Finiteness conditions on characteristic closures and cores of subgroups

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ABSTRACT. We characterise groups in which every abelian subgroup has finite index in its characteristic closure. In a group with this property *every* subgroup H has finite index in its characteristic closure and there is an upper bound for this index over all subgroups H of G. For every prime p we construct groups G with this property that are infinite nilpotent p-groups of class 2 and exponent p^2 in which $G' = Z(G)$ is finite and $\operatorname{Aut} G$ acts trivially on G/G'.

We also characterise abelian groups with the dual property that every subgroup has finite index over its characteristic core.

In 1955, in his paper [5], B.H. Neumann began a systematic study of finiteness conditions in group theory defined by restrictions on the conjugacy classes of subgroups. Among other results he proved that every subgroup H of a group G has finite index in its normal closure H^G if and only if G' is finite; as Tomkinson pointed out in [10] the same conclusion may be drawn if the finiteness of $|H^G : H|$ is required only for abelian subgroups. Several results of a similar nature, and many generalizations, have appeared in the literature. A dual condition was considered in [1]: a group G is called core-finite (or CF) if H/H_G is finite for all $H \leq G$; as usual H_G denotes the normal core of H in G. The class of CF-groups is rather more elusive than the class considered by B.H. Neumann. In fact the existence of many infinite groups all of whose proper subgroups are finite, like Tarski groups, makes clear that a simple description of CF-groups is out of range. Nonetheless, it was proved that CF-groups are abelian-by-finite under rather general hypotheses: for instance if they have no periodic quotient which is not locally finite (see [8]).

Here we shall consider another variation along the same lines. Instead of considering normal closures and cores of subgroups we deal with characteristic closures and cores. In other words, we deal with properties related to orbits under the action of the full automorphism group $\operatorname{Aut} G$ of a group G on subgroups, rather than (G-)conjugacy classes. This can be compared, for instance, to [4, 7], where groups G with finite ($\operatorname{Aut} G$)-orbits of subgroups are studied.

For every subgroup H of the group G let $H^\sharp$ and $H_\flat$ be the characteristic closure and the characteristic core of H in G, respectively, that is: $H^\sharp$ is the smallest characteristic subgroup of G containing H and $H_\flat$ is the largest characteristic subgroup of G contained in H. The absence of a reference to G in this notation will never cause ambiguity in what follows. We say that G has the property $\mathcal{P}^\sharp$ (respectively $\mathcal{P}^\flat$) if $|H^\sharp : H|$ (respectively $|H : H_\flat|$) is finite for every subgroup H of G. We define the property $\mathcal{P}^\sharp_{\mathrm{a}}$ by weakening the condition and requiring only that $|H^\sharp : H|$ is finite for all abelian subgroups H of G. We will also say that G satisfies $\mathcal{P}^\sharp$ (respectively $\mathcal{P}^\flat$) boundedly if there is a (finite) upper bound for $|H^\sharp : H|$ (respectively $|H : H_\flat|$), where H ranges over all subgroups of G.

It is clear that $\mathcal{P}^\sharp_{\mathrm{a}}$-groups have the property considered by B.H. Neumann and Tomkinson, hence they are finite-by-abelian, while $\mathcal{P}^\flat$-groups are core-finite, hence they are abelian-by-finite under suitable hypotheses. However, even in these classes it is not immediately obvious how to set apart groups which are $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$ from the others. For instance, the well-known examples of complicated abelian torsion-free groups in which the inversion map is the only nontrivial automorphism and so all subgroups are characteristic suggest that there is no hope of obtaining a completely explicit description of the groups in these classes. We shall classify abelian groups in $\mathcal{P}^\sharp$ or in $\mathcal{P}^\flat$ up to the description of the just-mentioned torsion-free groups (see Theorems 2.2 and 2.6). It turns out that the two properties are equivalent for periodic abelian groups, but in the nonperiodic case $\mathcal{P}^\flat$ is a stronger property than $\mathcal{P}^\sharp$. Another consequence of these results is that every abelian group satisfying either of the two properties satisfies it boundedly.

It is worth remarking that, unlike what happens in the abelian case, nonabelian $\mathcal{P}^\flat$-groups need not satisfy $\mathcal{P}^\sharp$. Indeed, if $G = U \rtimes \langle x \rangle$, where $|\operatorname{Aut} U| = 2$ and U is infinite, x has order 2 and $u^x = u^{-1}$ for all $u \in U$ (for instance, G might be the infinite dihedral group) then $G \in \mathcal{P}^\flat$ but $G' = U^2$ is infinite, hence $G \notin \mathcal{P}^\sharp$. Apart from this observation, we shall not here carry on the study of $\mathcal{P}^\flat$-groups to the nonabelian case, but we extend the results on abelian $\mathcal{P}^\sharp$-groups to arbitrary groups. We shall prove the following:

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Theorem. *For a group G the following are equivalent properties:*
(i) *G satisfies $\mathcal{P}^\sharp_{\mathrm{a}}$;*
(ii) *G satisfies $\mathcal{P}^\sharp$;*
(iii) *G satisfies $\mathcal{P}^\sharp$ boundedly.*

The proof will follow from a characterisation of $\mathcal{P}^\sharp_{\mathrm{a}}$-groups, which is split into Theorems 4.5 and 4.9. A remarkable feature is the following. In the case of the property considered by Neumann and Tomkinson, the fact that in a group G such that $|H^G : H|$ is finite for all $H \leq G$ these indices are boundedly finite follows from the existence of a finite subgroup (namely G') such that all subgroups containing it are normal—in this sense we can say that the only groups satisfying the property are those satisfying it trivially. The case of property $\mathcal{P}^\sharp$ is different in that there exist $\mathcal{P}^\sharp$-groups G in which no finite subgroup plays a rôle corresponding to that of G', since every finite subgroup is contained in a noncharacteristic one. It is noteworthy that there is nonetheless an upper bound for the indices considered.

Some interesting examples of periodic $\mathcal{P}^\sharp$-groups are those constructed in Theorem 3.1: they are nilpotent p-groups G of class 2 and exponent p^2 (p an arbitrary prime) such that $G' = Z(G)$ is finite and $[G, \operatorname{Aut} G] \leq G'$, that is, the only automorphisms of G are the central ones and $\operatorname{Aut} G$ is canonically isomorphic to $\operatorname{Hom}(G/G', G')$. Thus all subgroups of G containing G' are characteristic, so that G satisfies $\mathcal{P}^\sharp$ (boundedly) trivially, in the sense of the previous paragraph.

Besides what has been introduced in the previous paragraphs, notation is standard. Note that $\pi(G)$ is the set of all primes p such that the group G has an element of order p, and that 'H char G' means that H is a characteristic subgroup of G.

1. Some preliminary lemmas

Lemma 1.1. *Let G be a group with a direct product $\mathrm{Dr}_{i\in I} H_i$ as a subgroup, such that either*
(a) *G satisfies $\mathcal{P}^\sharp_{\mathrm{a}}$ and for all $i \in I$ the subgroup H_i is a direct factor of G; or*
(b) *G satisfies $\mathcal{P}^\flat$ and is abelian-by-finite, and for all finite subsets F of I the subgroup $\langle H_i \mid i \in F\rangle$ is a direct factor of G.*

Then, for all but finitely many $i \in I$, every subgroup of H_i is characteristic in H_i.

Proof. Let J be the set of all $i \in I$ such that H_i has a cyclic subgroup K_i which is not characteristic in H_i, and therefore not in G, because H_i is a direct factor of G. Let $K = \mathrm{Dr}_{i\in J} K_i$. For all $i \in J$ we have $K_i < (K_i)^\sharp \cap H_i$, hence $|K^\sharp : K| \geq |J|$ and J is finite if (a) holds. Now suppose that (b) holds. Since G is abelian-by-finite all but finitely many of the subgroups H_i are abelian, hence we may assume that $\langle H_i \mid i \in I\rangle$ is abelian. Let $n = |K/K_\flat|$ and suppose that J is infinite. Let F be an n-element subset of J, so that $H := \mathrm{Dr}_{i\in F} H_i$ is a direct factor of G and G has an automorphism α such that $H_i^\alpha = H_i$ and $K_i^\alpha \nleq K_i$ for all $i \in F$, and $[G, \alpha] \leq H$. If $L = \langle K_i \mid i \in F\rangle$ then $L^\alpha \cap K = L^\alpha \cap L$ and so $|L^\alpha K/K| = |L^\alpha : L^\alpha \cap L| > n$. On the other hand $|K^\alpha K/K| = |K^\alpha/K^\alpha \cap K| \leq |K^\alpha/K_\flat| = |K/K_\flat| = n$. This contradiction shows that J must be finite. □

We recall from the introduction that the hypothesis that G be abelian-by-finite is much weaker than it appears at first sight, since groups in $\mathcal{P}^\flat$ are core-finite and then they are abelian-by-finite under very general hypotheses. The conclusion of the previous lemma prompts the following remark. It is certainly well-known and we include it only for the sake of further reference.

