On Groups That Are Dominated by Countably Many Proper Subgroups

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In memoriam: Brian Hartley and David McDougall

Abstract

In this work we study groups for which there is a countable set of proper subgroups with the property that every proper subgroup is contained in some member of the set.

2010 Mathematics Subject Classification: Primary 20E15. **Key Words:** proper subgroup, countably dominated set.

1. Introduction

This article is the third in a series of studies of countability restrictions on the partially ordered set of subgroups of a group. In [16] and [6] the authors considered the property that a group have only countably many subgroups (**CMS**). This is a very strong property and its consequences for the group structure are considerable. For example, in [6] the authors were able to classify all soluble groups with **CMS**: they are precisely the soluble minimax groups without abelian factors of type $p^\infty \times p^\infty$ for any prime p.

In a subsequent paper [2] the present authors studied the much weaker property that a group have countably many maximal subgroups (**CG**). Modules and rings with countably many maximal submodules or right ideals respectively played an important part in

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the investigation. As a result numerous classes of soluble groups with **CG** were identified and several examples of finitely generated soluble groups with uncountably many maximal subgroups were described.

In the present work we study a property that is intermediate between the properties **CMS** and **CG**. Let $\mathcal{S}$ and $\mathcal{T}$ be non-empty sets of proper subgroups of a group G. Then $\mathcal{T}$ is said to *dominate* $\mathcal{S}$ if each member of $\mathcal{S}$ is contained in a member of $\mathcal{T}$. If there is a countable set of proper subgroups of G that dominates $\mathcal{S}$, then $\mathcal{S}$ is said to be *countably dominated* or **CD** in G. Finally, should the set of all proper subgroups of G be dominated by a countable subset $\mathcal{S}$, then we will say that G is *countably dominated* or a **CD**-group: we will also say that $\mathcal{S}$ dominates G. It is clear how the property **CD** should be defined for modules.

Evidently every **CMS**-group is a **CD**-group, and by Lemma 2.1 below every **CD**-group is a **CG**-group. In fact the three properties are distinct. Indeed the direct product G of groups of each prime order has **CD** since the set of subgroups of prime index dominates G. However, G is not a **CMS**-group, for plainly it has uncountably many subgroups. Also the group $p^\infty \times p^\infty$ has no maximal subgroups, so it is a **CG**-group, but it is not a **CD**-group by a simple direct argument (or Lemma 3.3 below). Hence the general situation is:

$$\mathbf{CMS} \subset \mathbf{CD} \subset \mathbf{CG}.$$

Lemma 2.2 also shows that for finitely generated groups the properties **CD** and **CG** are one and the same. On the other hand, the wreath product $\mathbb{Z}_p\ wr\ \mathbb{Z}$ is a finitely generated metabelian group, so it satisfies max-n and hence is a **CG**-group by [2: Theorem 6]. But the group does not have the property **CMS**, since it is not a minimax group. Indeed it appears that the property **CD** is much closer to **CG** than to **CMS**.

Results

In Section 2 we give a number of general results about the property **CD**; we also describe various sources of groups with this property. In Section 3 properties of p-adically irreducible modules are developed which are needed in the proofs of the main theorems in Sections 4 and 5. Virtually nilpotent **CD**-groups are characterized in Section 4. The approach adopted here involves modules over finite groups whose underlying abelian groups are divisible p-groups. The theory of such modules was developed by Zaicev [17] and Hartley [8]. Our first result is:

Theorem 4.1. *Let the group G have a nilpotent normal subgroup N of finite index and write $F = G/N$ and $\bar{N} = N^{ab}$. Then G is a* **CD***-group if and only if $\bar{N}$ has a finitely generated F-submodule X such that $(\bar{N}/X)_p$ is a Černikov group whose finite residual is a near direct sum of finitely many, pairwise non-near isomorphic, p-adically irreducible F-submodules for each prime p.*

The terminology employed here is as follows. A *near direct sum* of modules is a sum of submodules in which the intersection of any one submodule with the sum of all the others is bounded as a $\mathbb{Z}$-module, i.e., it has finite exponent. Two modules are said to be *near isomorphic* if they have isomorphic quotients modulo bounded submodules. Finally, if p is a prime, a module whose underlying abelian group is a p-group is said to be *p-adically irreducible* if it is unbounded, but every proper submodule is bounded.

There is little prospect of classifying soluble **CD**-groups, as there are too many different types. Indeed even the case of metanilpotent groups with min-n presents a significant challenge. Nevertheless our conclusions about these groups are quite complete.

Theorem 5.1. *Let G be a metanilpotent group satisfying min-n and let $A = \gamma_\infty(G)$, the last term of the lower central series of G. Then G has the property* **CD** *if and only if the following hold:*

(i) G/A is a nilpotent Černikov group whose finite residual is locally cyclic;

(ii) A^{ab} has the property **CD** *as a G-module.*

Some light on nature of the second condition in Theorem 5.1 is shed by two equivalent descriptions.

Theorem 5.7. *Let A be an artinian module over a nilpotent Černikov group Q. Then the following are equivalent.*

(i) $A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n + S$ where the A_i are pairwise non-near isomorphic, p-adically irreducible Q-modules for various primes p and S is a bounded Q-submodule.

(ii) A has countably many submodules.

(iii) A has the property **CD** *as a Q-module.*

Here the symbol $\dot{+}$ denotes a near direct sum. The proof of Theorem 5.7 calls for a detailed analysis of the structure of artinian modules over nilpotent Černikov groups. This in turn rests on important work of Hartley and McDougall [10] on the structure of such modules in the non-modular case. Notice the similarity between the module conditions in Theorems 4.1 and 5.7.

The situation for soluble groups satisying min-n with derived length > 2 is certainly more complex: for there are uncountable groups of this type (see [9]), whereas all **CD**-groups are countable.

The property **CD** is not inherited by subgroups, even in the finitely generated case, as is shown by the group $\mathbb{Z}_p$ *wr* $\mathbb{Z}$ – further examples are given at the end of Section 5. Thus we should expect the property that every subgroup of a group have **CD** – which will be denoted by **SCD** – to be much stronger than **CD**. As evidence of this we present a result which describes the soluble **SCD**-groups.

Theorem 6.1. *A soluble group G is an* **SCD**-*group if and only if it has finite abelian ranks and there are no factors in G of type* $p^\infty \times p^\infty$ *for any prime p.*

Recall that a soluble group has finite abelian ranks if it has a series of finite length whose factors are abelian groups with all ranks finite. Notice that the groups described in Theorem 6.1 are similar to the soluble **CMS**-groups, the difference being that infinitely many primes may occur as orders of elements. Moreover, Theorem 6.1 enables us to characterize virtually soluble **SCD**-groups as well, with the help of Lemma 2.4 below.

Our final result Theorem 6.2 characterizes the periodic generalized radical groups which are **SCD**-groups.

Notation

(i) **CD**: countably dominated.

(ii) **CMS**: countably many subgroups.

(iii) **CG**: countably many maximal subgroups.

(iv) H_K: the K-core of H.

(v) G^{ab}: the abelianization of G.

(vi) $\bar{Z}(G)$: the hypercentre of G.

(vii) $A[n]$: the subgroup of elements of order dividing n in an abelian group A.

(viii) $\pi(G)$: the set of primes dividing orders of elements of G.

(ix) $\gamma_i(G)$: a term of the lower central series of G.

(x) $A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n$: the near direct sum of submodules A_i.

(xi) $A \approx B$: modules A and B are near isomorphic.

(xii) $\mathrm{Der}(G, A)$, $\mathrm{Inn}(G, A)$: sets of derivations and inner derivations.

All modules are *right modules*.

2. General properties

Our first observation is that the property **CD** is closely related to that of having countably many maximal subgroups, denoted by **CG**. It will be convenient to establish this fact in a slightly more general form. First recall that two proper subgroups H and K of a group G are said to be *comaximal* if $G = \langle H, K\rangle$.

