)$, we conclude that all cohomology groups $H^j(\overline{G}, H^i(\langle x\rangle, M))$ are finite. Now the Lyndon-Hochschild-Serre spectral sequence shows that $H^n(G, M)$ is finite for all $n \in \mathbb{N}$. The lemma is proved. □

Corollary 2.6. *Let p be a prime and G a nilpotent group of finite rank acting on the group $P \simeq \mathcal{C}_{p^\infty}$ non-trivially. Then $H^n(G, P)$ is finite for all $n \in \mathbb{N}$, and $\mathrm{Der}(G, P)$ is countable.*

Proof. We have $C_P(G) < P$, hence $H^0(G, P) \simeq C_P(G)$ is finite. The first part of the corollary follows now from Lemma 2.5. $\mathrm{Der}(G, P)$ is an extension of the group of inner derivations $\mathrm{IDer}(G, P)$, which is isomorphic to $P/C_P(G) \simeq P$, by $H^1(G, P)$, hence $|\mathrm{Der}(G, P)| = \aleph_0$. □

We remark that it can be shown, by means of elementary, direct methods, that with the hypothesis of Corollary 2.6 every derivation $G \to P$ has finite image. Also note that if the hypothesis that G has finite rank is dropped in Lemma 2.5, then it is still true that the cohomology groups $H^n(G, M)$ have finite exponent ([7], Theorem 10.3.6) but they can have infinite rank, even if M is finite. Indeed, $\mathrm{Hom}(G, C_M(G))$ embeds in $\mathrm{Der}(G, M)$, so it is easy to arrange for $H^1(G, M)$ to be uncountable.

Theorem 2.7. *Let G be a soluble-by-finite group. Then the following are equivalent conditions:*

(i) $G \in$ (CMS);
(ii) $|\mathcal{L}(G)| < 2^{\aleph_0}$;
(iii) *G is a minimax group and, for all primes p, G has no section isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$;*
(iv) *G is a minimax group and, for all primes p, G has no subnormal section isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$.*

Proof. Assume $|\mathcal{L}(G)| < 2^{\aleph_0}$. By Lemma 2.3 all abelian sections of G are minimax—hence G is minimax—and none of them is isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$, for any prime p. This proves that (ii) implies (iii). That (i) and (iii) imply (ii) and (iv) respectively is obvious; to complete the proof we need to prove that (iv) implies (i). Assume that G is minimax and no subnormal section of G is isomorphic to the direct square of a Prüfer group. By [10], Theorem 10.33, if R is the finite residual of G and F/R is the Fitting subgroup of G/R, then R is divisible abelian, F/R is nilpotent and G/F is polycyclic-by-finite. In view of Lemma 2.1, it will be enough to prove that F has CMS. The hypothesis and Lemma 2.3 imply that all subnormal abelian sections of G, and hence all abelian section of F/R, satisfy CMS. Then an easy induction on the nilpotency class of F/R, based on Lemma 2.4, shows that $F/R \in$ (CMS).

Suppose that $F \notin$ (CMS). Then, by Lemma 2.2, F has a section V/U, where $U \leq R$, in which $(V \cap R)/U$ has uncountably many complements. This property is preserved if we replace V with VR (note that $U \lhd VR$), thus we may assume $R \leq V$. As $V \leq F$, this makes V subnormal in G. Using bars to denote images modulo U, fix K such that $\bar{V} = \bar{R} \rtimes \bar{K}$. Then $D := \mathrm{Der}(\bar{K}, \bar{R})$ is uncountable. Next, $\bar{R}$ is divisible; the hypothesis implies that it is the direct product of finitely many, pairwise nonisomorphic Prüfer groups. Thus $\bar{R} = \mathrm{Dr}_{p\in\pi} \bar{R}_p$, where π is a finite set of primes and $\bar{R}_p \simeq \mathcal{C}_{p^\infty}$ for all $p \in \pi$. Then $D \simeq \mathrm{Dr}_{p\in\pi} \mathrm{Der}(\bar{K}, \bar{R}_p)$, and $D_p := \mathrm{Der}(\bar{K}, \bar{R}_p)$ is uncountable for some $p \in \pi$. By Corollary 2.6, $\bar{K}$ acts trivially on $\bar{R}_p$. But then $D_p = \mathrm{Hom}(\bar{K}, \bar{R}_p)$. Since this group is uncountable, there exists $L \lhd K$ such that $U \leq L$ and $K/L \simeq \mathcal{C}_{p^\infty}$. If Q/U is the p'-component of $\bar{R}$ then V/LQ is a subnormal section of G isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$. This is a contradiction, which completes the proof. □

As a consequence of Theorem 2.7 and Lemma 2.3, *a soluble-by-finite group satisfies* CMS *if and only if all of its (subnormal) abelian sections satisfy* CMS.

Corollary 2.8. *Let G be an infinite soluble-by-finite group. Then the set of all subnormal subgroups of G has the same cardinality as $\mathcal{L}(G)$, and this cardinality is either $\aleph_0$ or $2^{|G|}$.*

Proof. If $|G| > \aleph_0$ the result is contained in [3]. Assume $|G| = \aleph_0$. It is clear that G has infinitely many subnormal subgroups. By Theorem 2.7, either $G \in$ (CMS) or G has a subnormal abelian section not in (CMS), and hence $2^{\aleph_0}$ subnormal subgroups. □

More information is readily available on soluble (or soluble-by-finite) groups with CMS. If G is such a group, let R and F be as in the proof of Theorem 2.7. If P is a nontrivial primary component of R, hence a Prüfer group, then either $P = [P, F]$ or $P \leq Z(F)$. Let S be the direct product of the primary components of R which are not central in F, thus $S = [S, F] = [R, F]$. It follows from [7], 10.1.15 and 10.3.6 that F nearly splits over S, so there exists a nilpotent subgroup $H \leq F$ such that $H \cap S$ is finite and SH has finite index in F. (Although we don't need this further observation, we remark that a simple argument involving [7], 10.3.9, together with the fact that S is divisible and locally cyclic shows that we may even choose H such that $SH = F$.)

Proposition 2.9. *Let G be a soluble-by-finite group with* CMS*. Then $\mathrm{r}_{p^\infty}(G) \leq 2$ for all primes p.*

Proof. Fix a prime p. Assume that we have proved our result for nilpotent groups. In the notation of the previous paragraph $\mathrm{r}_{p^\infty}(G) = \mathrm{r}_{p^\infty}(F) = \mathrm{r}_{p^\infty}(S) + \mathrm{r}_{p^\infty}(H)$; by our assumption $\mathrm{r}_{p^\infty}(H) \leq 2$, so we may assume that the p-component P of S is nontrivial, that is, $P \simeq \mathcal{C}_{p^\infty}$. As $[P, H'] = 1$ and G has no sections of type $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$, it is clear that H' has no section isomorphic to $\mathcal{C}_{p^\infty}$. Therefore $\mathrm{r}_{p^\infty}(H) = \mathrm{r}_{p^\infty}(H/H') \leq 1$ by Lemma 2.3, hence $\mathrm{r}_{p^\infty}(G) \leq 2$.