Lemma 1.2. *Let G be a group in which all subgroups are characteristic. Then G is abelian and its torsion subgroup is locally cyclic.*

Proof. That G is abelian is obvious, since hamiltonian groups have non-characteristic subgroups. If P is a primary component of $T = \operatorname{tor} G$ such that $\operatorname{rk} P > 1$ then it is possible to decompose G as $U \times V \times W$ where $U, V \leq P$ and $\operatorname{rk}(U) = \operatorname{rk}(V) = 1$; the fact that at least one of $\operatorname{Hom}(U, V)$ and $\operatorname{Hom}(V, U)$ is non-zero makes it impossible that both U and V are characteristic in G. □

Lemma 1.3. *Let G be a group and G/N a torsion-free abelian quotient of G. If G satisfies either $\mathcal{P}^\sharp_{\mathrm{a}}$ or $\mathcal{P}^\flat$, then every subgroup of G containing N is characteristic in G.*

Proof. Let H/N be a cyclic subgroup of G/N. It will be enough to show that H is characteristic in G. Suppose first that G satisfies $\mathcal{P}^\sharp_{\mathrm{a}}$. Then $N^\sharp/N$ is periodic, hence trivial, and N char G. We have $H = N\langle h\rangle$ for some h, hence $H^\sharp = N\langle h\rangle^\sharp$, so that $H^\sharp/H$ is finite. Since G/N is torsion-free it follows that $H^\sharp/N$ is cyclic. We deduce easily that H is characteristic. In the dual situation, when G satisfies $\mathcal{P}^\flat$, observe first that N is characteristic in G, because $N/N_\flat = \operatorname{tor}(G/N_\flat)$, and let $K/H_\flat = \operatorname{tor}(G/H_\flat)$, thus K/N is the pure closure of $H_\flat/N$ in G/N. Then K/N has rank 1, so that all subgroups of its periodic image $K/H_\flat$ are characteristic. Since K is characteristic in G and $H_\flat \leq H \leq K$ it follows that H is characteristic in G. □

Lemma 1.4. *Let G be a group and suppose that F is a torsion-free characteristic subgroup of G. For every odd integer n, if $Z(G)$ has an element of order n then $F \leq G'G^n$.*

Proof. It will be enough to show that F is contained in every normal subgroup N of G such that G/N is cyclic of prime-power odd order and $Z(G)$ has an element x of the same order $|G/N|$. Fix such N and x, and let $y \in G$ be such that $G = N\langle y\rangle$. We shall construct an automorphism α of G that centralises N and is such that $1 \neq [g, \alpha] \in \langle x\rangle$ for every $g \in G \setminus N$. It will follow that the characteristic closure of every subgroup of G not contained in N contains some nontrivial torsion elements, hence $F \leq N$.

If $G = N\langle x\rangle$ then $G = N \times \langle x\rangle$, and we can define α by letting $x^\alpha = x^{-1}$ (and $[N, \alpha] = 1$). Otherwise, $N\langle x\rangle < G$. Let ε be the endomorphism of G defined by $y^\varepsilon = x$ and $N^\varepsilon = 1$, and let $\alpha = 1 + \varepsilon$. Now α induces the identity on N and an automorphism on G/N (because $N\langle x\rangle < G$), hence $\alpha \in \operatorname{Aut} G$. Thus α satisfies the required properties, and this completes the proof. □

If n is even the proof breaks down in the case when $G = N\langle x\rangle$, because the automorphism α constructed in the proof centralises the subgroup of $\langle x\rangle$ of order 2 in this case. This is a genuine exception: let $G = \langle a\rangle \times \langle x\rangle$, where a has infinite order and x has order 2. Then $\langle a^2x\rangle$ is characteristic in G but not contained in $\langle a\rangle$.

Lemma 1.5. *Let $G = X \times Y$ be a group satisfying $\mathcal{P}^\flat$, where Y is abelian. Then $\operatorname{Hom}(X, Y)$ has finite exponent and it is finite if Y is finite.*

Proof. Every $\varepsilon \in \operatorname{Hom}(X, Y)$ yields an automorphism α of G defined by $[Y, \alpha] = 1$ and $x^\alpha = xx^\varepsilon$ for all $x \in X$; clearly $X \cap X^\alpha = \ker \varepsilon$. This shows that $\operatorname{Hom}(X, Y) \simeq \operatorname{Hom}(X/X_\flat, Y)$. The result follows. □

The centre of a group with the properties that we are considering is certainly subject to strong restrictions. One is the following:

Lemma 1.6. *Let G be a group satisfying $\mathcal{P}^\sharp_{\mathrm{a}}$ or $\mathcal{P}^\flat$, and assume that G is abelian-by-finite if it satisfies $\mathcal{P}^\flat$. Then $Z(G)$ does not contain any subgroup isomorphic to the direct product of two isomorphic Prüfer groups.*

Proof. Suppose that $C = P \times Q \leq Z(G)$ and $P \simeq Q \simeq \mathcal{C}_{p^\infty}$ for some prime p. As G is either abelian-by-finite or finite-by-abelian C has a near-complement in G (that is, a subgroup X such that both $C \cap X$ and $|G : CX|$ are finite). This implies that the cohomology class of the natural extension $C \hookrightarrow G \twoheadrightarrow G/C$ has finite order, necessarily a power q of p (see, for instance, [6], (2.5)). As a consequence, every automorphism of C of the form $1 + q\varepsilon$, where $\varepsilon \in \operatorname{End} C$, can be extended to an automorphism of G acting trivially on G/C (see [6], (4.3)). Let $\alpha\colon P \to Q$ be an isomorphism. Let $\varepsilon \in \operatorname{End} C$ be defined by $x^\varepsilon = x^\alpha$ for all $x \in P$ and $Q^\varepsilon = 1$. Then $\beta = 1 + q\varepsilon \in \operatorname{Aut} C$, and β can be extended to an automorphism of G. Since $[x, \beta] = x^{q\alpha}$ for all $x \in P$ we obtain that $Q \leq P^\sharp$ and $|P_\flat| \leq q$. This is a contradiction because G satisfies either $\mathcal{P}^\sharp_{\mathrm{a}}$ or $\mathcal{P}^\flat$. □

2. Abelian groups with $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$

The preparation in the previous section will help us in describing abelian groups satisfing $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$. We consider the periodic case first.

Lemma 2.1. *Let G be an abelian group satisfying either $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$. Set $T = \operatorname{tor} G$. Then:*
- (i) *for all primes p the primary component T_p of T is either finite or Prüfer-by-finite;*
- (ii) *for all but finitely many primes p the primary component T_p of T is locally cyclic.*

Proof. If H is a direct factor of a primary component of T and $\operatorname{rk}(H) > 1$ then H contains a direct factor of G of rank 2. It follows that if either T has infinitely many primary components of rank at least 2, or at least one of them has infinite rank, then it is possible to construct a subgroup of G that is the direct product of infinitely many rank-2 direct factors of G, also satisfying the extra condition in Lemma 1.1 (b). Thus Lemmas 1.1 and 1.2 yield a contradiction. Hence T has finite rank and all but finitely many of its many primary components have rank 1 at most—thus we obtain (ii). By Lemma 1.6, G cannot contain two isomorphic Prüfer subgroups, hence also (i) holds. □

Theorem 2.2. *For a periodic abelian group G the following conditions are equivalent:*
- (a) *G satisfies $\mathcal{P}^\sharp$;*
- (b) *G satisfies $\mathcal{P}^\flat$;*
- (c) *G has characteristic subgroups M and N such that both M and G/N are finite and every subgroup of G which either contains M or is contained in N is characteristic;*
- (d) *$G = F \times C$, where F is finite and C is locally cyclic.*

Proof. Each of (a) and (b) implies (d), by Lemma 2.1, and (d) implies (c): let $M = G[n]$, where $n = \exp F$, and $N = G^n$. Finally, it is obvious that (c) implies both (a) and (b). □

Now we consider the nonperiodic case. We begin with a very elementary observation.