Lemma 2.1. *Let G be a* **CD**-*group. Then G is a countable* **CG**-*group and every set of pairwise comaximal subgroups of G is countable.*

Proof. By hypothesis there is a countable set of proper subgroups $\mathcal{X}$ that dominates the set of all proper subgroups of G. Let $\mathcal{S}$ be a set of pairwise comaximal subgroups of G. Each $H \in \mathcal{S}$ is contained in some $H^* \in \mathcal{X}$. Moreover, if H and K are distinct members of $\mathcal{S}$, then $H^* \neq K^*$ since $G = \langle H^*, K^*\rangle$. Since $\mathcal{X}$ is countable, it follows that $\mathcal{S}$ is countable. Also distinct maximal subgroups are pairwise comaximal, so G has the property **CG**. It remains to prove that G is countable. For each $X \in \mathcal{X}$ choose $g_X \in G \setminus X$ and observe that $L = \langle g_X \mid X \in \mathcal{X}\rangle$ is not contained in any member of $\mathcal{X}$; therefore $L = G$ and G is countable. □

There is a partial converse of the last result.

Lemma 2.2. *A finitely generated* **CG**-*group is a* **CD**-*group.*

Proof. If G is finitely generated **CG**-group, then every proper subgroup of G is contained in a maximal subgroup. Hence the set of maximal subgroups, which is a countable set, dominates G. □

The next lemma will be used several times as a reduction tool.

Lemma 2.3. *Let N be a nilpotent normal subgroup of a group G. Then G satisfies* **CD** *if and only if G/N' does.*

Proof. Clearly a quotient of a **CD**-group is also a **CD**-group, so only the sufficiency requires proof. Now if $H < G$, then $HN' < G$; for otherwise $N = (H \cap N)N'$ and hence $N = H \cap N \leq H$, so that $H = G$. Since G/N' is a **CD**-group, it follows that G is too. □

Next we discuss closure properties of the class **CD**. It is a simple observation that $G = p^\infty \times p^\infty$ is not a **CD**-group. Indeed G has uncountably many subgroups of type p^∞ and these are pairwise comaximal, so Lemma 2.1 shows that G is not a **CD**-group. Thus the property **CD** is not closed under forming direct products or extensions.

As has already been observed, the property **CD** is not closed under the formation of subgroups. In fact a construction of Ol'šanskii [13: Theorem 1] demonstrates that *every countable group can be embedded in a finitely generated* **CD**-*group.* Also examples at the end of Section 5 tell us that the property **CD** is not even inherited by normal subgroups of finite index.

While **CD** is not closed under taking extensions, certain weak forms of extension closure are valid.

Lemma 2.4. *Let G be a group and let $N \triangleleft G$. If N has* **CD** *and G/N satisfies the maximal condition on subgroups, then G has* **CD**.

Proof. Let $\mathcal{X}$ be a countable set of subgroups which dominates N. Denote by $\mathcal{Y}$ the set consisting of all proper subgroups of G containing N and all subgroups of the form $X_S S$, where $X \in \mathcal{X}$ and S is a finitely generated subgroup of G. Then $\mathcal{Y}$ is countable since G/N satisfies max and N is countable by Lemma 2.1. We argue that $\mathcal{Y}$ dominates G. To this end let $H < G$. If $HN < G$ then $H \leq HN \in \mathcal{Y}$. Otherwise, $HN = G$ and $H \cap N < N$, so there exists $X \in \mathcal{X}$ such that $H \cap N \leq X$. Furthermore, since $H/H \cap N$ is finitely generated, there is a finitely generated subgroup S such that $H = (H \cap N)S$. Since S normalizes $H \cap N$, we have $H \cap N \leq X_S$ and $H \leq X_S S \in \mathcal{Y}$. Thus $\mathcal{Y}$ dominates G. □

The following proposition, which is a variation on Lemma 2.4, is analogous to certain results in [2], especially Theorems 2 and 9. The proof uses the fact that an infinite, virtually soluble group of finite exponent necessarily has uncountably many maximal subgroups [2: Theorem 4].

Proposition 2.5. *Let G be a group with a polycyclic normal subgroup N.*

(i) If G/N has **CG**, *then G has* **CG**.

(ii) If G/N has **CD**, *then G has* **CD**.

Proof. First note that polycyclic groups are **CMS**-groups and hence are **CD**-groups. In both parts of the proof induction on the derived length of N is used, which allows us to assume that N is abelian. Let $\mathcal{M}$ denote the set of maximal subgroups of G that do not contain N.

Assume that G/N, but not G, has **CG**; then $\mathcal{M}$ is uncountable. Since N has **CMS**, there exists $B < N$ such that $B = N \cap M$ for uncountably many $M \in \mathcal{M}$. Now $B \triangleleft NM = G$ for all such M, and it is easy to see that N/B is minimal normal in G/B. Since we can pass to G/B, we may assume that N is minimal normal in G, which means that it is a finite elementary abelian p-group for some prime p.

Uncountably many $M \in \mathcal{M}$ are a complements of N in G, so by the well known correspondence between derivations and complements of normal subgroups $\mathrm{Der}(M, N)$ is uncountable. Let $C = C_M(N)$. As N and M/C are finite, so is $\mathrm{Der}(M/C, N)$. In view of the exact sequence $0 \to \mathrm{Der}(M/C, N) \to \mathrm{Der}(M, N) \to \mathrm{Der}(C, N)$, it follows that $\mathrm{Der}(C, N) = \mathrm{Hom}(C, N)$ is uncountable. Consequently $C/C'C^p$ must be infinite. Now $M/C'C^p$ is abelian-by-finite and has finite exponent. In addition, $M \simeq G/N$, so that $M/C'C^p$ is a **CG**-group; therefore by [2: Theorem 4] it is finite, a contradiction which completes the proof of (i).

To prove (ii) assume that G/N has **CD**. By (i) and Lemma 2.1 the group G has **CG**. Set $\mathcal{S} = \{H < G \mid G = HN\}$; we will show that $\mathcal{S}$ is dominated by a countable set of proper subgroups. Let $H \in \mathcal{S}$. Then $H \cap N = (H \cap N)^H < N$ and $H \cap N$ is contained in a maximal H-invariant proper subgroup of N, say $B(H)$, since N is polycyclic. Thus $HB(H)$ is a maximal subgroup of G and it belongs to $\mathcal{S}$. Consequently $\{HB(H) \mid H \in \mathcal{S}\}$ is a countable set that dominates $\mathcal{S}$ and G has **CD**. □

We close the section by providing some further sources of **CD**-groups. For example, *finitely generated nilpotent-by-polycyclic-by-finite groups are* **CD***-groups.* To prove this reduce to the case of finitely generated abelian-by-polyclic-by-finite groups using Lemma 2.3. Such groups are well known to satisfy max-n, so they are **CG**-groups by [2: Theorem 6] and the claim is established.

Some classes of locally nilpotent **CD**-groups can be identified by using the next result.

Proposition 2.6. *Let G be a locally nilpotent group. Assume that G has a nilpotent normal subgroup A such that A^{ab} is finitely generated as a G-module and G/A has* **CD**. *Then G has* **CD**.

Proof. By Lemma 2.3 we may assume that A is abelian. By hypothesis the group $A/[A,G]$ is finitely generated and hence polycyclic, so $G/[A,G]$ has **CD** by Proposition 2.5(ii). Thus it is enough to show that there cannot exist $H < G$ such that $G = H[A,G]$. Assuming that H is such a subgroup, we have $B = H \cap A \triangleleft\ HA = G$ and, since A is finitely generated as a G-module, $B \leq C$ for some maximal submodule C of A. Then HC is a maximal subgroup of G. Since G is locally nilpotent, $[A,G] \leq G' \leq HC$, which leads to $G = H[A,G] \leq HC$, a contradiction. □

For example, if p is a prime and F is any finite p-group, the wreath product $F\ wr\ p^\infty$ is a **CD**-group. Finally, we record a very different source of **CD**-groups. Recall that a *barely transitive group* is a transitive permutation group on an infinite set such that the orbits of each proper subgroup are finite.