Therefore we may assume that G is nilpotent. We may also assume $m(G)$ minimal subject to G being a counterexample, where $m(G) = \mathrm{r}_0(G) + \sum_{q \text{ prime}} \mathrm{r}_{q^\infty}(G)$. Let p be a prime such that $\mathrm{r}_{p^\infty}(G) > 1$. The minimality of $m(G)$ implies that no subgroup of $Z(G)$ is infinite cyclic, and that G/G' has no quotient, and hence no section, isomorphic to $\mathcal{C}_{q^\infty}$ for any prime $q \neq p$. This latter property is inherited by finite tensor products of abelian minimax groups, so G has no section isomorphic to $\mathcal{C}_{q^\infty}$. Since $Z(G)$ is infinite, it has a subgroup isomorphic to $\mathcal{C}_{p^\infty}$. Let A be a maximal abelian normal subgroup of G and $C = C_G(A/Z(G))$. Now $\mathrm{r}_{p^\infty}(A) \leq 1$, because of Lemma 2.3, hence $A/Z(G)$ is finitely generated and so also G/C is finitely generated. Finally, C stabilizes the series $1 < Z(G) < A$, hence $C' \leq C_G(A) = A$ and $\mathrm{r}_{p^\infty}(C/A) \leq 1$, by Lemma 2.3 again. The result follows. □

In the next section we shall construct nilpotent groups G of class 2 with CMS such that $\mathrm{r}_{p^\infty}(G) = 2$ for some prime p. Clearly, soluble minimax groups with $\mathrm{r}_{p^\infty}(G) \leq 2$ for all p may fail to satisfy CMS (abelian groups providing easy counterexamples); but we record here an obvious consequence of Theorem 2.7:

Corollary 2.10. *Let G be a soluble-by-finite minimax group such that $\mathrm{r}_{p^\infty}(G) \leq 1$ for all primes p. Then $G \in$ (CMS).*

Next we look at the effect of CMS on more general classes of groups. For several classes of generalised soluble-or-finite group, CMS implies virtual solubility. For example, if G is a radical group with CMS, then every abelian section of G is minimax (Lemma 2.3) and so G is minimax and hence soluble-by-finite, by a theorem of Baer [1]. The same holds for the case G locally finite, by [16], so *a locally finite group satisfies* CMS *if and only if it is a Chernikov group with locally cyclic finite residual.* An extra argument allows us to extend further these results to wider group classes for which an analogue of Baer's theorem does not hold (Merzljakov [8] provides an example of insoluble locally soluble group in which all abelian sections are finitely generated).

Lemma 2.11. *Let $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ be a sequence of varieties, such that $\mathfrak{X}_n \subseteq \mathfrak{X}_{n+1}$ for all $n \in \mathbb{N}$, and let $\mathfrak{X} = \bigcup_{n\in\mathbb{N}} \mathfrak{X}_n$. Let G be a nontrivial locally-$\mathfrak{X}$ group, and assume that G has a chain $\mathcal{N}$ of normal subgroups such that $\bigcap \mathcal{N} = 1$. If no element of $\mathcal{N}$ is in $\mathfrak{X}$, then G has at least $2^{\aleph_0}$ subgroups.*

Proof. For all $K \in \mathfrak{X}$, let $\ell(K)$ (the length of K) be the least $n \in \mathbb{N}$ such that $K \in \mathfrak{X}_n$. Suppose that no element of $\mathcal{N}$ is in $\mathfrak{X}$. Then we can recursively construct a strictly decreasing sequence $(N_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{N}$ and, for all $i \in \mathbb{N}$, a finitely generated subgroup F_i of N_i in such a way that, after letting $H_i = \langle F_0, F_1, \dots, F_i\rangle$, we have $\ell_i := \ell(H_i) = \ell(H_iN_{i+1}/N_{i+1}) < \ell(F_{i+1})$ for all i. For, we start with any element of $\mathcal{N}$ as N_0 and $F_0 = \langle x\rangle$, for some nontrivial $x \in N_0$. Let $i \in \mathbb{N}$ and assume that the required subgroups have been constructed up to i. If $\ell_i = \ell(H_i)$, then there is some $N \in \mathcal{N}$ such that $N < N_i$ and $H_i/(H_i \cap N) \notin \mathfrak{X}_{\ell_i-1}$, because $\mathfrak{X}_{\ell_i-1}$ is a variety, hence $\ell(H_iN/N) = \ell_i$; we choose one such N as N_{i+1}. As $N \notin \mathfrak{X}_{\ell_i}$, we can then choose F_{i+1} as a finitely generated subgroup of N not in $\mathfrak{X}_{\ell_i}$. It is easy to check that the required conditions are satisfied by the sequences thus defined.

Now, for all $S \subseteq \mathbb{N}$ we let $F(S) = \langle F_i \mid i \in S\rangle$. If $T \subseteq \mathbb{N}$ and $T \neq S$, we claim that $F(S) \neq F(T)$. For, let j be the least integer in the symmetric difference $T \vartriangle S$ and let bars denote images modulo N_{j+1}. Without loss of generality, assume $j \in T$. Then (with the obvious proviso for the case $j = 0$), $\overline{F(S)} \leq \overline{H_{j-1}}$ and $\ell(\overline{H_{j-1}}) = \ell_{j-1} < \ell_j$, while $\langle \overline{F(T)}, \overline{H_{j-1}}\rangle = \overline{H_j}$ has length ℓ_j. This shows that $F(S) \neq F(T)$, as claimed. It follows that G has at least $2^{\aleph_0}$ subgroups, as required. □

Theorem 2.12. *Let G be a locally (soluble-by-finite) group. If $|\mathcal{L}(G)| < 2^{\aleph_0}$, then G is soluble-by-finite.*

Proof. Assume $|\mathcal{L}(G)| < 2^{\aleph_0}$. We start by taking on the case when G is locally soluble. For all $n \in \mathbb{N}^+$ we let $\mathfrak{X}_n$ be the variety of soluble groups of length at most n. Also let $\mathcal{N} = \{G^{(n)} \mid n \in \mathbb{N}^+\}$, so that $S := \bigcap \mathcal{N}$ is the soluble residual of G. Lemma 2.11 shows that G/S is soluble, so $S' = S$. Hence we may replace G with S and, arguing by contradiction, assume that G is perfect and non-trivial. After factoring out $Z(G) = \bar{Z}(G)$ we may also assume $Z(G) = 1$. We claim that every soluble normal section F of G is central. Since $G = G'$ it is enough to show that G acts nilpotently on F. Recall from Theorem 2.7 that F is minimax. For all $n \in \mathbb{N}^+$, F/F^n is finite, so $G/C_G(F/F^n)$ is finite, hence trivial because G is perfect. Thus G acts trivially on F/R, where R is the finite residual of F. Now, R is divisible abelian, by the already cited Theorem 10.33 of [10], so R is locally cyclic by Lemma 2.3. Then $G/C_G(R)$

is abelian, that is, $[R, G] = 1$. It follows that $[F, G] = 1$, as claimed. As a consequence, G has no minimal normal subgroups. Thus, by Zorn's Lemma, there exists a chain $\mathcal{M}$ of nontrivial normal subgroups of G such that $\bigcap \mathcal{M} = 1$. But then, by Lemma 2.11, some $F \in \mathcal{M}$ is soluble and hence central in G, a contradiction. We have proved that all locally soluble groups with less than $2^{\aleph_0}$ subgroups are soluble.