Lemma 2.3. *Let G be an abelian group and let x be an element of infinite order in G. Let $\{x_i \mid i \in I\}$ be a set of elements of G such that for every $i \in I$ there exists $\lambda_i \in \mathbb{N}$ such that $x_i^{\lambda_i} = x$, and λ_i and λ_j are coprime whenever $i \neq j$. Then $\langle x_i \mid i \in I\rangle$ is torsion-free.*

Proof. Let $X = \langle x_i \mid i \in I\rangle$, let p be a prime and let X_p be the p-component of X. Since the integers λ_i are pairwise coprime $X_p\langle x\rangle/\langle x\rangle \leq \langle x_i\rangle/\langle x\rangle$, for some $i \in I$. As $\langle x_i\rangle$ is torsion-free, $X_p = 1$. □

Lemma 2.4. *Let G be a nonperiodic abelian group satisfying either $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$. Then G has a free abelian characteristic subgroup F such that G/F is periodic.*

Proof. There exists a free abelian subgroup B of G such that G/B is periodic. If G satisfies $\mathcal{P}^\flat$ let $F = B_\flat$. If G satisfies $\mathcal{P}^\sharp$ then $B^\sharp/B$ is finite, of order t, say, and we set $F = (B^\sharp)^t$. In either case F has the properties required. □

Lemma 2.5. *Let G be a non-periodic abelian group satisfying either $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$, and let $T = \operatorname{tor} G$. Then:*

(i) *for every $n \in \mathbb{N}$ the group G/T has a quotient of exponent n;*
(ii) *every primary component of T is finite.*

Proof. Let p be a prime. Then $G = T_p \times U$ for some subgroup U, as follows from Lemma 2.1. If $U^p = U$ then let α be the automorphism of G defined by $[T_p, \alpha] = 1$ and $u^\alpha = u^p$ for all $u \in U$. Then α induces on $\bar{G} := G/T$ an automorphism which does not fix all subgroups of $\bar{G}$. This is a contradiction, by Lemma 1.3. Hence $U^p < U$. Now, U is an extension of T/T_p—which is p-divisible—by $\bar{G}$, hence the latter cannot be p-divisible. Since $\bar{G}$ is torsion-free and so $\bar{G} \simeq (\bar{G})^{p^\lambda}$ for every positive integer λ, it follows that (i) holds if n is assumed to be a power of p. The general case is an easy consequence.

Assume now that G has a Prüfer subgroup, say $P \simeq \mathcal{C}_{p^\infty}$. Then $G = T_p \times U$, as above, and since U has p-quotients of arbitrarily high exponent it follows that, for every $n \in \mathbb{N}$, there is a homomorphism $\varepsilon_n \colon U \to P$ whose image has order p^n and hence an automorphism α_n of G such that $[T_p, \alpha_n] = 1$ and $u^{\alpha_n} = uu^{\varepsilon_n}$ for all $u \in U$. Then $U/U_\flat$ is infinite, and $P \leq U^\sharp$, because $P = \bigcup_{n\in\mathbb{N}}[U, \alpha_n]$. This yields a contradiction which, together with Lemma 2.1, proves (ii). □

Theorem 2.6. *Let G be a non-periodic abelian group. Then:*

(a) *G satisfies $\mathcal{P}^\sharp$ if and only if $G = T \times U$ for some finite subgroup T and some torsion-free subgroup U such that $|\operatorname{Aut} U| = 2$.*
(b) *G satisfies $\mathcal{P}^\flat$ if and only if $G = T \times U$ as in* (a) *with the extra condition that U/U^n is finite, where $n = \exp T$.*

Proof. Suppose that G satisfies either $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$. Let F be a torsion-free characteristic subgroup of G such that G/F is periodic (see Lemma 2.4), and let x be any nontrivial element of F. Assume that $T = \operatorname{tor} G$ is infinite. Then, by Lemmas 2.1 and 2.5, the set of all odd primes p such that T_p is cyclic and nontrivial is infinite. Write this set as $\{p_i \mid i \in \mathbb{N}\}$, where $p_i \neq p_j$ if $i \neq j$. For every $i \in \mathbb{N}$ let $\langle a_i\rangle = T_{p_i}$and $q_i = |T_{p_i}|$, then $G = \langle a_i\rangle \times G_i$ for some $G_i \leq G$. Because of Lemma 1.4 we have $F \leq G^{q_i} = G_i^{q_i}$, so there exists $x_i \in G_i$ such that $x_i^{q_i} = x$. Let $H = \langle a_ix_i \mid i \in \mathbb{N}\rangle$. We have $(a_ix_i)^{q_i} = x$ for all $i \in \mathbb{N}$; it follows from Lemma 2.3 that H is torsion-free. For all $i \in \mathbb{N}$ consider the automorphism α_i defined by $a_i^{\alpha_i} = a_i^{-1}$ and $[G_i, \alpha_i] = 1$. Then $[a_ix_i, \alpha_i] = a_i^{-2}$ generates $\langle a_i\rangle$, so $a_i \in HH^{\alpha_i}$; as H and H^{α_i} are torsion-free it follows that q_i divides both $|HH^{\alpha_i}/H|$ and $|H/H \cap H^{\alpha_i}| = |HH^{\alpha_i}/H^{\alpha_i}|$. Therefore both $H^\sharp/H$ and $H/H_\flat$ are infinite; this contradiction shows that T is finite. Now G splits over T, so $G = T \times U$ for some torsion-free U. Since every automorphism of U extends to G Lemma 1.3 shows that $|\operatorname{Aut} U| = 2$. If G has $\mathcal{P}^\flat$ then $\operatorname{Hom}(U, T)$ is finite by Lemma 1.5, hence $U/U^{\exp T}$ is finite. Thus G must have the structure described in (a) or in (b), according to which of $\mathcal{P}^\sharp$ or $\mathcal{P}^\flat$ it satisfies.

Conversely, let $G = T \times U$ as in (a). Then every subgroup of G containing T is characteristic in G, hence $H^\sharp \leq HT$ for all $H \leq G$, thus showing that G satisfies $\mathcal{P}^\sharp$. Also, if $H \leq G$ and U/U^n is finite, where $n = \exp T$, then $H/H \cap U^n$ is finite. Now, $U^n = G^n$ char G, and $U^n \simeq U$. Therefore every subgroup of U^n is characteristic in U^n and hence also in G. This proves (b). □

It is not hard to see that the groups described in (b) have finitely many automorphisms only. Therefore if G is a nonperiodic abelian group satisfying $\mathcal{P}^\flat$ then $\operatorname{Aut} G$ is finite; note that the converse does not hold: there are torsion-free abelian groups with finitely many, but more than two, automorphisms; a detailed discussion of such groups is in [3], Section 116.

There exist torsion-free groups U such that $|\operatorname{Aut} U| = 2$ and U/U^p is infinite for every prime p; for instance they can be constructed as in [3], Lemma 88.3 (but also see [7], p. 281), by starting from an infinite rigid system of groups with two automorphisms only and extending their direct product with an elementary abelian group without elements of order 2. This shows that properties $\mathcal{P}^\flat$ and $\mathcal{P}^\sharp$ are not equivalent for mixed abelian groups, unlike what happens in the periodic case. What remains true in comparison with Theorem 2.2 is that an abelian group G satisfies $\mathcal{P}^\sharp$ (respectively $\mathcal{P}^\flat$) if and only if it has a subgroup N which is finite (respectively of finite index) and is such that every subgroup of G containing (respectively contained in) it is characteristic.