Proposition 2.7. *Let G be a barely transitive permutation group which is not finitely generated. Then G is a* **CD***-group.*

Proof. Let H be a point stabilizer in G. It is well-known that G is countable and that its proper subgroups are residually finite – see [1]. Hence G is locally graded and [1: Lemma 2.1] shows that H is not contained in a maximal subgroup of G. It follows that G can be written as the union of a strictly increasing sequence $(X_i)_{i\in\mathbb{N}}$ of proper subgroups of G containing H each of which is finitely generated modulo H. Let $U < G$. By bare transitivity $|U : H \cap U|$ is finite, whence $\langle H, U\rangle = \langle H, F\rangle$ for some finite subset F of G. It follows that $U \leq X_i$ for some $i \in \mathbb{N}$. Therefore the set $\{X_i \mid i \in \mathbb{N}\}$ dominates G and G has **CD**. □

3. Near direct sums and p-adically irreducible modules

In this section we present certain module theoretic results that are needed in Sections 4 and 5. Two modules A and B over the same ring are said to be *near isomorphic*, in symbols $A \approx B$, if there exist submodules $A_0 \leq A$ and $B_0 \leq B$ which are bounded as abelian groups and are such that $A/A_0 \simeq B/B_0$ as modules. If A and B are ($\mathbb{Z}$-) divisible, it is easy to show that $A \approx B$ if and only if $A \simeq B/B_0$ for some bounded submodule B_0 of B, by using the fact that $A \simeq A/A[n]$ for $n > 0$.

A critical concept for us is that of a near direct sum. Let A be a module and $\{A_i | i \in I\}$ a family of submodules of A. Then A is said to be the *near direct sum* of the submodules A_i if $A = \sum_{i \in I} A_i$ and $A_i \cap (\sum_{j \neq i \in I} A_j)$ is bounded for each $i \in I$. If all these intersections are annihilated by a fixed positive integer t, the sum is called a *t-near direct sum*. The notation

$$A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n$$

will be used to indicate that A is the near direct sum of submodules $A_1, A_2, \ldots, A_n$.

Let p be a prime and let A be a p-torsion module, i.e., its underlying abelian group is a p-group. Then A is *p-adically irreducible* if all its proper submodules are bounded, but A itself is unbounded; in this event A is clearly divisible. A case of special interest is that of a divisible p-torsion module over $\mathbb{Z}F$ where F is a finite group. Zaicev [17] (see also [8: p.218]) proved that *if A is a such module with B a submodule, then there is a submodule C such that A is the $|F|$-near direct sum of B and C.* Zaicev drew the conclusion that A is an $|F|$-near direct sum of p-adically irreducible submodules.

Near direct sums of pairwise non-near isomorphic modules that are p-adically irreducible feature prominently in this work. In certain respects they behave like semisimple modules, as the next result shows.

Lemma 3.1. *Let A be a divisible p-torsion F-module where F is a finite group. Assume that A is the near direct sum of a family $\{S_i | i \in I\}$ of pairwise non-near isomorphic, p-adically irreducible submodules. If B is a divisible submodule of A, then $B = \sum_{i \in J} S_i$ for some $J \subseteq I$.*

Proof. There is a positive integer ℓ such that $A/A[p^\ell] = \bigoplus_{i \in I}(S_i +$

$A[p^\ell])/A[p^\ell]$. Let S be a p-adically irreducible submodule of A. For each $i \in I$ composition of the natural homomorphism $S \to A/A[p^\ell]$ with the projection $A/A[p^\ell] \to (S_i + A[p^\ell])/A[p^\ell]$ yields a homomorphism $\theta_i \colon S \to (S_i + A[p^\ell])/A[p^\ell]$. Since $S \nleq A[p^\ell]$, there is a $j \in I$ for which $\theta_j \neq 0$. Thus $(S)\theta_j$ is a non-zero divisible submodule of the p-adically irreducible module $(S_j + A[p^\ell])/A[p^\ell]$. Therefore θ_j is surjective. Since S is p-adically irreducible, $\mathrm{Ker}(\theta_j)$ is bounded and $S \approx (S_j + A[p^\ell])/A[p^\ell] \approx S_j$. Since the modules S_i are pairwise non-near isomorphic, $\theta_i = 0$ for all $i \in I \setminus \{j\}$, which means that $S \leq S_j + A[p^\ell]$. Hence $S = p^\ell S \leq S_j$, so that $S = S_j$.

Now consider a non-zero divisible submodule B of A. By Zaicev's theorem quoted above, B is a near direct sum of p-adically irreducible submodules. By the first part of the proof each of these p-adically irreducible submodules is one of the S_i. □

The next result is surely known.

Lemma 3.2. *Let p be a prime and F a finite group. Then the number of near-isomorphism classes of p-adically irreducible $\mathbb{Z}F$-modules is at most $|F|$.*

Proof. Let A be a divisible p-torsion $\mathbb{Z}F$-module with finite $\mathbb{Z}$-rank r. Let $\hat{\mathbb{Z}}_p$ and $\hat{\mathbb{Q}}_p$ denote the ring of p-adic integers and the field of p-adic numbers respectively. The dual of A is $A^* = \mathrm{Hom}(A, p^\infty)$, which is a free $\hat{\mathbb{Z}}_p$-module of rank r. Set $A^\circledast = A^* \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathbb{Q}}_p$, which is a $\hat{\mathbb{Q}}_p$-vector space of dimension r. Clearly A is p-adically irreducible if and only if $A^\circledast$ is $\hat{\mathbb{Q}}_p$-simple.

Let A and B be p-adically irreducible F-modules; then A and B have finite rank since F is finite. Clearly $A \approx B$ if and only if $A^\circledast \overset{\hat{\mathbb{Q}}_pF}{\simeq} B^\circledast$. A count of irreducible characters shows that there are at most $|F|$ isomorphism classes of simple $\hat{\mathbb{Q}}_pF$-modules, so the lemma follows. □

We will have several opportunities to use the next observation.

Lemma 3.3. *Let F be a group, p a prime and A a $\mathbb{Z}F$-module. Assume that $A = A_1 \dot{+} A_2 \dot{+} C$ where A_1 and A_2 are near isomorphic, p-adically irreducible submodules and C is a submodule of A. Then there is an uncountable set $\mathcal{B}$ of pairwise comaximal submodules of A such that A/B is p-adically irreducible for $B \in \mathcal{B}$. Hence A does not have the property* **CD** *as a $\mathbb{Z}F$-module.*

Proof. In the first place A has a quotient $\bar{A} = U \oplus V$ where U and V are isomorphic p-adically irreducible $\mathbb{Z}F$-modules. Note that $\mathrm{End}_{\mathbb{Z}F}(U)$ is uncountable, since it has a subring isomorphic with $\hat{\mathbb{Z}}_p$. Thus U has uncountably many complements in $\bar{A}$: let $X \neq Y$ be two of them. Then $\bar{A}/Y \overset{\mathbb{Z}F}{\simeq} U$ is p-adically irreducible, while $X + Y/Y$ is unbounded since $X \neq Y$; hence $\bar{A} = X + Y$. It follows that the complements of U in $\bar{A}$ form an uncountable set of pairwise comaximal submodules of A. The final statement follows by an argument in the proof of of Lemma 2.1. □

We end the section with an example showing that near isomorphic, p-adically irreducible modules need not be isomorphic.

Example. Let $P = A \oplus A$ where $A \simeq 2^\infty$, and consider the automorphisms α, β of P defined by

$$\alpha\colon (a,b) \mapsto (b,a) \quad \text{and} \quad \beta\colon (a,b) \mapsto (a,-b)$$

where $a, b \in A$. Thus α and β have order 2 and $F = \langle \alpha, \beta \rangle$ is a dihedral group of order 8. Observe that P is 2-adically irreducible as a $\mathbb{Z}F$-module. Now let u denote the element of order 2 in A. Then $x = (u,u)$ is a fixed point of F and $P \approx P/\langle x \rangle$. A simple calculation shows that P and $P/\langle x \rangle$ are not $\mathbb{Z}F$-isomorphic.