Now let G be a locally (soluble-by-finite) group with the same property. By the first part of the proof, all locally soluble subgroups of G are soluble, hence of finite rank by Theorem 2.7. Then the main theorem in [5] (also see [6]) shows that G has a locally soluble subgroup of finite index. The result follows. □

We close this section with another consequence of Theorem 2.7, which is of some relevance in view of the forthcoming Theorem 3.1.

Corollary 2.13. *Suppose that all nilpotent subgroups of the soluble-by-finite group G satisfy* CMS*. Then $G \in$ (CMS).*

Proof. G is minimax, because all abelian subgroups of G satisfy CMS. Let R be the finite residual of G and F/R be the Fitting subgroup of G/R. Suppose $G \notin$ (CMS), then $F \notin$ (CMS), so F has a section $X/Y \simeq \mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ for some prime p. Then $X \notin$ (CMS) and we may well assume $G = X$. The near-splitting argument in the paragraph following Corollary 2.8 still applies. So there exist $H \leq G$ and $S \triangleleft G$ such that $H \cap S$ and $|G : SH|$ are finite, H is nilpotent and $S = [S, G] \leq G' \leq Y$. But this implies that $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ is an epimorphic image of H, so $H \notin$ (CMS), a contradiction. □

3. Counterexamples

It is fairly common that a group-theoretical property must hold in a soluble group if it is satisfied by all of its abelian subgroups; this is most often the case for finiteness conditions. The property CMS is not of this kind. In the previous section we noticed that soluble groups in which all abelian sections satisfy CMS must themselves satisfy CMS; this sentence becomes false if we substitute 'subgroups' for 'sections'. Indeed, there exists a *nilpotent* group in which all abelian subgroups satisfy CMS, and yet the group itself does not. Slightly more than that:

Theorem 3.1. *For all primes p and all integers $n > 2$ there exists a nilpotent p-minimax group G of class 2 and torsion-free rank n without* CMS *but with just countably many abelian subgroups.*

The group that we shall construct to justify Theorem 3.1 also turns out to have subgroups proving:

Proposition 3.2. *For all primes p and all integers $n > 1$ there exists a nilpotent p-minimax group G of class 2 satisfying* CMS *such that $\mathrm{r}_{p^\infty}(G) = 2$ and $\mathrm{r}_0(G) = n$.*

Recall that a p-minimax soluble group is a soluble minimax group G with no section isomorphic to $\mathcal{C}_{q^\infty}$ for any prime $q \neq p$. This section mostly consists in the proof of these two results. The construction is an expansion of one carried out in [4], Example 3.1.

We start by constructing the groups that will be the abelianisations of the groups in Theorem 3.1. We fix some notation. Let r be a positive integer and β_{ji} and γ_{ji} integers, where j and i range in $\{1, 2, \ldots, r\}$ and $\mathbb{N}$ respectively. Let A be the abelian group generated by elements $a_1, a_2, \ldots, a_r$ and b_i, c_i, where i ranges in $\mathbb{N}$, subject to the relations

$$\forall i \in \mathbb{N} \qquad pb_{i+1} = b_i + \sum_{j=1}^{r} \beta_{ji} a_j, \qquad pc_{i+1} = c_i + \sum_{j=1}^{r} \gamma_{ji} a_j. \tag{$\mathcal{R}$}$$

It is clear that the subgroup $U = \langle a_1, a_2, \ldots, a_r \rangle$ of A is free abelian of rank r and $A/U \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$ (as usual, $\mathbb{Q}_p$ is the subgroup $\{np^m \mid n, m \in \mathbb{Z}\}$ of the rational group); thus A has rank $r + 2$.

For all $j \in \{1, 2, \ldots, r\}$, define the p-adic integers $\beta_j = \sum_{i \in \mathbb{N}} \beta_{ji} p^i$ and $\gamma_j = \sum_{i \in \mathbb{N}} \gamma_{ji} p^i$. We shall show that choosing these p-adic integers algebraically independent forces infinitely generated subgroups of A to have rank greater than r. We start from a simple case, which is exactly what is needed to prove the main results of this section restricted to the case $n = 3$.

Lemma 3.3. *Assume that $r = 1$ and $\{1, \beta_1, \gamma_1\}$ is a linearly independent set (over $\mathbb{Z}$) of cardinality 3. Then all subgroups of rank 1 in A are cyclic.*

Proof. Since A is p-minimax a subgroup of rank 1 in A must be either cyclic or isomorphic to $\mathbb{Q}_p$. Assume that the latter case occurs. Then there exists a sequence $(x_i)_{i \in \mathbb{N}}$ of nonzero elements of A such that $px_{i+1} = x_i$ for all $i \in \mathbb{N}$. Now $x_0 = \lambda b_{1n} + \mu c_{1n} + \nu a_1$ for some positive integer n and $\lambda, \mu, \nu \in \mathbb{Z}$. Elements of $A/U = A/\langle a_1 \rangle$ are uniquely divisible by p; it follows that for all $i \in \mathbb{N}$ there exists $\nu_i \in \mathbb{Z}$ such that $x_i = \lambda b_{1,n+i} + \mu c_{1,n+i} + \nu_i a_1$. Then

$$x_i = px_{i+1} = \lambda pb_{1,n+i+1} + \mu pc_{1,n+i+1} + p\nu_{i+1} a_1 = \lambda b_{1,n+i} + \mu c_{1,n+i} + (\lambda \beta_{1,n+i} + \mu \gamma_{1,n+i} + p\nu_{i+1}) a_1$$

and so

$$\nu_i - p\nu_{i+1} = \lambda \beta_{1,n+i} + \mu \gamma_{1,n+i}$$

for all $i \in \mathbb{N}$. Then $\nu = \nu_0 = \sum_{i\in\mathbb{N}}(\nu_i - p\nu_{i+1})p^i = \lambda\sum_{i\in\mathbb{N}}\beta_{1,n+i}\,p^i + \mu\sum_{i\in\mathbb{N}}\gamma_{1,n+i}\,p^i$, hence

$$p^n\nu = \lambda\Big(\beta_1 - \sum_{i=0}^{n-1}\beta_{1i}\,p^i\Big) + \mu\Big(\gamma_1 - \sum_{i=0}^{n-1}\gamma_{1i}\,p^i\Big).$$

Since 1, β_1 and γ_1 are linearly independent this yields $\lambda = \mu = 0 = \nu$, that is, $x_0 = 0$, a contradiction. □

We mention that the converse of Lemma 3.3 also holds (if 1, β_1 and γ_1 are linearly dependent then A has a subgroup isomorphic to $\mathbb{Q}_p$), although we will not make use of this observation.