As a consequence, we have:

Corollary 2.7. *Let G be an abelian group satisfying $\mathcal{P}^\sharp$ (respectively $\mathcal{P}^\flat$). Then G satisfies $\mathcal{P}^\sharp$ (respectively $\mathcal{P}^\flat$) boundedly.*

We shall see that this Corollary holds true for nonabelian groups as well.

3. p-groups with few automorphisms

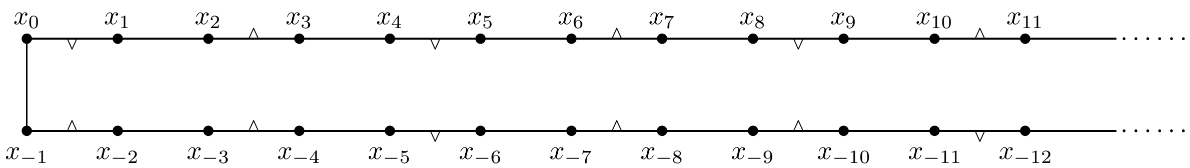
For every prime p, every infinite nilpotent p-group G must have $2^{|G|}$ automorphisms (see [2]). In the case when G is not abelian and has finite exponent this is an immediate consequence of the fact that the group of all automorphisms of G acting trivially on $G/G' \cap Z(G)$ is isomorphic to $\mathrm{Hom}\,(G/G', G' \cap Z(G))$. In this section we shall construct a family of examples of p-groups of class 2 and finite exponent having no automorphisms other than these.

Theorem 3.1. *For every prime p there exist countably infinite p-groups G of nilpotency class 2 and exponent p^2 such that $G' = Z(G)$ is elementary abelian of rank 3 and $\mathrm{Aut}\,G$ acts trivially on G' and on G/G'.*

These groups are relevant in our context because they obviously satify $\mathcal{P}^\sharp$ (boundedly): in such a group G all subgroups containing G' are characteristic and $H^\sharp \leq HG'$ for all $H \leq G$. Examples sharing some of these properties are constructed in [7], Proposition 3.

Note that easier examples of nonabelian groups satisfying $\mathcal{P}^\sharp$ can be obtained starting from abelian groups with the same property. In fact, it follows from Theorem 2.2 and Theorem 2.6 that every extension of a finite group by an abelian group with $\mathcal{P}^\sharp$ still satisfies $\mathcal{P}^\sharp$. As is clear, the groups in Theorem 3.1 are not of this kind.

To construct a group G as in Theorem 3.1 we start with the 2-regular tree whose edges are coloured by three colours, indicated here by flags (or their absence):



Thus, every second edge has the no-flag colour; the flag-down colour appears at every second flagged edge on one of the two branches rooted on the vertex x_0, and appears at every third flagged edge on the other branch. This coloured tree has no nontrivial automorphism; more than that, it is not isomorphic to any tree obtained by permuting the colours nontrivially and then applying an automorphism of the underlying (non-coloured) tree.

Fix a prime p and consider the variety $\mathcal{V}$ consisting of all groups V such that $V'V^p \leq Z(V)$ and $V^{p^2} = 1$. We use the coloured tree just described to define the commutator relations of a group $G \in \mathcal{V}$ generated by elements a, b, c and x_i, where i ranges over the integers, and with the following extra relations: for all i, $j \in \mathbb{Z}$,

$$x_i^p = w_i; \qquad [x_i, x_j] = 1, \quad \text{if } |i-j| \neq 1; \qquad [x_i, x_{i+1}] = \begin{cases} a, & \text{if } i \in A \\ b, & \text{if } i \in B \\ c, & \text{if } i \in C \end{cases}, \tag{Rel}$$

where each w_i is a word in a, b, c and $\{A, B, C\}$ is the partition of $\mathbb{Z}$ defined thus: C is the set of all negative multiples of 6 and all nonnegative multiples of 4; B is the set of all remaining even integers and $A = 1 + 2\mathbb{Z}$ is the set of all odd integers. Thus $[x_i, x_{i+1}]$ is a, b or c according to whether the edge joining x_i and x_{i+1} has the no-flag colour, the flag-up colour or the flag-down colour respectively.

It is clear that $Z := \langle a, b, c \rangle = Z(G) = G'$; also, G can be realised easily as a quotient of a free group in $\mathcal{V}$, thus showing that the elements x_i are $\mathbb{Z}_p$-independent modulo Z and Z is elementary abelian of rank 3.

Our first claim is that, in such a group G, all subgroups $\langle x_i \rangle Z$ are characteristic. Thereafter we will see that for some suitable choice of the power relations in (Rel) (that is, a choice of the words w_i) G has the property required by Theorem 3.1.

Throughout this section we consider fixed the notation used so far for G and later for its distinguished subgroups and elements as introduced. Every $g \in G$ can be written (uniquely) as

$$g = z \prod_{i \in \mathbb{Z}} x_i^{\lambda_i} \tag{$*$}$$

for some $z \in Z$ and integers $\lambda_i \in \{0, 1, 2, \dots, p-1\}$, where all but finitely many of the λ_i are zero and the factors in the product are ordered according to the standard ordering of the integers. We shall often use this notation too.

Lemma 3.2. *If $g \in G \setminus Z$ then $a \in [g, G]$.*

Proof. Suppose that $g = z \prod_{i\in\mathbb{Z}} x_i^{\lambda_i}$ as in $(*)$, and let $r = \max\{i \in \mathbb{Z} \mid \lambda_i \neq 0\}$. If r is odd, then $[g, x_{r+1}] = [x_r^{\lambda_r}, x_{r+1}] = a^{\lambda_r}$, so we may assume that r is even. Let s be the least even integer such that $\lambda_s \neq 0$. Then $\lambda_{s-2} = 0$, hence $[x_{s-1}, g] = [x_{s-1}, x_s^{\lambda_s}] = a^{\lambda_s}$ and the proof is complete. □

Once this has been proved it is also easy to see that, still with reference to the notation in $(*)$, we have $b \in [g, G]$ (respectively $c \in [g, G]$) if and only if $\lambda_i \neq 0$ or $\lambda_{i+1} \neq 0$ for some $i \in B$ (respectively $i \in C$). For, if $j \in \mathbb{Z}$ then $[g, x_j] = [x_{j-1}, x_j]^{\lambda_{j-1}}[x_{j+1}, x_j]^{\lambda_{j+1}}$, and one of $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ is a (which is in $[g, G]$ anyway as long as $g \notin Z$), the other is either b or c depending on the value of j. By considering the various possible cases the claim is established. It follows that:

$$X := \{g \in G \mid [g, G] = \langle a, b\rangle\} \cup Z = \langle x_i, x_{i+1} \mid i \in B\rangle Z$$
$$Y := \{g \in G \mid [g, G] = \langle a, c\rangle\} \cup Z = \langle x_i, x_{i+1} \mid i \in C\rangle Z$$

and

$$[g, G] = Z \quad \text{for all } g \in G \setminus (X \cup Y).$$

Therefore

$$\langle a\rangle = \bigcap_{g\in G\setminus Z} [g, G] \ \text{char}\ G.$$

Also, an automorphism of G can either fix X and Y or interchange them. But $X' = \langle a, b\rangle$, while $Y' = \langle c\rangle$, so X and Y are not isomorphic. Hence X and Y are also characteristic in G, and the same holds for $\langle c\rangle = Y'$.