4. Virtually nilpotent groups with CD

In this section we give a complete characterization of the virtually nilpotent groups which have the property **CD**. What emerges is a criterion that largely hinges on the structure of certain associated modules.

Theorem 4.1. *Let the group G have a nilpotent normal subgroup N of finite index and write $F = G/N$ and $\bar{N} = N^{ab}$. Then G is a* **CD**-*group if and only if $\bar{N}$ has a finitely generated F-submodule X such that $(\bar{N}/X)_p$ is either finite or a Černikov group whose finite residual is a near direct sum of finitely many, pairwise non-near isomorphic, p-adically irreducible F-submodules for each prime p.*

Proof. In the first place Lemma 2.3 shows that we can take N to be abelian. Starting with the necessity, we assume that G is a **CD**-group. Initially we will suppose that G is periodic and we can then assume that N is a p-group. Let A denote the maximum divisible subgroup of N. Since G/N^p is a **CD**-group of finite exponent, it is finite by [2: Theorem 4]. It follows that N/A, and hence G/A, is finite because it is reduced.

We view A as a ZF-module. By Zaicev's theorem A is the near direct sum of a family $\{A_i \mid i \in I\}$ of p-adically irreducible F-submodules. Suppose that $A_i \approx A_j$ for distinct $i, j \in I$. Then by Lemma 3.3 there is an uncountable set $\mathcal{Y}$ of F-submodules of A that are pairwise comaximal in A and are such that A/Y is p-adically irreducible for all $Y \in \mathcal{Y}$. Choose a transversal T to A in G and let $\mathcal{Y}_1 = \{\langle T\rangle Y \mid Y \in \mathcal{Y}\}$. If $Y \in \mathcal{Y}$, the group A/Y is not finitely generated, whence neither is G/Y. Therefore $G \notin \mathcal{Y}_1$ and $\mathcal{Y}_1$ is a set of pairwise comaximal subgroups of G. Now the mapping $Y \in \mathcal{Y} \mapsto \langle T\rangle Y \in \mathcal{Y}_1$ is injective; for, if $\langle T\rangle Y = \langle T\rangle Y^*$ with distinct Y, Y^* in $\mathcal{Y}$, then $YY^* = A$ and hence $G = \langle T\rangle Y \in \mathcal{Y}_1$, a contradiction. It follows that $\mathcal{Y}_1$ is uncountable, which is in contradiction to Lemma 2.1. Hence the modules A_i are pairwise non-near isomorphic and by Lemma 3.2 the index set I must be finite. As a consequence N is a Černikov group. Since A is the finite residual of N, we see that N has the required structure.

At this point we drop the hypothesis of periodicity. There is a free abelian subgroup X such that N/X is periodic. Since G/N is finite, we can replace X by its core in G and assume that $X \lhd G$. Let p be any prime; then G/X^p is a periodic **CD**-group, so the first part of the proof shows that X/X^p is finite. Thus X is finitely generated and also $(N/X)_p$ is a Černikov group. Moreover, if D/X^p is the finite residual of $(N/X^p)_p$, then DX/X is the finite residual of $(N/X)_p$, which completes the proof of necessity.

Next we establish sufficiency; assume that G satisfies the conditions. First of all assume that $X = 1$, so that G is periodic. Note that as a consequence G is countable. We will construct a countable set that dominates G as follows. Let R denote the finite residual of N. For each $p \in \pi(N)$ let $\mathcal{C}_p$ be the (finite) set of all proper subgroups of G containing $N_{p'}R_p$. Furthermore, if $R_p \neq 1$, then R_p is the near direct product of a finite family $\{S_i | i \in I\}$ of pairwise non-near isomorphic, p-adically irreducible F-modules. Denote by $\mathcal{M}_p$ the set of all F-submodules that have the form $\langle S_j \mid j \neq i, j \in I\rangle$ for some $i \in I$. Now let $\mathcal{X}_p$ be the set of all subgroups $ELN_{p'}$, where E is a finite subgroup of G and $L \in \mathcal{M}_p$; note that subgroups in X_p are proper. Define $\mathcal{X}$ be the union of all the sets $\mathcal{C}_p$ and $\mathcal{X}_p$ just defined. Clearly $\mathcal{X}$ is countable; we will prove that it dominates G. To this end let $U < G$; we will argue that $U \leq X$ for some $X \in \mathcal{X}$.

In the first place we may assume that $N \not\leq U$, since otherwise $U \in \mathcal{X}$. Choose $p \in \pi(N/N \cap U)$; then $N \cap (UN_{p'}) = (N \cap U)N_{p'} < N$, so that $UN_{p'} < G$ and, on replacing U by $UN_{p'}$, we may assume that $N_{p'} \leq U$. If $UR_p < G$, take X to be $UR_p \in \mathcal{C}_p$. Therefore we suppose that $UR_p = G$, so that $R_p \neq 1$ and $U \cap R_p \lhd G$. Let S be the finite residual of $U \cap R_p$. Then S is a proper divisible F-submodule of R_p and thus Lemma 3.1 shows that $S \leq L$ for some $L \in \mathcal{M}_p$. Moreover, $U/SN_{p'}$ is finite since $G/N_{p'}$ is Černikov. Therefore $U = ESN_{p'}$ for some finite $E \leq G$. If $X = ELN_{p'}$, then $U \leq X \in \mathcal{X}_p \subseteq \mathcal{X}$ and therefore $\mathcal{X}$ dominates G and G has **CD**.

Finally, consider the general case, so X is the finitely generated submodule of N specified in the hypothesis. Then G/X is a periodic **CD**-group by the previous part of the proof. It now follows from Proposition 2.5 that G is a **CD**-group. □

The full structural consequences of the condition **CD** are shown in the next result.

Corollary 4.2. *Let G, N and F be as in the theorem and assume that G has* **CD**. *Then:*

(i) the group N has finite abelian ranks;

(ii) there is a finitely generated subgroup X of N such that G/X^G is periodic and $(N/X^G)_p$ is a Černikov group whose finite residual is the near direct sum of finitely many, pairwise non-near isomorphic, p-adically irreducible F-submodules for all primes p.

Proof. By the theorem N^{ab} has finite ranks, so by the well known tensor product property of the lower central series the lower central factors of N also have finite ranks; therefore N has finite abelian ranks. Consequently, N has a finitely generated subgroup X such that each factor $X\gamma_i(N)/X\gamma_{i+1}(N)$ is periodic. Therefore G/X^G is periodic, so we may assume that G is periodic; we can also take N to be a p-group. Since N is a nilpotent Černikov group, N' is finite and central in N. If R denotes the finite residual of N, then RN'/N' is the finite residual of N^{ab}. Furthermore $R \approx RN'/N'$. Since G/N' is a **CD**-group, it follows from the theorem that R inherits the relevant properties of the module RN'/N'. □

We single out two special cases of Theorem 4.1 in which the characterization takes a simpler form.

Corollary 4.3.

(i) A periodic nilpotent group is a **CD***-group if and only if each of its infinite primary components is a finite extension of a Prüfer group.*

(ii) A nilpotent Černikov group is a **CD***-group if and only if its finite residual is locally cyclic (in which case the group is a* **CMS***-group).*

Proof. Let G be a periodic nilpotent **CD**-group. Applying Corollary 4.2 with $N = G$, we conclude that each primary component of G is Černikov, so that the finite residual R of G is central. Hence the p-adically irreducible subgroups of R_p are p^∞-groups and R is locally cyclic. The converse follows at once from Theorem 4.1. It is immediate that (ii) is valid. □

Observe that the class of nilpotent **CD**-groups is not subgroup closed. For example, let $G = \langle x \rangle \ltimes A$ where $A \simeq 2^\infty \oplus 2^\infty$ and $(a_1, a_2)x = (a_1, a_1 + a_2)$; then G is a nilpotent **CD**-group of class 2, but A is not a **CD**-group. On the other hand it follows from Corollary 4.3 and Proposition 2.5 that the class of polycyclic-by-periodic nilpotent **CD**-groups is subgroup closed.