Lemma 3.4. *Assume that $\{\beta_1, \beta_2, \dots, \beta_r, \gamma_1, \gamma_2, \dots, \gamma_r\}$ is algebraically independent (over $\mathbb{Z}$) and has cardinality $2r$. Then all subgroups of rank r in A are finitely generated.*

Proof. To start with, we claim that if V is a proper, pure subgroup of U the group A/V still satisfies the hypothesis of the lemma, for suitably defined parameters. For, denote by bars images modulo V. Then $\bar{U}$ is freely generated by elements $u_1, u_2, \dots, u_s$, say, and $\bar{a}_j = \sum_{k=1}^{s} t_{jk}u_k$ for all j, where (t_{jk}) is an integer matrix of type $r\times s$ and rank s. At the expense of permuting the subscripts j in $\{1, 2, \dots, r\}$ and suitably redefining the basis elements u_i, we may assume that the first s rows of (t_{jr}) form a lower triangular matrix of rank s. For all $i \in \mathbb{N}$

$$p\bar{b}_{i+1} = \bar{b}_i + \sum_{j=1}^{r}\beta_{ji}\bar{a}_j = \bar{b}_i + \sum_{j=1}^{r}\beta_{ji}\Big(\sum_{k=1}^{s}t_{jk}u_k\Big) = \bar{b}_i + \sum_{k=1}^{s}\sum_{j=1}^{r}\beta_{ji}t_{jk}u_k = \bar{b}_i + \sum_{k=1}^{s}\beta^*_{ki}u_k,$$

having set $\beta^*_{ki} = \sum_{j=1}^{r}\beta_{ji}t_{jk}$. For all $k \in \{1, 2, \dots, s\}$ let $\beta^*_k = \sum_{i\in\mathbb{N}}\beta^*_{ki}p^i$; then $\beta^*_k = \sum_{j=1}^{r}t_{jk}\beta_j$. In a similar fashion we define the integers $\gamma^*_{ki} = \sum_{j=1}^{r}\gamma_{ji}t_{jk}$ and the p-adic integers $\gamma^*_k = \sum_{i\in\mathbb{N}}\gamma^*_{ki}p^i$ in such a way that $p\bar{c}_{i+1} = \bar{c}_i + \sum_{k=1}^{s}\gamma^*_{ki}u_k$ for all $i \in \mathbb{N}$. We have to prove that $\{\beta^*_1, \beta^*_2, \dots, \beta^*_s, \gamma^*_1, \gamma^*_2, \dots, \gamma^*_s\}$ is an algebraically independent set (over $\mathbb{Z}$, or equivalently $\mathbb{Q}$) of cardinality $2s$. To this end, assume the equality $\sum c_{\mathbf{uv}}\prod_{k=1}^{s}\beta^{*u_k}_k\gamma^{*v_k}_k = 0$, where the sum is intended to be taken over a finite set of pairs $(\mathbf{u}, \mathbf{v})$, each $\mathbf{u} = (u_1, \dots, u_s)$ and $\mathbf{v} = (v_1, \dots, v_s)$ being an s-tuple of non-negative integers, and the $c_{\mathbf{uv}}$ are integers. We have to check that all $c_{\mathbf{uv}}$ are zero. We have

$$0 = \sum c_{\mathbf{uv}}\prod_{k=1}^{s}\beta^{*u_k}_k\gamma^{*v_k}_k = \sum c_{\mathbf{uv}}\prod_{k=1}^{s}\Big(\sum_{j=1}^{r}t_{jk}\beta_j\Big)^{u_k}\Big(\sum_{j=1}^{r}t_{jk}\gamma_j\Big)^{v_k}. \tag{1}$$

Arguing by contradiction, assume that $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ is maximal subject to $c_{\tilde{\mathbf{u}}\tilde{\mathbf{v}}} \neq 0$, with respect to lexicographic order. Look at the occurrences of "monomials" $\lambda\prod_{j=1}^{s}\beta_j^{\tilde{u}_j}\gamma_j^{\tilde{v}_j}$ (with $\lambda \in \mathbb{Z}$) in the formal expansion of the right-hand side of (1). Assume that $(\mathbf{u}, \mathbf{v})$ is such that this monomial occurs in $c_{\mathbf{uv}}\prod_{k=1}^{s}(\sum_{j=1}^{r}t_{jk}\beta_j)^{u_k}(\sum_{j=1}^{r}t_{jk}\gamma_j)^{v_k}$ and $c_{\mathbf{uv}} \neq 0$. Recalling that $t_{jk} = 0$ if $j < k \le s$ we observe that β_1 does not explicitly appear (with nonzero coefficient) beyond the first factor, and this shows that $u_1 \ge \tilde{u}_1$. By the maximality of $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ it follows that $u_1 = \tilde{u}_1$. This also means that the first factor does not give any contribution to our monomial except for the power of β_1. Then $\beta_2^{\tilde{u}_2}$ must occur in $\prod_{k=2}^{s}(\sum_{j=1}^{r}t_{jk}\beta_j)^{u_k}$. As in the previous case, we deduce that $u_2 \ge \tilde{u}_2$, and hence $u_2 = \tilde{u}_2$ by maximality. By iterating this argument, we prove that $u_j = \tilde{u}_j$ for all $j \le s$, hence $\mathbf{u} = \tilde{\mathbf{u}}$. The same argument shows that $\mathbf{v} = \tilde{\mathbf{v}}$. Therefore our monomial occurs in the expansion of (1) exactly once, with coefficient $\lambda = c_{\tilde{\mathbf{u}}\tilde{\mathbf{v}}}\prod_{j=1}^{s}t_{jj}^{\tilde{u}_j+\tilde{v}_j}$. Since all t_{jj} are nonzero, the algebraic independence of $\{\beta_1, \beta_2, \dots, \beta_r, \gamma_1, \gamma_2, \dots, \gamma_r\}$ shows that $c_{\tilde{\mathbf{u}}\tilde{\mathbf{v}}} = 0$, a contradiction. Now our claim (in the first line of our proof) is justified; it shows that when needed in the proof we can factor out from A proper, pure subgroups of U.

Another reduction argument is as follows. Let $n, m \in \mathbb{N}$ and $\mu, \lambda \in \mathbb{Z} \smallsetminus \{0\}$. For all $i \in \mathbb{N}$, let $b^*_i = \lambda b_{i+n}$ and $c^*_i = \mu c_{i+m}$, and for all $j \in \{1, 2, \dots, r\}$ let $\beta^*_{ji} = \lambda\beta_{j,i+n}$ and $\gamma^*_{ji} = \mu\gamma_{j,i+n}$. Then $pb^*_{i+1} = b^*_i + \sum_{j=1}^{r}\beta^*_{ji}a_j$ and $pc^*_{i+1} = c^*_i + \sum_{j=1}^{r}\gamma^*_{ji}a_j$ for all $i \in \mathbb{N}$. Also, for all j, $\beta^*_j := \sum_{i\in\mathbb{N}}\beta^*_{ji}p^i = (\lambda/p^n)\left(\beta_j - \sum_{t=0}^{n-1}\beta_{jt}p^t\right)$ and, similarly, $\gamma^*_j := \sum_{i\in\mathbb{N}}\gamma^*_{ji}p^i = (\mu/p^m)\left(\gamma_j - \sum_{t=0}^{m-1}\gamma_{jt}p^t\right)$. It follows that $\{\beta^*_1, \beta^*_2, \dots, \beta^*_r, \gamma^*_1, \gamma^*_2, \dots, \gamma^*_r\}$ is algebraically independent of cardinality $2r$. Therefore we may replace A with $A^* = U + \langle b^*_i, c^*_i \mid i \in \mathbb{N}\rangle$, obtaining a group in which λb_n and μc_m play the roles of b_0 and c_0—note that A/A^* is finite. Furthermore, if $u = \sum_{j=1}^{r}u_ja_j, v = \sum_{j=1}^{r}v_ja_j \in U$ then we may replace b_0 with $b_0 + u$ and c_0 with $c_0 + v$ leaving the remaining b_i and c_i unaltered. The only change needed is redefining β_{j0} as $\beta_{j0} - u_j$ and γ_{j0} with $\gamma_{j0} - v_j$ for each j, and hence β_j and γ_j as $\beta_j - u_j$ and $\gamma_j - v_j$ respectively.