For all $i \in \mathbb{Z}$ let $L_i = \langle x_j \mid i \geq j \in \mathbb{Z}\rangle Z$ and $R_i = \langle x_j \mid i \leq j \in \mathbb{Z}\rangle Z$. We shall show that these subgroups are characteristic in G. Observe first that, for all $i \in \mathbb{Z}$,

$$L_{i-2} = C_G(R_i) \qquad \text{and} \qquad R_{i+2} = C_G(L_i); \qquad\qquad \text{also:} \qquad C_G(x_i) = L_{i-2}\langle x_i\rangle R_{i+2}.$$

Another pair of relevant centralizers, which certainly are characteristic, are

$$C_G(Y) = \langle x_i, x_{i+1} \mid 0 > i \equiv 3 \pmod 6\rangle Z \qquad \text{and}$$
$$C_G\big(C_G(Y)\big) = \langle x_i, x_{i+1} \mid 0 > i \equiv 0 \pmod 6\rangle R_0 = (Y \cap L_{-1})R_0.$$

These equalities are checked as follows: $C_G(Y)$ is the intersection of the centralizers $L_{i-2}R_{i+3}$ of all subgroups $\langle x_i, x_{i+1}\rangle$ where i ranges over C; this explains the first equality, the second one is proved similarly. Now

$$X^+ := X \cap R_0 = \langle x_i, x_{i+1} \mid 0 < i \in B\rangle Z = X \cap C_G\big(C_G(Y)\big) \ \text{char}\ G,$$

and so, by arguing as for $C_G(Y)$, we arrive at a key step:

$$L_0 = C_G(X^+) \ \text{char}\ G.$$

Lemma 3.3. *Let (u, U) be one of (a, A), (b, B) and (c, C), and let $i \in U$. Then each of L_i and R_{i+1} is the centralizer of the other modulo $\langle u\rangle$ in G:*

i) $\{g \in G \mid [g, R_{i+1}] \leq \langle u\rangle\} = L_i$;
ii) $\{g \in G \mid [g, L_i] \leq \langle u\rangle\} = R_{i+1}$;

Proof. Let $g = z \prod_{j\in\mathbb{Z}} x_j^{\lambda_j}$, as in $(*)$, and suppose that $[g, R_{i+1}] \leq \langle u\rangle$. Let $r = \max\{j \in \mathbb{Z} \mid \lambda_j \neq 0\}$. Assume, for a contradiction, that $r > i$. Then $x_{r+1} \in R_{i+1}$ and $[g, x_{r+1}] = [x_r, x_{r+1}]^{\lambda_r} \in \langle u\rangle$, hence $r \in U$. Since $i \in U$ and U does not contain consecutive integers, this implies that $r \neq i+1$ and so $x_{r-1} \in R_{i+1}$; moreover, $v := [x_{r-1}, x_r] \neq u$. Then $[g, x_{r-1}] = [x_{r-2}, x_{r-1}]^{\lambda_{r-2}} v^{-\lambda_r} \in \langle u\rangle$. However, this is impossible because v is linearly independent (over $\mathbb{Z}_p$) from u and $[x_{r-2}, x_{r-1}]$. This shows that $r \leq i$, that is, $g \in L_i$. On the other hand L_i centralises R_{i+1} modulo $\langle u\rangle$, hence (i) is proved. The proof of (ii) is dual. □

Lemma 3.4. *For all nonnegative integers i both L_i and R_{i+2} are characteristic in G. Therefore $\langle x_i\rangle Z$ char G for all integers $i \geq 2$.*

Proof. The second half of the statement follows from the first since $\langle x_i\rangle Z = L_i \cap R_i$. To prove the first half, suppose that L_i char G for some nonnegative, even integer i. Then $R_{i+2} = C_G(L_i)$ char G. Since $\langle a\rangle$ char G and $i + 1 \in A$ Lemma 3.3 shows that L_{i+1}, the centralizer of R_{i+2} modulo $\langle a\rangle$ in G, is also characteristic. Now, let K be that one of X and Y such that $x_i \in K$, and let $K^* = K \cap R_{i+2}$. Then $x_{i+4} \in K^* \leq R_{i+4}$ (we are using here the fact that $i \geq 0$) and one can see that $C_G(K^*) = L_{i+2}$. Therefore L_{i+2} char G. Since we already know that L_0 char G this is enough to prove the Lemma. □

A consequence of Lemma 3.4 is that $\langle b\rangle$ char G, since $\langle b\rangle = \big[\langle x_i\rangle Z, \langle x_{i+1}\rangle Z\big]$ for any positive $i \in B$. This allows us to use Lemma 3.3 in its full strength.

Lemma 3.5. *For all integers i, the subgroup $\langle x_i\rangle Z$ is characteristic in G.*

Proof. Let $i \in \mathbb{Z}$ and suppose that L_i char G. Then R_{i+1} char G, because of Lemma 3.3 and the fact that each of $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$ is characteristic in G. Therefore $L_{i-i} = C_G(R_{i+1})$ char G. By an obvious induction argument it follows that all subgroups L_i are characteristic, and the same is true of all $R_i = C_G(L_{i-2})$ and $\langle x_i\rangle Z = L_i \cap R_i$. □

To complete the proof of Theorem 3.1 we have to make sure that it is possible to choose the words w_i in (Rel) in such a way that $[\operatorname{Aut} G, G] \le Z$. There are many such choices, as the following lemma makes clear.

Lemma 3.6. *If $x_0^p = x_1^p = x_2^p = x_3^p \ne 1$ then $[\operatorname{Aut} G, G] \le Z$.*

Proof. Let $y = x_0^p$ and let $\alpha \in \operatorname{Aut} G$. For all $i \in \mathbb{Z}$ let λ_i be that integer such that $0 < \lambda_i < p$ and $x_i^\alpha \equiv x_i^{\lambda_i}$ modulo Z; the existence of λ_i is granted by Lemma 3.5. Then $y^\alpha = y^{\lambda_0}$; if $i \in \{1, 2, 3\}$ we also have $y = x_i^p$, hence $y^\alpha = y^{\lambda_i}$ and $\lambda_i = \lambda_0$. Therefore $c^\alpha = [x_0, x_1]^\alpha = [x_0^{\lambda_0}, x_1^{\lambda_0}] = c^{\lambda_0^2}$ and, similarly, $a^\alpha = a^{\lambda_0^2}$ and $b^\alpha = b^{\lambda_0^2}$. Thus $y^{\lambda_0} = y^\alpha = y^{\lambda_0^2}$ and $\lambda_0 = 1$. Therefore α acts trivially on G'. Finally, for all $i \in \mathbb{Z}$ we have $[x_i, x_{i+1}] = [x_i, x_{i+1}]^\alpha = [x_i, x_{i+1}]^{\lambda_i\lambda_{i+1}}$, hence $\lambda_i\lambda_{i+1} \equiv_p 1$. It follows that $\lambda_i = 1$ for all $i \in \mathbb{Z}$. This proves the result. □

At this point the proof of Theorem 3.1 is complete. We may observe that if $p = 2$ then any choice for the words w_i in (Rel) yields a group G such that $[\operatorname{Aut} G, G] \le Z$, but this is not true of any other prime p. However, even for any odd prime, our construction gives rise to $2^{\aleph_0}$ pairwise nonisomorphic examples. The following two remarks justify these comments.

Remark 3.7. Let p be an odd prime. Define G by letting, in (Rel), $w_i = 1$ for all $i \in \mathbb{Z}$. Then G has an automorphism defined by $x_i \mapsto x_i$ if i is even and $x_i \mapsto x_i^{-1}$ if i is odd, and $z \mapsto z^{-1}$ for all $z \in Z$. This automorphism does not fix $\langle x_i x_{i+1}\rangle Z$, for any $i \in \mathbb{Z}$. Hence G does not satisfy the conclusion of Theorem 3.1. More than that, G does not even satisfy $\mathcal{P}^\sharp$.

Remark 3.8. The construction carried on in this section provides, for each prime p, $2^{\aleph_0}$ pairwise nonisomorphic p-groups G with the properties listed in Theorem 3.1. To prove this, suppose that G is a p-group defined as in this section, that is on generators x_i, where $i \in \mathbb{Z}$, a, b, c and relations (Rel) in the variety $\mathcal{V}$, and let $\bar{G}$ be defined similarly, on generators $\bar{x}_i$, $\bar{a}$, $\bar{b}$, $\bar{c}$ in place of x_i, a, b, c and by choosing words $\bar{w}_i$ on $\{\bar{a}, \bar{b}, \bar{c}\}$ to replace the words w_i. Suppose that $\theta\colon G \to \bar{G}$ is an isomorphism. Then, by repeating the arguments used above to show that various subgroups of G are characteristic one can see that θ maps each of Z, X, Y, L_i and R_i, for all $i \in \mathbb{Z}$, onto the analogously defined subgroups of $\bar{G}$. Thus, if $\bar{Z} := \langle \bar{a}, \bar{b}, \bar{c}\rangle$, we have $(\langle x_i\rangle Z)^\theta = \langle \bar{x}_i\rangle \bar{Z}$ for all $i \in \mathbb{Z}$, which implies $\langle w_i\rangle^\theta = \langle \bar{w}_i\rangle$ (by considering p-th powers); moreover $\langle a\rangle^\theta = \langle \bar{a}\rangle$, $\langle c\rangle^\theta = \langle \bar{c}\rangle$ and $\langle b\rangle^\theta = [\langle x_2\rangle Z, \langle x_3\rangle Z]^\theta = \langle \bar{b}\rangle$.