It is worthwhile recording the situation for abelian groups in regard to the property **CD**.

Corollary 4.4. *An abelian group A is a* **CD***-group if and only if there is a finitely generated subgroup B such that A/B is periodic and each of its infinite primary components is a finite extension of a Prüfer group.*

5. Metanilpotent groups with min-n and CD

Our aim in this section is to give necessary and sufficient conditions for a metanilpotent group satisfying min-n to have the property **CD**. In this endeavour it is essential to keep in mind that a metanilpotent group with min-n is locally finite and countable by work of Baer [4] and McDougall [12] respectively. It was shown in [2] that all metanilpotent groups with min-n are **CG**-groups. However, the question of which of these groups have the property **CD** is more subtle and progress requires detailed knowledge of the structure of artinian modules over nilpotent Černikov groups. Our principal aim is to establish the following result.

Theorem 5.1. *Let G be a metanilpotent group satisfying min-n and let $A = \gamma_\infty(G)$, the last term of the lower central series of G. Then G has the property* **CD** *if and only if the following hold:*

(i) G/A is a nilpotent Černikov group whose finite residual is locally cyclic.

(ii) A^{ab} has the property **CD** *as a G-module.*

Proof. First note that A and G/A are nilpotent and by Lemma 2.3 we can assume that A is abelian. To prove necessity assume that G has **CD** and note that (i) holds by Corollary 4.3. Let $\mathcal{S}$ denote the set of all proper G-submodules of A and suppose that A does not have **CD** as a G-module. Write $A_0 = A \cap \bar{Z}(G)$ and note that $\bar{Z}(G)$ is a Černikov group by a theorem of Baer [3] – see also [14: Theorem 5.22]. For the moment let $A_0 = 1$. Since $H^0(G/A, A) = 0$, we have $H^2(G/A, A) = 0$ by [11: 10.3.2], so that G splits over A, say $G = H \ltimes A$. Notice that $HS < G$ for $S \in \mathcal{S}$. As G is a **CD**-group, there is a countable set of proper subgroups $\mathcal{T}$ such that for each $S \in \mathcal{S}$ we have $HS \leq T(S)$ for some $T(S) \in \mathcal{T}$. Then $S \leq A \cap T(S)$ and $A \cap T(S)$ is a proper G-submodule of A. Hence $\mathcal{S}$ is dominated by $\{A \cap T(S) | S \in \mathcal{S}\}$, which is a countable set of proper G-submodules of A. Therefore A has the property **CD** as a G-module.

Returning to the general case, we note that by the previous paragraph the set of $S \in \mathcal{S}$ such that $SA_0 < A$ is **CD** in G. What remains to be proved is that the set $\mathcal{S}_1 = \{S \in \mathcal{S} | SA_0 = A\}$ is **CD** in G. Let $S \in \mathcal{S}_1$. Then A/S is a Černikov group since $A/S \simeq A_0/S \cap A_0$. Evidently we may assume that A/S is either finite or else a divisible abelian p-group of finite rank which is p-adically irreducible as a G-module. Since A is an artinian G-module, it has finitely many submodules of finite index. Thus it is enough to show that the set $\mathcal{S}_2$ of all $S \in \mathcal{S}_1$ for which A/S is a p-adically irreducible G-module is **CD** in the module A. Assume that this is false.

Write $C = C_G(A_0)$; then $F = G/C$ is finite by a result of Baer [3] – see also [14: Theorem 3.29.2]. Since $A/S \overset{G}{\simeq} A_0/S \cap A_0$, the subgroup C centralizes A/S, so the latter is an F-module. By

Lemma 3.2 there are finitely many near isomorphism types of p-adically irreducible F-modules, from which it follows that there is an uncountable subset $\mathcal{S}_3 \subseteq \mathcal{S}_2$ such that if $S, T \in \mathcal{S}_3$, then A/S and A/T are near isomorphic as F-modules.

Suppose that $S + T = A$ for some $S, T \in \mathcal{S}_3$. Then $A/S \overset{G}{\simeq} T/S \cap T$ and $A/T \overset{G}{\simeq} S/S \cap T$. But then $G/(S \cap T)$ is a Černikov **CD**-group whose finite residual contains two near isomorphic, p-adically irreducible submodules $S/(S \cap T)$ and $T/(S \cap T)$. Since this contradicts Theorem 4.1, we have $S + T \neq A$ for all $S, T \in \mathcal{S}_3$.

Now fix $S \in \mathcal{S}_3$ and let $T \in \mathcal{S}_3$ be arbitrary. Since $S+T < A$ and A/T is p-adically irreducible, $(S+T)/T \overset{G}{\simeq} S/S \cap T$ is finite. Since S is G-artinian, it has finitely many G-submodules of finite index. Hence there exist uncountably many $T \in \mathcal{S}_3$ such that $S \cap T = U$ is fixed. But G/U is a Černikov group, so it has countably many finite subgroups. This gives the contradiction that there are countably many T's; thus necessity is established.

Turning to sufficiency, we suppose that properties (i) and (ii) hold, but the set $\mathcal{H}$ of all proper subgroups of G is not **CD** in G. Now G/A is a **CD**-group by Corollary 4.3, so the set $\{H \in \mathcal{H} | HA < G\}$ is **CD** in G. Therefore the set $\mathcal{H}_1 = \{H \in \mathcal{H} | HA = G\}$ is not **CD** in G.

For any $H \in \mathcal{H}_1$ we have $H \cap A \triangleleft HA = G$; thus $H \cap A$ is a proper submodule of A. By condition (ii) the set of proper submodules of A is dominated by some countable set $\mathcal{S}$ of proper submodules. For each $S \in \mathcal{S}$ define $\mathcal{H}_1(S) = \{H \in \mathcal{H}_1 | H \cap A \leq S\}$. Since $\mathcal{S}$ is countable, there must exist $S \in \mathcal{S}$ such that $\mathcal{H}_1(S)$ is not **CD** in G. Factoring out by S, we reach the situation where $G = H \ltimes A$ for $H \in \mathcal{H}_2$, a subset of $\mathcal{H}_1$ which is not **CD** in G and is therefore uncountable. Notice that H is a Černikov group.

Let $A_0 = A \cap \bar{Z}(G)$ and recall that $\bar{Z}(G)$ is a Černikov group. For the moment suppose that $A_0 = 1$. Then $H^1(G/A, A) = 0$ by [11: 10.3.2]. Hence $\mathrm{Der}(G/A, A) = \mathrm{Inn}(G/A, A)$, which is countable. It follows that there are only countably many complements H in $G = H \ltimes A$, which gives the contradiction that $\mathcal{H}_2$ is countable and completes the proof in this case.

We now proceed to the general case. Since $(A/A_0) \cap \bar{Z}(G/A_0) = 1$, the group G/A_0 is a **CD**-group by the previous argument, so the set of $H \in \mathcal{H}_2$ such that $HA_0 < G$ is **CD** in G. It follows that the set $\mathcal{H}_3 = \{H \in \mathcal{H}_2 | HA_0 = G\}$ cannot be **CD** in G and thus is uncountable. For $H \in \mathcal{H}_3$ we have $A = A \cap (HA_0) = A_0$, from which it follows that A, and hence G, is Černikov. In addition $A = [A, G]$, which implies that $H^1(G/A, A)$ has finite exponent by [11: 10.3.6]. From this it is routine to deduce that $H^1(G/A, A)$ is finite. Therefore $\mathrm{Der}(G/A, A)$ is countable and again there are countably many complements H. However, this means that $\mathcal{H}_3$ is countable, a final contradiction. □

Artinian modules over centre-by-finite groups

Let G be a metanilpotent group with min-n and write $A = \gamma_\infty(G)$ and $Q = G/A$. Then A^{ab} is an artinian module over the nilpotent Černikov group Q; note also that Q is centre-by-finite. Theorem 5.1 makes it clear that the key to determining whether G is a **CD**-group is the Q-module structure of A^{ab}. Thus such modules merit our attention. In particular, we need to understand the effect of the module property **CD** on A^{ab}.