Now let H be a subgroup of A which is not finitely generated. We have to show that $\mathrm{r}(H) > r$ or, equivalently, $\mathrm{r}_0(A/H) \le 1$. There is no loss in replacing H with its pure closure in A, so we shall assume that H is pure. It is obvious that $\mathrm{r}_0(A/H) \le 1$ if $U \le H$, so we may exclude this case. In view of the previous paragraphs, we may factor out $H \cap U$, so we assume $H \cap U = 0$. Then H is isomorphic to a subgroup of $A/U \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$. Let X be a subgroup of rank 1 in H. Choose any $V \le U$ such that $U/V \simeq \mathbb{Z}$; then A/V satisfies the hypotheses of Lemma 3.3 and so $X \simeq (X+V)/V$ is cyclic. Thus all rank-1 subgroups of H are cyclic. We deduce that $\mathrm{r}(H) = 2$; more precisely H is an extension of a cyclic group by $\mathbb{Q}_p$. It also follows that $A/(U+H)$ is periodic. Now we have $\mathrm{r}_0(A/H) = \mathrm{r}(U) = r$, so we only have to show that $r = 1$.

Since $A/(U+H)$ is periodic there exist $\lambda \in \mathbb{Z}$ and $u \in H$ such that $\lambda \neq 0$ and $x := \lambda c_0 + u \in H$. Now let X be the pure closure of $\langle x \rangle$ in H. Write $|X : \langle x \rangle|$ as $p^n t$ where $n, t \in \mathbb{N}$ and p does not divide t. Then there exists $h \in X$ such that $x = h^{p^n}$ and $\langle h \rangle$ is p-pure in H and hence in A. By uniqueness of division by p^n in A/U, we have $h \equiv_U \lambda c_n$. Therefore $h = \lambda c_n + u'$ for some $u' \in U$. At the expense of substituting h for c_0 (see the reduction arguments above), we may assume that $\langle c_0 \rangle$ is p-pure in A and contained in H; the same applies for $\langle b_0 \rangle$.

Thus the elements of $H/\langle c_0 \rangle$ are uniquely divisible by p. Therefore there exists a sequence $(h_i)_{i\in\mathbb{N}}$ of elements of H (uniquely defined modulo $\langle c_0 \rangle$) such that $h_0 = b_0$ and $ph_{i+1} \equiv_{\langle c_0 \rangle} h_i$ for all $i \in \mathbb{N}$. Now, for all $i \in \mathbb{N}$ we have $p^i(h_i - b_i) \in U + \langle c_0 \rangle$, and the set of all elements of A of order at most p^i modulo $U + \langle c_0 \rangle$ is $U + \langle c_i \rangle$. It follows that, for all $i \in \mathbb{N}$, we have:

$$h_i = b_i + \mu_i c_i + u_i; \qquad\qquad ph_{i+1} = h_i + \tau_i c_o;$$

for some integers μ_i, τ_i and $u_i \in U$. Expanding the equation on the right yields:

$$p(b_{i+1} + \mu_{i+1} c_{i+1} + u_{i+1}) = b_i + \mu_i c_i + u_i + \tau_i c_o$$

and hence

$$(\mu_{i+1} - \mu_i)c_i + \sum_{j=1}^{r} (\beta_{ji} + \mu_{i+1}\gamma_{ji} + pu_{j,i+1} - u_{ji})a_j = \tau_i c_o, \tag{2}$$

where we let $u_i = \sum_{j=1}^{r} u_{ji} a_j$ for all i. Projecting this equality modulo U gives $(\mu_{i+1} - \mu_i)c_i \equiv_U \tau_i p^i c_i$, that is, $\mu_{i+1} - \mu_i = \tau_i p^i$ for all $i \in \mathbb{N}$. Now, $\mu_0 = 0$, as $h_0 = b_0$, hence

$$\mu_{i+1} = \sum_{t=0}^{i} \tau_t p^t.$$

Going back to (2) we now have $(\mu_{i+1} - \mu_i)c_i = \tau_i p^i c_i = \tau_i(c_0 + \sum_{j=1}^{r} \sum_{t=0}^{i-1} \gamma_{jt} p^t a_j)$. Then, for each i and j, after letting $\hat{\gamma}_{ji} = \sum_{t=0}^{i-1} \gamma_{jt} p^t$ we have:

$$\tau_i \hat{\gamma}_{ji} + \beta_{ji} + \mu_{i+1}\gamma_{ji} + pu_{j,i+1} - u_{ji} = 0. \tag{3}$$

Now,

$$\sum_{i\in\mathbb{N}} \mu_{i+1}\gamma_{ji}\, p^i = \sum_{i\in\mathbb{N}} \left(\sum_{t=0}^{i} \tau_t p^t \right) \gamma_{ji}\, p^i = \sum_{t,i\in\mathbb{N};\, t\le i} \tau_t p^t \gamma_{ji}\, p^i = \sum_{t\in\mathbb{N}} \tau_t(\gamma_j - \hat{\gamma}_{jt})p^t,$$

hence, by multiplying the left-hand side of (3) by p^i and summing over $\mathbb{N}$ we obtain:

$$0 = \sum_{i\in\mathbb{N}} \tau_i \hat{\gamma}_{ji} p^i + \beta_j + \sum_{i\in\mathbb{N}} \mu_{i+1}\gamma_{ji} p^i + \sum_{i\in\mathbb{N}} (pu_{j,i+1} - u_{ji})p^i = \tau\gamma_j + \beta_j - u_{j0}. \tag{4}$$

Assume for a contradiction that $r > 1$. From the two equations obtained from (4) for $j = 1$ and $j = 2$ we deduce:

$$(\beta_1 - u_{10})\gamma_2 = (\beta_2 - u_{20})\gamma_1,$$

which contradicts the algebraic independence of $\{\beta_1, \beta_2, \ldots, \beta_r, \gamma_1, \gamma_2, \ldots, \gamma_r\}$. This contradiction shows that $r = 1$, and ultimately that A has the required property. The proof is complete. □

Lemma 3.5. *Let p be a prime and G a direct product of a finite number $n > 1$ of copies of $\mathcal{C}_{p^\infty}$. Let $\mathcal{P}$ be a countable set of subgroups of G isomorphic to $\mathcal{C}_{p^\infty}$ and let X be a set of elements of the socle G which are linearly independent over $\mathbb{Z}_p$. Then G has $2^{\aleph_0}$ subgroups K such that $K \cap X = \varnothing$ and $G = PK$ for all $P \in \mathcal{P}$.*