This proves that, given a group G presented as in (Rel), if we replace some of the words w_i with some $\bar{w}_i \in Z$ such that $\langle \bar{w}_i\rangle \ne \langle w_i\rangle$ then we obtain a group which is not isomorphic to G. Together with Lemma 3.6 this is enough to justify our remark.

4. Nonabelian groups with $\mathcal{P}^\sharp$

The examples of $\mathcal{P}^\sharp$-groups described so far all share the property of satisfying $\mathcal{P}^\sharp$ (boundedly) for a very straightforward reason: each has a finite subgroup such that every subgroup containing it is characteristic. As we shall see there exist groups with no such finite subgroups, but satisfying $\mathcal{P}^\sharp$ for less obvious reasons—our characterisation will show that they all satisfy $\mathcal{P}^\sharp$ boundedly nonetheless.

As stated in the Introduction, it also turns out that $\mathcal{P}^\sharp$ is equivalent to the seemingly weaker property $\mathcal{P}^\sharp_{\mathrm{a}}$. Thus the general hypothesis will be that our groups satisfy $\mathcal{P}^\sharp_{\mathrm{a}}$. Our strategy in studying such groups consists in finding information on the structure of the centre and the torsion part. The starting-point is the observation, already made, that the derived subgroup of a $\mathcal{P}^\sharp_{\mathrm{a}}$-group G must be finite. Therefore the periodic elements of G form a subgroup. Another useful consequence is that $G/Z(G)$ has finite exponent. This fact plays a key rôle in the proofs of this section. Also, $G/Z(G)$ is residually finite. Better than that, we have:

Lemma 4.1. *Let G be a group satisfying $\mathcal{P}^\sharp_{\mathrm{a}}$. Then $Z(G)$ is the intersection of characteristic subgroups of finite index in G.*

Proof. For each $x \in G$ let $C_x = C_G(\langle x\rangle^\sharp)$. Since $\langle x\rangle^\sharp$ is cyclic-by-finite and finite-by-abelian, hence finite-by-cyclic, $\operatorname{Aut}(\langle x\rangle^\sharp)$ is finite and so G/C_x is finite. As $Z(G) = \bigcap_{x\in G} C_x$ the result follows. □

Therefore, if G satisfies $\mathcal{P}^\sharp_{\mathrm{a}}$ then every divisible subgroup of G lies in $Z(G)$. Thus Lemma 1.6 shows that G has at most one Prüfer p-subgroup for each prime p.

Lemma 4.2. *Let G be a $\mathcal{P}_{\mathrm{a}}^{\sharp}$-group. Then all but finitely many primary components of $Z(G)$ are locally cyclic and the torsion subgroup of $Z(G)$ has finite rank.*

Proof. First we prove that for every prime p the p-primary component Z_p of $Z(G)$ has finite rank. To this end, assume that Z_p has infinite rank for some prime p, let S be the socle of Z_p and let L be an infinite subgroup of S such that S/L is also infinite. Suppose first that G has a subgroup M of index p and containing $G'L$. Let $x \in G \setminus M$. For every $y \in L$ there exists an automorphism of G centralising M and mapping x to xy. Therefore $L \leq \langle x \rangle^{\sharp}$, a contradiction. Thus there exists no such subgroup M, which amounts to saying that G/LG' is p-divisible. Let P/G' be the p-component of G/G', so that P/LG' is divisible. Also, $SG'/LG' \simeq S/L(G' \cap S)$ has infinite rank, so P/LG' also has infinite rank. It follows that P^pG'/G' is divisible of infinite rank. Since G' is finite this shows that P^p has a divisible p-subgroup of rank 2, and this is impossible in view of Lemma 1.6. Therefore Z_p has finite rank.

Suppose that p is a prime not in $\pi := \pi\big(G/Z(G)\big)$. Then Z_p is a direct factor of G, because it is a direct factor of $Z(G)$. It follows from Lemmas 1.1 and 1.2 that there is a cofinite subset ψ of the complement π' of π (in the set of all primes) such that the ψ-component of $Z(G)$ is locally cyclic. Since π is finite the lemma follows. □

Lemma 4.3. *Let G be a $\mathcal{P}_{\mathrm{a}}^{\sharp}$-group. Then G has a characteristic subgroup V of finite index such that $Z(G) \leq V$ and all subgroups of V containing $Z(G)$ are characteristic in G. Also, G has a characteristic subgroup J containing $Z(G)$ such that $J/Z(G)$ is finite and all subgroups of G containing J are characteristic.*

Proof. Assume that there is no such subgroup V, and let N be any characteristic subgroup of finite index in G containing $Z(G)$. Then N has an element x_1 such that $K := \langle x_1 \rangle Z(G)$ is not characteristic in G. Let $S = K^{\sharp}$, then $S/Z(G)$ is finite (recall that $G/Z(G)$ is periodic) and $S \leq N$; by Lemma 4.1 and since G' is finite $C_N(S)$ contains a subgroup N_1 which is both characteristic and of finite index in G, such that $S \cap N_1 = Z(G)$. Similarly, N_1 has an element x_2 such that $\langle x_2 \rangle^{\sharp} \nleq \langle x_2 \rangle Z(G)$, and a subgroup N_2 which is of finite index and characteristic in G, centralises $\langle x_1, x_2 \rangle^{\sharp}$ and is such that $\langle x_1, x_2 \rangle^{\sharp} \cap N_2 = Z(G)$. By iterating this construction it is possible to produce a sequence $(x_i)_{i \in \mathbb{N}}$ of noncentral, pairwise commuting elements of G such that none of the subgroups $K_i = \langle x_i \rangle Z(G)$ is characteristic in G and the characteristic closures of these subgroups generate their direct product modulo $Z(G)$. This is impossible, because of property $\mathcal{P}_{\mathrm{a}}^{\sharp}$. Thus we have proved the existence of V. The second statement follows easily: let T be a transversal for V in G and let $J = \langle T \rangle^{\sharp} Z(G)$; then $J/Z(G)$ is finite, because $G/Z(G)$ is periodic and $\langle T \rangle^{\sharp} = \langle \langle t \rangle^{\sharp} \mid t \in T \rangle$ is finitely generated. Moreover, G/J is $(\operatorname{Aut} G)$-isomorphic to $V/V \cap J$, so J has the required property. □

As we did in Section 2 for abelian groups, we shall discuss periodic and nonperiodic $\mathcal{P}_{\mathrm{a}}^{\sharp}$-groups separately, starting with the periodic case. If G is a periodic group with a direct, coprime factorisation $G = X \times Y$, in the sense that $\pi(X) \cap \pi(Y) = \varnothing$, then there is a natural isomorphism between $\operatorname{Aut} G$ and $\operatorname{Aut} X \times \operatorname{Aut} Y$, so G satisfies $\mathcal{P}_{\mathrm{a}}^{\sharp}$ (or $\mathcal{P}^{\sharp}$, or $\mathcal{P}^{\flat}$) if and only if both X and Y have the same property. Also, G satisfies the property boundedly if and only if the same holds for X and Y. Therefore the next lemma (together with Theorem 2.2) reduces the study of periodic groups G satisfying $\mathcal{P}_{\mathrm{a}}^{\sharp}$ to the case when $\pi(G)$ is finite.

Lemma 4.4. *Let G be a periodic $\mathcal{P}_{\mathrm{a}}^{\sharp}$-group and let $\pi = \pi(G') \cup \pi\big(G/Z(G)\big)$, a finite set of primes. Then $G = P \times A$, where P is a π-group and A is the π'-component of $Z(G)$.*

Proof. The π'-elements of G lie in $Z(G)$, hence they form a subgroup A. Since G/A is a π-group and $A \cap G' = 1$, A has a complement in G and the result follows. □

Theorem 4.5. *Let G be a periodic group. Then G satisfies $\mathcal{P}_{\mathrm{a}}^{\sharp}$ if and only if $G = P \times A$, where $\pi(P) \cap \pi(A) = \varnothing$ and:*

(a) *A is an abelian group satisfying $\mathcal{P}^{\sharp}$;*
(b) *P' is finite, and P has a Černikov subgroup J containing $Z(P)$ such that the finite residual of J is locally cyclic and every subgroup of P containing J is characteristic in P.*

If G satisfies $\mathcal{P}_{\mathrm{a}}^{\sharp}$ then it satisfies $\mathcal{P}^{\sharp}$ boundedly.