Firstly, we note that if Q is a nilpotent Černikov group and A an artinian Q-module, then $G = Q \ltimes A$ is a metanilpotent group satisfying min-n. By the results of Baer and McDougall G is locally finite and countable, from which it follows that A is countable and periodic.

The structure of artinian modules over centre-by-finite groups has been analyzed in the non-modular case by B. Hartley and D. McDougall in the important paper [10]. We will describe their results in some detail since they are critical for this investigation.

For any prime p let Q be a countable, locally finite p'-group. Let $\{M_\lambda | \lambda \in \Lambda\}$ denote a complete set of non-isomorphic, simple $\mathbb{Z}_p Q$-modules and let the rank of M_λ be n_λ. If V is a divisible abelian p-group of rank n_λ, we can endow $V[p]$ with a Q-module structure by identifying it with the abelian group M_λ. Since Q is a locally finite p'-group, the Q-module structure of $V[p]$ can be extended to $V[p^n]$ – see for example [10: Lemma 3.2]. In this way we obtain Q-modules $V_\lambda(n) = V[p^n]$, $n = 1, 2, \ldots$. Let $V_\lambda(\infty) = \bigcup_{n=1,2,\ldots} V_\lambda(n)$; this is the Q-injective hull of $V[p]$. Then $V_\lambda(n+1)/V_\lambda(n) \overset{Q}{\simeq} M_\lambda$. It is easy to show that $V_\lambda(\infty)$ is a uniserial, p-adically irreducible Q-module. Moreover, the $V_\lambda(n)$ are the only proper submodules of $V_\lambda(\infty)$. Finally, note that the Q-modules $V_\lambda(n)$ are noetherian and artinian, while $V_\lambda(\infty)$ is artinian.

Example (i). Let Q be a group of type q^∞ and let M be the simple *Čarin Q-module* over $\mathbb{Z}_p$ where $p \neq q$ – see [5] or else [11: 1.5.1]. Recall that M is the field obtained by adjoining q^nth roots of unity for $n = 1, 2, \ldots$ to the field of p elements, and giving M the natural Q-module structure. Then there is a corresponding p-adically irreducible Q-module V such that $M \overset{Q}{\simeq} V[p]$.

The main result of [10] is a Krull-Schmidt Theorem for artinian Q-modules (Theorems A and C). In the following theorem $V_\lambda(n)$ and $V_\lambda(\infty)$ have the above meanings.

Theorem. *Let p be a prime, Q a countable centre-by-finite p'-group and A an artinian Q-module which is a p-group. Then A is the direct sum of finitely many indecomposable submodules each of which is isomorphic to some $V_\lambda(n)$ or $V_\lambda(\infty)$. Moreover the decomposition is unique up to an automorphism of A.*

Another result that will be important here is:

Lemma 5.2. *Let A be an artinian module over a nilpotent Černikov group Q. Then $Q_p/C_{Q_p}(A_p)$ is finite for all primes p.*

This may be deduced from [12: Theorem 3.2] and it is not hard to prove directly. We remark that $Q/C_Q(A_p)$ might not be a p'-group, so we could be faced with a modular situation – see the example (v) at the end of this section. Another useful fact is:

Lemma 5.3. *Let Q be a nilpotent Černikov group and A an artinian Q-module. If A is bounded, then it is noetherian.*

Proof. We may assume that Q acts faithfully on A and that A is a p-group. Lemma 5.2 shows that $Q = P \times R$ where P is a finite p-group and R is a p'-group. Since Q/R is finite, A is R-artinian. Now A being bounded, the Hartley-McDougall decomposition shows that A is the direct sum of finitely many R-submodules of types $V_\lambda(i)$, $i = 1, 2, \cdots$. Since each $V_\lambda(i)$ is noetherian, the result follows. □

Corollary 5.4. *Let G be a metanilpotent group satisfying min-n and set $A = \gamma_\infty(G)$. If A has finite exponent, then G has the property* **CD**.

Proof. First note that G/A is a Černikov group. By Lemma 5.3 the G/A-module A^{ab} is noetherian. Since A is countable, it has countably many submodules and the result follows by Theorem 5.1.

□

Next we establish a basic result about artinian modules over nilpotent Černikov groups that will allow us to analyse the condition (ii) in Theorem 5.1 in the modular case.

Proposition 5.5. *Let Q be a nilpotent Černikov group and A an artinian Q-module which is a p-group. Then there is a near direct decomposition*

$$A = (A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n) + A[p^\ell],$$

where ℓ is a natural number and the A_i are p-adically irreducible submodules.

Proof. We may assume without loss that Q acts faithfully on A. By Lemma 5.2 we have $Q = P \times R$, where P is a finite p-group, of order p^k say, and R is a Černikov p'-group. Of course, A may be assumed to be unbounded.

Since A is artinian, there is a Q-submodule A_1 which is minimal subject to being unbounded. Then A_1 is a p-adically irreducible Q-module. Since A_1 is divisible, it is R-injective by [10: Lemma 2.3] and we can write $A = A_1 \oplus B$ for some R-submodule B. Let $\pi : A \to A_1$ be the canonical R-projection and define a map $\bar{\pi} : A \to A_1$ by $(a)\bar{\pi} = \sum_{x \in P} (ax)\pi x^{-1}$, $a \in A$. By a standard calculation $\bar{\pi}$ is a Q-homomorphism and $(A_1)\bar{\pi} = p^k A_1 = A_1$. Since $\bar{\pi}^2 = p^k \bar{\pi}$, we have for any $a \in A$ that $(p^k a - (a)\bar{\pi})\bar{\pi} = 0$, so that $p^k a - (a)\bar{\pi} \in K_1 = \mathrm{Ker}(\bar{\pi})$ and thus $A/(A_1 + K_1)$ is bounded. Notice that $A_1 \cap K_1 = A_1[p^k]$, so $A_1 + K_1 = A_1 \dot{+} K_1$. If K_1 is unbounded, we repeat the argument for K_1 to get a p-adically irreducible submodule A_2 and a submodule K_2 such that $K_1/(A_2 + K_2)$ is bounded and $A_2 + K_2 = A_2 \dot{+} K_2$.

Continuing in this manner, we obtain a sequence of p-adically irreducible submodules $A_1, A_2, \ldots$ and submodules $K_1, K_2, \ldots$ such that $A/(A_1+A_2+\cdots+A_n+K_n)$ is bounded and $A_1+A_2+\cdots+A_n+K_n$ is near direct. Since A is artinian, this procedure must terminate finitely, which means that some K_n is bounded. Hence $A/(A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n)$ is bounded. Write $B = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n$. Since B is a divisible subgroup, we have $A = B \oplus S$ where S is a bounded subgroup. If $p^\ell S = 0$, then $S \leq A[p^\ell]$ and hence $A = B + A[p^\ell]$. □

From the last result we obtain some useful structural information about artinian modules over nilpotent Černikov groups.

Proposition 5.6. *Let Q be a nilpotent Černikov group and A an artinian Q-module. Then there is a bounded Q-submodule K such that $A/K \overset{Q}{\simeq} A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where the A_i are p-adically irreducible Q-modules for various primes p. Hence $A \approx A_1 \oplus A_2 \oplus \cdots \oplus A_n$.*

Proof. Evidently we may assume that A is a p-group. By Proposition 5.5 we have $A = (A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n) + A[p^\ell]$ for some $\ell > 0$ where the A_i are p-adically irreducible submodules. Choose $m \geq \ell$ so large that $A[p^m]$ contains all the intersections $A_i \cap (\sum_{j \neq i} A_j)$, $i = 1, 2, \ldots, n$. Then

$$A/A[p^m] \overset{Q}{\simeq} A_1/A_1[p^m] \oplus A_2/A_2[p^m] \oplus \cdots \oplus A_n/A_n[p^m].$$

Finally, $A_i/A_i[p^m] \overset{Q}{\simeq} A_i$ via multiplication by p^m. □

We are now in a position to characterize those artinian modules over nilpotent Černikov groups which have the module property **CD**.