Proof. The proof is by induction on n. If X has at least two distinct elements, a and b, then let $y = a^{-1}b$, otherwise we define y as an element in the socle of G not in $\langle X \rangle$. In either case G has $2^{\aleph_0}$ subgroups $Y \simeq \mathcal{C}_{p^\infty}$ such that $y \in Y$, and hence $Y \cap X = \varnothing$. Let $\mathcal{Y}$ be the set of all such Y which are not in $\mathcal{P}$, hence $|\mathcal{Y}| = 2^{\aleph_0}$. If $n = 2$ then all elements of $\mathcal{Y}$ satisfy the requirement for K, so we may assume $n > 2$. Fix one $Y \in \mathcal{Y}$. Use bars to denote images modulo Y. Then $\bar{X}$ is a linearly independent subset of the socle of $\bar{G}$ (with $|\bar{X}| = |X| - 1$ if $|X| > 1$). Also, $\bar{P} \simeq \mathcal{C}_{p^\infty}$ for all $P \in \mathcal{P}$. Therefore, by induction hypothesis, there are $2^{\aleph_0}$ subgroups K/Y of $\bar{G}$ such that $\bar{K} \cap \bar{X} = \varnothing$ and $\bar{K}\bar{P} = \bar{G}$ for all $P \in \mathcal{P}$. For each such K we have $K \cap X = \varnothing$; moreover $G/K \simeq \mathcal{C}_{p^\infty}$ and, for all $P \in \mathcal{P}$, $P \nleq K$ and hence $PK = G$. The proof is now complete. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let A be a torsion-free abelian group of rank n which is an extension of a free abelian group of rank $r := n - 2$ by $\mathbb{Q}_p \oplus \mathbb{Q}_p$ and such that all subgroups of rank $n - 2$ in A are finitely generated. Lemma 3.4 (or, for $n = 3$, Lemma 3.3) makes sure that such groups do exist for every choice of $n > 2$. Any such group A has a presentation of the form described in the first part of this section, with relations as in $(\mathcal{R})$, so we retain the same notation introduced there. We also let $F = \langle a_j, b_0, c_0 \mid 1 \le j \le r \rangle \le A$. Then F is free abelian of rank n and $A/F \simeq \mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$.

If B is a pure subgroup of rank $n - 1$ in A then $\mathrm{r}_{p^\infty}(B) = 1$, for $\mathrm{r}_{p^\infty}(B) = 2$ would imply that B has a subgroup of rank $n - 2$ which is not finitely generated. Then B satisfies CMS by Lemma 2.3. Now, every subgroup of A of rank $n-1$ is contained in a pure subgroup of the same rank. But A has countably many pure subgroups only, each being the pure closure of a finite subset of A. It follows that *the set of all subgroups of rank $n-1$ in A is countable.*

We shall compute the Schur multiplier $M(A)$ of A. As usual, we identify $M(A)$ with $A \wedge A = (A \otimes A)/D_A$, where $D_A = \langle x \otimes x \mid x \in A \rangle \leq A \otimes A$, and for all $x, y \in A$ we let $x \wedge y = (x \otimes y) + D_A$. For all $j, h \in \{1, 2, \ldots, r\}$ and $i, k \in \mathbb{N}$ we let

$$u_{ji} := a_j \wedge b_i; \qquad v_{ji} := a_j \wedge c_i; \qquad w_{ik} := b_i \wedge c_k; \qquad z_{jh} := a_j \wedge a_h.$$

These elements generate $M(A)$: note that, for all $i, k \in \mathbb{N}$, if $i > k$ we have $b_k = p^{i-k} b_i - u$ for some $u \in U = \langle a_1, a_2, \ldots, a_r \rangle$, hence $b_i \wedge b_k = u \wedge b_i$ and a similar argument works for $c_i \wedge c_k$. More explicitly, we will make use of the formulae:

$$b_i \wedge b_0 = b_i \wedge \Big(p^i b_i - \sum_{j=1}^{r} \theta_{bji} a_j \Big) = \sum_{j=1}^{r} \theta_{bji} u_{ji} \qquad \text{and, similarly,} \qquad c_i \wedge c_0 = \sum_{j=1}^{r} \theta_{cji} v_{ji}, \tag{5}$$

where we have let $\theta_{bji} = \sum_{t=0}^{i-1} \beta_{jt} p^t$ and $\theta_{cji} = \sum_{t=0}^{i-1} \gamma_{jt} p^t$.

Since A is torsion-free, for all $B \leq A$ the natural map $M(B) \to M(A)$ induced by inclusion is injective (see [2], p. 520, Corollary), which allows us to identify $M(B)$ with the subgroup $\langle x \wedge y \mid x, y \in B \rangle$ of $M(A)$, as we shall do from now on. After this identification we have that $M(F)$ is free abelian on the basis $\{u_{j0}, v_{j0}, u_{r0}, v_{r0}, w_{00}, z_{jh} \mid 1 \leq j < h \leq r\}$.

Let $Z = \langle z_{jh} \mid 1 \leq j < h \leq r \rangle$ and $E = Z + \langle u_{ji}, v_{ji} \mid i, j \in \mathbb{N},\ 1 \leq j \leq r \rangle$. As before, for all $i, k, \ell \in \mathbb{N}$, from $p^\ell b_{i+\ell} \equiv_U b_i$ we obtain $p^\ell w_{i+\ell,k} = (p^\ell b_{i+\ell}) \wedge c_k \equiv_E b_i \wedge c_k = w_{ik}$ and, similarly, $p^\ell w_{i,k+\ell} \equiv_E w_{ik}$; thus

$$p^\ell w_{i+\ell,k} \equiv_E w_{ik} \equiv_E p^\ell w_{i,k+\ell}.$$

As a further consequence, each of w_{ik}, w_{ki} and $w_{i+k,0}$ is congruent to $p^{i+k} w_{i+k,i+k}$ modulo E, hence

$$w_{ik} \equiv_E w_{ki} \equiv_E w_{i+k,0}.$$

It follows that $M(A) = E + \langle w_{i0} \mid i \in \mathbb{N} \rangle$. Now, the usual relations easily give, for all $i \in \mathbb{N}$ and $j \in \{1, 2, \ldots, r\}$,

$$p u_{j,i+1} \equiv_Z u_{ji}; \qquad p v_{j,i+1} \equiv_Z v_{ji}; \qquad p w_{i+1,0} \equiv_E w_{i0}.$$

Since the $2r+1$ elements u_{j0}, v_{j0} and w_{00} are independent modulo Z, it follows that E/Z is the direct sum of $2r$ copies of $\mathbb{Q}_p$ and $M(A)/E \simeq \mathbb{Q}_p$. But $\operatorname{Ext}(\mathbb{Q}_p, \mathbb{Q}_p) = 0$, hence $M(A)$ splits over E modulo Z. Therefore $M(A)/Z$ is the direct sum of $2r+1$ copies of $\mathbb{Q}_p$. Now, $Z \leq M(F)$, and it follows that $M(A)/M(F)$ is the direct sum of $2r+1$ copies of $\mathcal{C}_{p^\infty}$.

Let $P \simeq \mathcal{C}_{p^\infty}$, viewed as a trivial A-module, assume that a subgroup K of $M(A)$ is given such that $M(A)/K \simeq \mathcal{C}_{p^\infty}$ and fix an epimorphism $\xi\colon M(A) \twoheadrightarrow P$ such that $K = \ker \xi$. The Universal Coefficients Theorem (see [12], 11.4.18; [15], Ch. 5) yields a natural isomorphism $H^2(A, P) \xrightarrow{\sim} \operatorname{Hom}(M(A), P)$, so there exists a central extension $P \hookrightarrow G \overset{\nu}{\twoheadrightarrow} A$ where the commutator map $A \times A \to P$ induces ξ. More explicitly the resulting group G will be such that $[x, y] = (x^\nu \wedge y^\nu)^\xi$ for all $x, y \in G$. Thus $P = G' \leq Z(G)$, and the choice of K in $M(A)$ determines centralizers in G. We shall see that it is possible to choose K in such a way that all abelian subgroups of G are finitely generated modulo P and, as a consequence, that there are only countably many of them.