Proof. Suppose that G satisfies $\mathcal{P}_{\mathrm{a}}^{\sharp}$. Let P and A be defined as in Lemma 4.4, so that $G = P \times A$ and $\pi(P) \cap \pi(A) = \varnothing$. Moreover, both P and A satisfy $\mathcal{P}_{\mathrm{a}}^{\sharp}$, hence (a) holds and P' is finite. By Lemma 4.3 we know that there exists $J \leq P$ such that $Z(P) \leq J$, all subgroups of P containing J are characteristic in P and $J/Z(P)$ is finite. By Lemma 4.2 and since $\pi(P)$ is finite $Z(P)$ is a Černikov group, so J is a Černikov group. Its finite residual is contained in $Z(P)$, thus (b) follows from Lemma 1.6.

Conversely, suppose that G has the structure specified. In view of the remarks preceding Lemma 4.4 and by Corollary 2.7, to prove that G satisfies $\mathcal{P}^{\sharp}$ boundedly it will be enough to prove the same of P. Let C be the finite residual of J, thus J/C is finite. Then $C \leq Z(P)$, because P' is finite; for the same reason $P/Z(P)$ has finite exponent. Moreover, P/P' splits over CP'/P', hence P has a subgroup K such that $P = KC$ and $K \cap C$ is finite. Thus $n = \exp K$

is finite, and also $F := J \cap K$ is finite. We may replace K with $\{x \in P \mid x^n \in P'\}$ (and then redefine n), to have K char P also. Let $H \leq P$. If $kc \in H_1 := HF$, where $k \in K$ and $c \in C$ then $c^n \in H_1$, so $c \in S := \{x \in C \mid x^n \in H_1\}$ and $H_1 \leq KS$. Note that $|H_1S : H_1| = |S : H_1 \cap S| \leq n$, as S is locally cyclic. Since C is characteristic and locally cyclic, S too is characteristic in P. Now $J \cap KS = FS \leq H_1S \leq KS$, hence $H_1S = H_1J \cap KS$. Both H_1J and KS are characteristic, so H_1S char P. Now $|H_1S : H| = |H_1S : H_1||H_1 : H| \leq n|F|$ is finite. Thus every subgroup of P has index at most $n|F|$ in its characteristic closure (in P, therefore in G). Since this bound is independent of H the proof is complete. □

In Theorem 4.5 we can choose P and A in such a way that A is locally cyclic, as follows from Theorem 2.2. In this case all subgroups of G containing the subgroup J referred to in (b) are characteristic, not only those in P. Thus every periodic $\mathfrak{P}^\sharp$-group has a Černikov subgroup J such that H char G for all H with $J \leq H \leq G$. It can happen that no such J is finite. An example is given here.

Example 4.6. *Let $K = G \times C$, where G is a p-group as in Theorem 3.1, for an odd prime p, and $C \simeq \mathcal{C}_{p^\infty}$. Then K satisfies $\mathfrak{P}^\sharp$ and every finite subgroup of K is contained in some subgroup of K that is not characteristic.*

Proof. That K satisfies $\mathfrak{P}^\sharp$ follows from Theorem 4.5, as $Z(K) = CG'$ has the property required for J. Suppose that K has a finite subgroup F such that H char K whenever $F \leq H \leq K$. Let z be an element of C of order greater than $\exp F$. Let $\{x_i \mid i \in \mathbb{Z}\}$ be the generating set used to define G in Section 3. For all $i \in \mathbb{Z}$ we have $\langle zx_i\rangle F$ char K. An automorphism of K is defined by letting $g \mapsto g$ and $h \mapsto h^{-1}$ for all $g \in G$ and $h \in C$. It follows that $z^{-1}x_i \in \langle zx_i\rangle F$, hence $x_i \in \langle zx_i\rangle F$ and there exists $t_i \in \mathbb{N}$ such that $z^{t_i}x_i^{t_i-1} \in F$. Since F is finite there exist two different integers, i, j such that $z^{t_i}x_i^{t_i-1} = z^{t_j}x_j^{t_j-1}$, that is, $z^{t_i} = z^{t_j}$ and $x_i^{t_i-1} = x_j^{t_j-1}$. This latter equality implies that $t_i \equiv_p 1$. But then $z^{t_i}x_i^{t_i-1}$ has the same order as z, which is impossible since it belongs to F. □

The nonperiodic case is harder to deal with, and the description that we shall give in this case is less explicit than the one in Theorem 4.5. The key piece of information is the following.

Lemma 4.7. *Let G be a nonperiodic $\mathfrak{P}^\sharp_{\mathrm{a}}$-group. Then $\operatorname{tor}(Z(G))$ is finite and $\operatorname{tor} G$ has finite exponent.*

Proof. Let $T = \operatorname{tor} G$ and $Z = Z(G)$. Since $\exp(G/Z)$ is finite it is clear that $\exp T$ is finite if $\operatorname{tor} Z = T \cap Z$ is finite, hence it will be enough to prove this latter fact. We begin by showing that G has no Prüfer subgroups. Suppose that $\mathcal{C}_{p^\infty} \simeq P \leq G$. Then $G = PK$ for some subgroup K such that $P \cap K \leq G'$. Let bars denote images modulo T. Since $P \leq Z$ every homomorphism $\varepsilon\colon \bar{G} = G/T \to P$ gives rise to an automorphism of G, defined by $x \mapsto x\bar{x}^\varepsilon$ for all $x \in G$, hence $\bar{U}^\varepsilon \leq U^\sharp$ for every $U \leq K$. If the abelian torsion-free group $\bar{G}$ is p-divisible then also $\bar{Z}$ is p-divisible, because G/Z has finite exponent. Then there exists $S \leq TZ$ such that $T \leq S$ and $TZ/S \simeq \mathcal{C}_{p^\infty}$. As TZ is a direct factor in G modulo S there exists $\varepsilon \in \operatorname{Hom}(\bar{G}, P)$ such that $\bar{Z}^\varepsilon = P$. But $\bar{Z} = (Z \cap K)T/T$, hence $P \leq (Z \cap K)^\sharp$. If, instead, $\bar{G}$ is not p-divisible and $\bar{x} \in \bar{G} \setminus \bar{G}^p$ then x has order p^n modulo TG^{p^n}, for every $n \in \mathbb{N}$; it follows that $P \leq \langle x\rangle^\sharp$. In either case we obtain a contradiction. Therefore T is reduced, and, in view of Lemma 4.2, to establish the lemma we only need to prove that $\pi(Z)$ is finite.

Let π be the set of all odd primes $p \in \pi(Z) \setminus \pi(G/Z)$ such that the p-component Z_p of Z is cyclic. By Lemma 4.2, and since G/Z has finite exponent, $\pi(Z) \setminus \pi$ is finite. To complete the proof we shall assume that π is infinite and derive a contradiction. For all $p \in \pi$ let $\langle a_p\rangle = Z_p$ and note that $G = \langle a_p\rangle \times K_p$ for some $K_p \leq G$; let α_p be the automorphism of G defined by $a_p^{\alpha_p} = a_p^{-1}$ and $[K_p, \alpha_p] = 1$.

If Z has infinite torsion-free rank, we can choose in it an (infinite) independent subset $\{x_p \mid p \in \pi\}$ (here $x_p \neq x_q$ if $p \neq q$). At the expense of replacing x_p by a_px_p where needed, we can also assume that $x_p \notin K_p$ for all $p \in \pi$, hence $1 \neq [x_p, \alpha_p] \in \langle a_p\rangle$. Then $S := \langle x_p \mid p \in \pi\rangle$ is a torsion-free abelian subgroup such that $S^\sharp$ has infinite torsion subgroup. This contradiction shows that Z has finite torsion-free rank.