Theorem 5.7. *Let A be an artinian module over a nilpotent Černikov group Q. Then the following statements are equivalent.*

(i) $A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n \dot{+} S$ where the A_i are non near isomorphic, p-adically irreducible Q-modules for various primes p and S is a bounded Q-submodule.

(ii) A has countably many submodules.

(iii) A has the property **CD** *as a Q-module.*

Proof. Evidently we can assume that A is a p-group for some prime p.
(i) ⇒ (ii). Let A have the decomposition in (i). Note that S is noetherian by Lemma 5.3, so it has countably many submodules; thus we can assume that $n > 0$. Let $B = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_{n-1} + S \neq A$. By induction on n the submodule B has countably many submodules and the same is true of A/B since its proper submodules are bounded and hence noetherian.

Assume that nevertheless A has uncountably many submodules $\{S_\lambda | \lambda \in \Lambda\}$. Then there are submodules C and D such that $S_\lambda \cap$

$B = C$ and $S_\lambda + B = D$ with C, D fixed for uncountably many λ. Then $D/C = (S_\lambda/C) \oplus (B/C)$. If D/B is bounded, it is noetherian and hence $\mathrm{Hom}_Q(D/B, B/C)$ is countable. Otherwise $D = A$ and $\mathrm{Hom}_Q(A/B, B/C) = 0$ since A/B cannot be near Q-isomorphic with a submodule of B/C. This gives the contradiction that there are countably many complements S_λ/C and hence countably many S_λ's.

(ii) $\Rightarrow$ (iii). This implication is obvious.

(iii) $\Rightarrow$ (i). Assume that A has **CD** as a Q-module. By Proposition 5.5 there is a decomposition $A = (A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n) + A[p^\ell]$ where the A_i are p-adically irreducible and $\ell > 0$. Now apply Lemma 3.3 to show that no two of the A_i can be near isomorphic Q-modules.

□

Combining Theorems 5.1 and 5.7, we obtain a satisfying description of the metanilpotent groups with min-n that have **CD**.

Corollary 5.8. *Let G be a metanilpotent group satisfying min-n and let $A = \gamma_\infty(G)$. Then G is a* **CD***-group if and only if G/A is a nilpotent Černikov group whose finite residual is locally cyclic and A^{ab} has countably many G-submodules.*

It is easy to find examples of metanilpotent groups with min-n that are not **CD**-groups – and even some that are Černikov groups.

Example (ii). Consider the group $G = Q \ltimes A$ where $Q = \langle x \rangle$ has order 2 and $A = A_1 \times A_2$ with $A_i \simeq 2^\infty$. Let Q act on A via $a^x = a^{-1},\ a \in A$. Then $\gamma_\infty(G) = A$ has uncountably many Q-submodules, so G is not a **CD**-group by Corollary 5.8. On the other hand, if we make x act trivially on A_1 and invert in A_2, then G is a **CD**-group since A_1 and A_2 are not near isomorphic as Q-modules. Notice that G is a hypercentral group and A is not a **CD**-group.

Example (iii). In this example $A = A_1 \oplus A_2$ where $A_i \simeq 5^\infty$. Then A_i has an automorphism α of order 4. Let $Q = \langle x \rangle$ have order 4 and make A into a Q-module by defining $a_1 x = (a_1)\alpha$ and $(a_2)x = a_2\alpha^{-1}$, where $a_i \in A_i$. Set $G = Q \ltimes A$. Evidently $\gamma_\infty(G) = A$ and G is a **CD**-group since A_1 and A_2 are not near isomorphic as Q-modules. In this case G has trivial centre.

Example (iv). To obtain non-Černikov examples let Q be a group of type q^∞ and let A be the p-adically irreducible Q-module (where

$p \neq q$) derived from a simple Čarin $\mathbb{Z}_p Q$-module – see Example (i) above. Set $G = Q \ltimes A$; this is a metabelian group with min-n that is not a Černikov group. Evidently $\gamma_\infty(G) = A$ has only countably many submodules, so G is a **CD**-group. On the other hand, the group $Q \ltimes (A \oplus A)$ is not a **CD**-group because the Q-module $A \oplus A$ has uncountably many submodules.

From examples in (ii) and (iii) we can see that if a metanilpotent group with min-n is a **CD**-group, its finite residual need not have **CD**. It is more challenging to show that the Černikov residual need not inherit the property **CD**.

Example (v). There is a metabelian group with min-n which has **CD**, *but whose Černikov residual does not have* **CD**.

Let $Q = R \times \langle z \rangle$ where R is a 2^∞-group and $|z| = 3$. Let $A = A_1 \oplus A_2$ where A_i is the injective hull of the Čarin 3-module over R; thus R acts on A_i via the field multiplication. Make A into a Q-module via the actions

$$(a_1, a_2)r = (a_1 r, a_2 r),\ (a_1, a_2)z = (a_2, -a_1 - a_2),$$

where $r \in R$ and $a_i \in A_i$. Clearly A is R-artinian.

We prove that A is a 3-adically irreducible Q-module. If this is false, then A has a proper unbounded submodule B. First note that $B \not\geq A_1$; for otherwise we will also have $B \geq A_2$ and $B = A$. Factoring out by a suitable $A[3^m]$, we can assume that $B \cap A_1 = 0$. Since B is unbounded, $A = B \oplus A_1$ and $B \overset{R}{\simeq} A/A_1 \overset{R}{\simeq} A_2$. Let $1 \neq b \in B[3]$; then $\langle b \rangle \langle z \rangle$ is a finite elementary abelian 3-group and hence z fixes some non-trivial element of it. Since $B[3]$ is R-isomorphic with a simple Čarin module and $b \mapsto b(z-1)$ is an R-module endomorphism, we conclude that $B[3](z-1) = 0$.

Suppose that $B(z-1) \neq 0$ and write $B_i = B[3^i]$. There is a least $n \geq 2$ such that $B_n(z-1) \neq 0$. Since $B_{i+1}(z-1) \leq B_i$, we have $B_{n+1}(z-1)^3 \leq B_{n-1}(z-1) = 0$. Since $z^3 = 1$, it follows that

$$0 = B_{n+1}(z-1)^3 = B_{n+1}(3z(1-z)) = B_n(z-1),$$

a contradiction which shows that $B(z-1) = 0$. Let $(a_1, a_2) \in B$; then $(a_1, a_2) = (a_2, -a_1 - a_2)$, so that $3a_1 = 0 = 3a_2$. Hence $3B = 0$, a contradiction which proves A to be 3-adically irreducible.

In conclusion define $G = Q \ltimes A$, which is a metabelian group satisfying min-n. Then $A = \gamma_\infty(G)$ has **CD** as a G-module by

Theorem 5.7 and Theorem 5.1 shows that G is a **CD**-group. The finite residual of G is RA and the Černikov residual is A, neither of which is a **CD**-group. □

6. Groups whose subgroups are CD-groups.

As was mentioned in the introduction, soluble groups with **CD** are much harder to deal with than nilpotent groups. It is noteworthy that a periodic soluble **CD**-group can have abelian p-subgroups of infinite rank: for example the wreath product $\mathbb{Z}_p \; wr \; p^\infty$ is a **CD**-group, yet the base group is infinite elementary abelian. This observation suggests that one of the difficulties in dealing with soluble **CD**-groups is the failure of subgroup closure for this property. With this in mind, we strengthen our hypothesis in this section and consider groups all of whose subgroups have **CD**: this property will be denoted by **SCD**.