If H is a subgroup of G with torsion-free rank less than $n-1$, then HP/P is finitely generated, because of the properties required for A. Next we consider the case when H has torsion-free rank $n-1$. In $M(A)$, let $\mathcal{S}$ be the set of all subgroups of the form $M(B)$ where B is a subgroup of A which has rank $n-1$ and is not finitely generated. If we assume that K is chosen in such a way that the following condition is satisfied:

($\mathcal{K}_1$): $S + K = M(A)$ for all $S \in \mathcal{S}$,

then, for all $H \leq G$ such that HP/P is not finitely generated and has rank $n-1$, we have $H' = P$, so that H is not abelian. Indeed, if H is such a subgroup, then $M(H^\nu) \in \mathcal{S}$, because $H^\nu \simeq HP/P$, so that if ($\mathcal{K}_1$) holds then $H' = M(H^\nu)^\xi = M(A)^\xi = P$ (recall that $K = \ker \xi$).

To discuss subgroups of rank n modulo P we observe that if the following holds:

($\mathcal{K}_2$): $M(F) + \langle u_{j1} \mid 1 \leq j \leq r \rangle \leq K$ and $w_{10} \notin K$,

then $C_G(F_0) = F_0$, where F_0 is the preimage of F under ν. For, let $X = \{x \in A \mid x \wedge F \subseteq K\}$. Then $X = \big(C_G(F_0)\big)^\nu$, so the required conclusion is equivalent to $X = F$. Assume that ($\mathcal{K}_2$) holds. Then $F \subseteq X$ and we only need to check the reverse inclusion. Let $x \in X$. Then $x = f + \lambda b_i + \mu c_i$ for some $f \in F$ and $i, \lambda, \mu \in \mathbb{N}$. To prove that $x \in F$ we may well assume $f = 0$. Also, at the expense of redefining i, we may assume that at least one of λ and μ is not divisible by p. Since A/F is a p-group, we may further assume that λ and μ are coprime. The following elements are in K for all j; we use (5) to compute them:

$$a_j \wedge x = \lambda u_{ji} + \mu v_{ji}; \qquad b_0 \wedge x = -\lambda \sum_{j=1}^{r} \theta_{bji} u_{ji} + \mu w_{0i}; \qquad c_0 \wedge x = -\lambda w_{i0} - \mu \sum_{j=1}^{r} \theta_{cji} v_{ji}. \tag{6}$$

Now,

$$w_{0i} = b_0 \wedge c_i = \Big(p^i b_i - \sum_{j=1}^{r} \theta_{bji} a_j\Big) \wedge c_i = (b_i \wedge p^i c_i) - \sum_{j=1}^{r} \theta_{bji} v_{ji}$$
$$= \Big(b_i \wedge \Big(c_0 + \sum_{j=1}^{r} \theta_{cji} a_j\Big)\Big) - \sum_{j=1}^{r} \theta_{bji} v_{ji} = w_{i0} - \sum_{j=1}^{r} (\theta_{cji} u_{ji} + \theta_{bji} v_{ji}).$$

Modulo K, the equalities in (6) yield $\lambda u_{ji} \equiv -\mu v_{ji}$ for all j and hence

$$0 \equiv b_0 \wedge x \equiv \mu\Big(\sum_{j=1}^{r} \theta_{bji} v_{ji} + w_{0i}\Big) \equiv \mu\Big(w_{i0} - \sum_{j=1}^{r} \theta_{cji} u_{ji}\Big)$$
$$0 \equiv -c_0 \wedge x \equiv \lambda\Big(w_{i0} - \sum_{j=1}^{r} \theta_{cji} u_{ji}\Big).$$

Now, λ and μ are coprime, therefore $w_{i0} \equiv_K \sum_{j=1}^{r} \theta_{cji} u_{ji}$. But, by $(\mathcal{K}_2)$, w_{i0} has order p^i modulo K, while the element on the right has smaller order, unless $i = 0$. Therefore $i = 0$, that is, $x \in F$. Thus our claim is established.

Next we show that the consequence $F_0 = C_G(F_0)$ of $(\mathcal{K}_2)$ in turn implies that all abelian subgroups H of G with torsion-free rank n are finitely generated modulo P. For, assume $H \leq G$ and $\mathrm{r}_0(H) = n$. Then $F_0^\lambda \leq H$ for some positive integer λ, hence $[H^\lambda, F_0] = [H, F_0^\lambda] = 1$. Therefore $H^\lambda \leq C_G(F_0) = F_0$ and so HP/P is finitely generated.

Summing up the argument thus far, we have proved that if K is chosen in such a way that both $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ are satisfied, then all abelian subgroups H of G are finitely generated modulo P. In this case there are just countably many abelian subgroups of G containing P. Moreover, each of them satisfies CMS, by Lemma 2.3, and it follows that the set of all abelian subgroups of G is countable. Clearly, $G \notin (\mathrm{CMS})$, because $\mathrm{r}_{p^\infty}(G) = 3$.

Therefore, to complete the proof of Theorem 3.1, we only need to show that $M(A)$ has a subgroup K such that $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ hold. To this aim, we first look at the structure of the subgroups of $M(A)$ in the set $\mathcal{S}$. We know that $M(A)$ is an extension of the free abelian group $W := M(F) + \langle u_{j1} \mid 1 \leq j \leq r\rangle$ by the direct sum of $2r+1$ copies of $\mathcal{C}_{p^\infty}$. Let B be a subgroup of A which has rank $n-1$ and is not finitely generated. As is well-known, $M(B)$ has rank $\binom{n-1}{2}$, hence it is not trivial. All subgroups of rank $n-2$ in B are finitely generated, hence B has no infinite cyclic quotients, that is, $\mathrm{Hom}(B, \mathbb{Z}) = 0$. Therefore $\mathrm{Hom}(B \otimes B, \mathbb{Z}) \simeq \mathrm{Hom}(B, \mathrm{Hom}(B, \mathbb{Z})) = 0$, and so $\mathrm{Hom}(M(B), \mathbb{Z}) = 0$, since $M(B)$ is an epimorphic image of $B \otimes B$. It follows that $M(B)$ is not finitely generated, hence $M(B) + W/W$ has at least one subgroup isomorphic to $\mathcal{C}_{p^\infty}$. Having chosen one such subgroup for each B, let $\mathcal{P}$ be the set that they form. Note that $\mathcal{P}$ is countable, because A has only countably many subgroups of rank $n-1$. Now apply Lemma 3.5 to the group $M(A)/W$ to produce a subgroup K of $M(A)$ containing W such that $w_{01} \notin K$ and $P(K/W) = M(A)/W$ for all $P \in \mathcal{P}$. Then K satisfies $(\mathcal{K}_2)$ and also $(\mathcal{K}_1)$, since every $S \in \mathcal{S}$ contains the preimage modulo W of some $P \in \mathcal{P}$. Thus the proof of Theorem 3.1 is complete. □

It turns out that *all* subgroups of torsion-free rank $n-1$ in the group G just constructed (in the case when both $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ hold) satisfy CMS.