In general one cannot expect to say much about **SCD**-groups – after all Tarski p-groups have this property. However, we are able to characterize virtually soluble groups with **SCD** in terms of their abelian sections. By Lemma 2.4 it is sufficient to do this for soluble groups.

Theorem 6.1. *A soluble group G is an* **SCD**-*group if and only if it has finite abelian ranks and there are no factors in G of type $p^\infty \times p^\infty$ for any prime p.*

Proof. The necessity of the conditions follows from Corollary 4.4. Assume that the conditions hold in G. Since these are inherited by subgroups, it is enough to prove that G is a **CD**-group.

We argue by induction on the derived length d that G is a **CD**-group. Let $d > 1$ and put $A = G^{(d-1)}$. Then G/A is a **CD**-group, so it suffices to show that the set $\mathcal{S} = \{H < G \mid G = HA\}$ has a countable cofinal subset. By Corollary 4.4 the subgroup A has **CD**; let $\mathcal{X}$ be a countable dominating set of proper subgroups in A. For any $X \in \mathcal{X}$, the group A/X_G has a non-trivial G-invariant quotient which is either divisible or finite elementary abelian. Since G has no $p^\infty \times p^\infty$ factors, there is a G-invariant subgroup $B(X) < A$ such that $X_G \leq B(X)$ and $A/B(X)$ is isomorphic to $\mathbb{Q}$ or p^∞ or a finite elementary abelian p-group for some prime p. Let $\mathcal{C} = \{B(X) \mid X \in \mathcal{X}\}$, which is a countable set.

Let $H \in \mathcal{S}$. Then $D = H \cap A \lhd HA = G$ and $D < A$; hence $D \leq X$ for some $X \in \mathcal{X}$, whence $D \leq X_G \leq B = B(X)$. Now

$(HB) \cap A = (H \cap A)B = B$, so that $HB < G$ and $G/B = (HB/B) \ltimes (A/B)$.

It follows that the set of all subgroups K of G such that K/B is a complement of A/B in G/B for some $B \in \mathcal{C}$ is cofinal in $\mathcal{S}$. To complete the proof it is enough to show that for each $B \in \mathcal{C}$ there are only countably many such complements. We can factor G by such a B, so let $B = 1$. Hence A is finite or $A \simeq \mathbb{Q}$ or $A \simeq p^\infty$. We will show that A has countably many complements in G.

Let H be a complement of A in G and set $C = C_H(A) \lhd G$. We wish to show that $\mathrm{Der}(H, A)$ is countable. There is an exact sequence

$$0 \to \mathrm{Der}(H/C, A) \to \mathrm{Der}(H, A) \to \mathrm{Hom}(C, A),$$

so it is enough to prove that $\mathrm{Der}(H/C, A)$ and $\mathrm{Hom}(C, A)$ are countable. This is clear if A is finite, so assume it is infinite.

First suppose that $A \simeq \mathbb{Q}$. Then $\mathrm{Aut}(\mathbb{Q})$ is a direct product of cyclic groups and it has finite torsion subgroup, while H/C has finite abelian ranks. It follows that H/C is finitely generated. Therefore $\mathrm{Der}(H/C, A)$ is countable. Since $\mathrm{Hom}(C, A)$ is a finite dimensional $\mathbb{Q}$-space, it too is countable.

Now let A be a p^∞-group; then $\mathrm{Aut}(A)$ and hence H/C, is abelian and residually finite. There is a finitely generated subgroup X/C such that H/X is periodic. If $X \neq C$, then $C_A(X)$ is finite and thus $\mathrm{Der}(H/X, C_A(X))$ is finite. Also $\mathrm{Der}(X/C, A)$ is clearly countable, which implies that $\mathrm{Der}(H/C, A)$ is countable by the cohomology sequence. If $X = C$, then H/C is periodic and hence finite, since it is residually finite. Thus $\mathrm{Der}(H/C, A)$ is countable. Finally, consider $\mathrm{Hom}(C, A)$. There cannot be a surjective homomorphism $C \to A$; for if there were, there would be a p^∞- quotient C/N and then the factor CA/N would be of type $p^\infty \times p^\infty$. Consequently a homomorphism from C to A has finite image, from which it follow that $\mathrm{Hom}(C, A)$ is countable. □

It is worth noting that a soluble group whose abelian subgroups have **CD** need not have this property. Indeed, the nilpotent group constructed in [6: Theorem 3.1] has a quotient of type $p^\infty \times p^\infty$, so it does not satisfy **CD**, yet all its abelian subgroups have **CMS** and hence **CD**. On the other hand, a *periodic* soluble group whose abelian subgroups are **CD**-groups does in fact have **CD**. We will put this result in a more general setting. First recall that a *generalized* *radical* group is a group with an ascending series whose infinite factors are locally nilpotent.

Theorem 6.2. *Let G be a a periodic generalized radical group. Then the following conditions on G are equivalent:*

(i) G is an **SCD**-*group;*

(ii) all abelian subgroups of G are **CD**-*groups;*

(iii) for each prime p the Sylow p-subgroups of G are either finite or finite extensions of a p^∞-group.

Proof. Certainly (i) implies (ii). Assume that (ii) holds and let P be a Sylow p-subgroup of G. Then P is a locally finite p-group whose abelian subgroups have finite rank. Thus P is a Černikov group (see [7: Theorem 1.6.9]). In addition the finite residual of P can have rank at most 1, so (iii) follows.

Next assume that (iii) holds. Since the hypothesis is inherited by subgroups, it is enough to prove that G is a **CD**-group. The group G is locally finite and satisfies min-p for all primes p, so a theorem of Belyaev – see [7: Theorem 3.5.15] – implies it has a locally soluble subgroup of finite index. By Lemma 2.4 we may assume that G is locally soluble and hence is radical. By [7: Proposition 5.4.8] the group G is countable and it cannot contain a proper subgroup isomorphic to itself. Furthermore, for every prime p the quotient $G/O_{p'}(G)$ is a Černikov group [7: Corollary 2.5.13].

We claim that G has no proper subgroup H such that $G = HO_{p'}(G)$ for every prime p. For, let H be such a subgroup. By [7: Corollary 3.1.6], if P is a Sylow p-subgroup of H, then $H = PQ$ for some p'-subgroup Q of G, which implies that $G = P(QO_{p'}(G))$, whence it follows that P is a Sylow p-subgroup of G. In the terminology of [7] this means that H is a basic subgroup of G relative to the set of all primes. But then $H \simeq G$ by [7: Theorem 5.3.10], so that $H = G$, thus justifying the claim.

It follows that every proper subgroup fails to contain $O_{p'}(G)$ for some prime p. Finally, $G/O_{p'}(G)$ is a **CD**-group by Theorem 4.1 and it follows that G is a **CD**-group. □

There is some similarity between the structures of soluble **CMS**-groups and soluble **SCD**-groups. However, this does not extend to locally soluble groups. Indeed, while locally soluble **CMS**-groups are soluble by [6: Theorem 2.12], there are periodic locally nilpotent **SCD**-groups which are insoluble. For example, let F_p be a finite p-group of derived length d_p where the d_p are unbounded. Then $G = \mathrm{Dr}_p\ F_p$ is a periodic, residually finite, locally nilpotent group which is an **SCD**-group by Theorem 6.2. Of course G is insoluble.

Concluding remarks.

The proof of Theorem 6.2 shows that a locally finite group with **CD** satisfies the equivalent conditions (ii) and (iii) in the theorem. It is tempting to conjecture that the converse is true, i.e., (iii) implies the property **CD** for locally finite groups. However, this is disproved by the existence of uncountable locally finite groups with finite Sylow subgroups [7: 5.4.11]: recall that all **CD**-groups are countable. Of course one can still ask whether a countable locally finite group G satisfying min-p and having no subgroups of type $p^\infty \times p^\infty$ for all p must have **CD** and hence **SCD**. This question remains open. It is not hard to see that it would be sufficient to prove it for countable locally finite groups with finite Sylow subgroups.

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