For, let $H \leq G$ and assume $\mathrm{r}_0(H) = n-1$. If $L \leq H$ and L' is infinite then $P = L' \leq L$. As remarked in the course of the proof, all subgroups of rank $n-1$ in A satisfy CMS, hence $H/P \in (\mathrm{CMS})$. Thus H has at most countably many subgroups with infinite derived subgroup. Now assume that $L \leq H$ and L' is finite. Then $L/Z(L)$ is finite. We showed that all abelian subgroups of G are finitely generated modulo P, hence LP/P is finitely generated. As $\mathrm{r}_{p^\infty}(LP) = 1$, we also have $LP \in (\mathrm{CMS})$. It follows that there are only countably many possible choices for L, therefore $H \in (\mathrm{CMS})$.

Now, G has subgroups H such that $\mathrm{r}_0(H) = n-1$ and $\mathrm{r}_{p^\infty}(H) = 2$, for instance those such that $P < H$ and H/P is a pure subgroup of G/P of rank $n-1$. This proves Proposition 3.2. □

We close the paper by observing that the groups constructed to prove Theorem 3.1 when $n = 3$ are of the least possible ranks. We make use of an elementary and certainly known lemma.

Lemma 3.6. *Let G be a nilpotent group. If $G/Z(G)$ has finite rank and G' is finitely generated, then $G/Z(G)$ is finitely generated.*

Proof. Arguing by induction on the nilpotency class of G, we may assume that $G/Z_2(G)$ is finitely generated. Assume that $G/Z(G)$ is not finitely generated, then we can let $\bar{A} := A/Z(G)$ be a subgroup of minimal rank in $Z_2(G)/Z(G)$ subject to not being finitely generated. Then $\bar{A}$ has no infinite cyclic quotients. Let $x \in G$. Since $A/C_A(x) \simeq [A, x]$ is finitely generated and isomorphic to a quotient of $\bar{A}$, we see that $[A, x]$ is finite. Hence $[A^n, x] = [A, x]^n = 1$, where n is the order of torsion subgroup of G'. But then $A^n \leq Z(G)$, so that $\bar{A}$ has finite exponent and is therefore finite. This contradiction completes the proof. □

Proposition 3.7. *Let G be a soluble-by-finite in which all abelian subgroups satisfy* CMS. *If $\mathrm{r}_0(G) \leq 2$ or $\mathrm{r}_{p^\infty}(G) \leq 2$ for all primes p, then $G \in (\mathrm{CMS})$.*

Proof. By Corollary 2.13, there is no loss in assuming G nilpotent. The already quoted theorem of Baer, shows that G is minimax; we may further assume that it is a counterexample with the least value of $m(G) = \mathrm{r}_0(G) + \sum_{q \text{ prime}} \mathrm{r}_{q^\infty}(G)$. Next, G has a section isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ for some prime p; from our minimality assumption it follows that G has a quotient, say G/N, of this type and that G is p-minimax. Of course, N is infinite, hence $N \cap Z(N)$ has a subgroup U, which is either infinite cyclic or isomorphic to $\mathcal{C}_{p^\infty}$. But $G/U \notin$ (CMS), hence, by minimality, G/U has an abelian subgroup $A/U \notin$ (CMS). We may well substitute A for G and assume $G' \leq U$. If G' is cyclic then $G/Z(G)$ is finitely generated by Lemma 3.6, then $G = NZ(G)$ and $Z(G)$ has a quotient isomorphic to $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$, which is excluded by the hypothesis. Therefore $G' \simeq \mathcal{C}_{p^\infty}$, hence $\mathrm{r}_{p^\infty}(G) > 2$ and then $\mathrm{r}_0(G) \leq 2$. Let H be an infinite cyclic subgroup of G and K be the isolator of H in G. Then $G' \leq K$. If V/G' is a locally cyclic subgroup of K/G' then V is abelian, hence V satisfies CMS and $\mathrm{r}_{p^\infty}(V) = 1$; but this means that V/G' is cyclic. It follows that K/HG' is finite, so that $\mathrm{r}_{p^\infty}(G/K) > 1$. But G/K is a torsion-free abelian group and $\mathrm{r}_0(G/K) \leq 1$; thus we have a contradiction and the proof is complete. $\square$

Acknowledgement

The authors thank the referee for some helpful suggestions.

References

[1] R. Baer, *Polyminimaxgruppen*, Math. Ann. **175** (1968), 1–43.
[2] N. Bourbaki, *Algebra I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation.
[3] R.G. Burns, F. Okoh, H. Smith and J. Wiegold, *On the number of normal subgroups of an uncountable soluble group*, Arch. Math. (Basel) **42** (1984), no. 4, 289–295.
[4] G. Cutolo, *On groups satisfying the maximal condition on non-normal subgroups*, Riv. Mat. Pura Appl. (1991), no. 9, 49–59.
[5] M.R. Dixon, M.J. Evans and H. Smith, *Locally (soluble-by-finite) groups of finite rank*, J. Algebra **182** (1996), no. 3, 756–769.
[6] ———, *Locally (soluble-by-finite) groups with various restrictions on subgroups of infinite rank*, Glasg. Math. J. **47** (2005), no. 2, 309–317.
[7] J.C. Lennox and D.J.S. Robinson, *The theory of infinite soluble groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2004.
[8] J.I. Merzljakov, *On the theory of locally polycyclic groups*, J. London Math. Soc. (2) **30** (1984), no. 1, 67–72.
[9] A.Y. Ol′shanskiĭ, *Geometry of defining relations in groups*, Mathematics and its Applications (Soviet Series), vol. 70, Kluwer Academic Publishers Group, Dordrecht, 1991, Translated from the 1989 Russian original by Yu. A. Bakhturin.
[10] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups, Part 2*, Springer-Verlag, New York, 1972.
[11] ———, *On the cohomology of soluble groups of finite rank*, J. Pure Appl. Algebra **6** (1975), no. 2, 155–164.
[12] ———, *A course in the theory of groups*, second ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996.
[13] S.V. Rychkov and A.A. Fomin, *Abelian groups with a countable number of subgroups*, Abelian groups and modules, No. 10 (Russian), Tomsk. Gos. Univ., Tomsk, 1991, pp. 99–105, 185, 191.
[14] S. Shelah, *Uncountable groups have many nonconjugate subgroups*, Ann. Pure Appl. Logic **36** (1987), no. 2, 153–206.
[15] U. Stammbach, *Homology in group theory*, Springer-Verlag, Berlin, 1973, Lecture Notes in Mathematics, Vol. 359.
[16] V.P. Šunkov, *Locally finite groups of finite rank*, Algebra i Logika **10** (1971), 199–225, English translation: Algebra and Logic 10 (1971), 127–142.
[17] J.S. Wilson, *Large hereditarily just infinite groups*, J. Algebra **324** (2010), no. 2, 248–255.

Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Via Cintia — Monte S. Angelo, I-80126 Napoli, Italy
E-mail address: cutolo@unina.it
URL: http://www.dma.unina.it/~cutolo/

c/o Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837, USA.
E-mail address: howsmith@bucknell.edu