Now let $p \in \pi$ and suppose that $Z(K_p)$ is p-divisible. The six-term homology sequence (see [9], Theorem V.2.2) shows that if X is a centre-by-finite group of finite exponent e then the Schur multiplier of X has finite exponent dividing e^2. Hence the Schur multiplier of $K_p/Z(K_p)$ has finite exponent, involving primes in $\pi(G/Z)$ only. By the Universal Coefficients Theorem ([9], section II.5), the cohomology group $H^2\big(K_p/Z(K_p), Z(K_p)\big)$ also has finite exponent involving primes in $\pi(G/Z)$ only. Thus, as $p \notin \pi(G/Z)$, there exists $n \in \mathbb{N}$ such that $q := p^n \equiv 1$ modulo the order of the cohomology class of the extension $Z(K_p) \hookrightarrow K_p \twoheadrightarrow K_p/Z(K_p)$. It follows that K_p has an automorphism inducing the identity on $K_p/Z(K_p)$ and the mapping $x \mapsto x^q$ on $Z(K_p)$. This automorphism extends to an automorphism of G that does not induce a power automorphism on G/T. This is a contradiction, by Lemma 1.3, hence $Z(K_p)$ is not p-divisible. Since $\operatorname{tor}(Z(K_p))$ is a p'-group this means that $Z/\operatorname{tor} Z \simeq Z(K_p)/\operatorname{tor}(Z(K_p))$ is not p-divisible. Therefore, for every $p \in \pi$ we can choose $g_p \in Z \setminus TZ^p$.

Next we prove that no element x of Z has finite p-height for infinitely many $p \in \pi$; this is clear if x is periodic. Let x be not periodic and let ψ be the set of all $p \in \pi$ such that x has finite p-height. For each $p \in \psi$ there exists

$y_p \in Z \smallsetminus Z^p$ such that $x = y_p^{r_p}$ for some power r_p of p. By Lemma 2.3, $S := \langle y_p \mid p \in \psi\rangle$ is torsion-free. Fix $p \in \psi$ and set $y = y_p$. Note that $y \notin G'G^p$, because G/Z is a p'-group and $y \notin Z^p$. Recall the automorphism α_p defined earlier by using the direct decomposition $G = \langle a_p\rangle \times K_p$. If $y \notin K_p$ then $[y, \alpha_p]$ is a nontrivial p-element in $\langle y\rangle^\sharp$. If $y \in K_p$ then $y \notin \langle a_p\rangle G'G^p$. Therefore a suitable homomorphism from $G/\langle a_p\rangle G'G^p$ to $\langle a_p\rangle$ gives rise to an automorphism of G mapping y to yb, where b is an element of order p in $\langle a_p\rangle$, thus $b \in \langle y\rangle^\sharp$. In either case $\langle y\rangle^\sharp$ has nontrivial p-elements. Thus $S^\#$ contains elements of order p for all $p \in \psi$, and this shows that ψ is finite, as claimed.

Going back to the nonperiodic elements g_p fixed earlier (where $p \in \pi$), we are now able to define a sequence $(q_i)_{i\in\mathbb{N}}$ of (pairwise distinct) primes in π such that, for all $i \in \mathbb{N}$, the element g_{q_i} has infinite q_j-height for all integers $j > i$. Since Z has finite torsion-free rank there exist $n, m \in \mathbb{N}$ such that $1 \neq g_{q_n}^m \in H := \langle g_{q_1}, g_{q_2}, \dots, g_{q_{n-1}}\rangle$. But the elements of H have infinite q_n-height, while g_{q_n} is not q_n-divisible modulo $T \cap Z$. This is the final contradiction, and the proof is now complete. □

For every group G let $\mathfrak{L}G$ denote the set of all subgroups of G. If $U \le G$ we say that a mapping $\theta\colon \mathfrak{L}U \to \mathfrak{L}G$ is *join-preserving* if $\langle X \mid X \in S\rangle^\theta = \langle X^\theta \mid X \in S\rangle$ for all $S \subseteq \mathfrak{L}U$.

Lemma 4.8. *Let $U \le G$, where G is an FC-group, and let $\theta\colon \mathfrak{L}U \to \mathfrak{L}G$ be a join-preserving mapping. If A^θ is finite for all countable abelian subgroups A of U then U^θ is finite.*

Proof. Suppose that U^θ is infinite. Let H be a finitely generated subgroup of U. Then H^θ is generated by finitely many images (under θ) of cyclic subgroups, hence it is finite. Moreover, $U = KC_U(H)$ for some finitely generated $K \le U$, hence K^θ is finite and it follows that $\big(C_U(H)\big)^\theta$ is infinite. Suppose that we have chosen elements $x_1, x_2, \dots, x_n \in U$ such that $H := \langle x_1, x_2, \dots, x_n\rangle$ is abelian and $\langle x_{i+1}\rangle^\theta \nleq \langle x_1, x_2, \dots, x_i\rangle^\theta$ for all positive integers $i < n$. Then, since $\big(C_U(H)\big)^\theta$ is infinite, there exists $x_{n+1} \in C_U(H)$ such that $\langle x_{n+1}\rangle^\theta \nleq H^\theta$. This suggests how to construct an infinite sequence of elements of U generating a (countable) abelian subgroup A such that A^θ is infinite. Thus we obtain a contradiction; the result follows. □

Theorem 4.9. *Let G be a nonperiodic group. Then G satisfies $\mathcal{P}_{\mathrm{a}}^\sharp$ if and only if one of the following holds:*

(a) *there exists $F \le G$ such that F is finite and H char G for all H such that $F \le H \le G$;*
(b) *both G' and $\operatorname{tor}(Z(G))$ are finite, $Z(G)$ has finite rank and G has a subgroup J containing $Z(G)$ as a subgroup of finite index such that every subgroup of G containing either $\operatorname{tor} G$ or J is characteristic in G.*

If G satisfies $\mathcal{P}_{\mathrm{a}}^\sharp$ then it satisfies $\mathcal{P}^\sharp$ boundedly.

Proof. Let $T = \operatorname{tor} G$ and $Z = Z(G)$. Let G satisfy $\mathcal{P}_{\mathrm{a}}^\sharp$. We know from Lemma 1.3 that H char G for all H such that $T \le H \le G$, so, to prove that either (a) or (b) holds, we may assume that T is infinite. We can also assume that (b) does not hold; in view of Lemmas 4.3 and 4.7 this means that Z has infinite torsion-free rank. Let $\Gamma := C_{\operatorname{Aut} G}(G/T)$. Since $\operatorname{Aut} G$ acts on G/T by means of power automorphisms either $\operatorname{Aut} G = \Gamma$ or $|\operatorname{Aut} G : \Gamma| = 2$ and every $\alpha \in \operatorname{Aut} G \smallsetminus \Gamma$ induces the inversion map on G/T. Define a map $\theta\colon \mathfrak{L}G \to \mathfrak{L}G$ by setting, for each $X \le G$, $X^\theta = G'[X, \Gamma]\langle xx^\alpha \mid x \in X,\ \alpha \in \operatorname{Aut} G \smallsetminus \Gamma\rangle$. Then θ is join-preserving; moreover $X^\theta \le T$ and $X^\sharp G' = XX^\theta$ for all $X \le G$. It is apparent that G^θ char G and $\operatorname{Aut} G$ acts on G/G^θ by means of power automorphisms. Hence to prove that (a) holds it will suffice to prove that G^θ is finite and set $F = G^\theta$.

If A is an abelian subgroup of G and $\operatorname{tor} A$ is finite then $A^\theta \le T \cap A^\sharp G' = G' \operatorname{tor}(A^\sharp)$, which is finite because of $\mathcal{P}_{\mathrm{a}}^\sharp$. Since $\exp T$ is finite by Lemma 4.7, there exists $K \le G$ such that $G' = K \cap T$ and $G = KT$. By the previous remark A^θ is finite for all abelian subgroups A of K, hence K^θ is finite by Lemma 4.8 (applied to a restriction of θ). Next, $K \cap Z$ has infinite torsion-free rank, because $Z/K \cap Z$ is periodic . Let $\{a_i \mid i \in \mathbb{N}\}$ be an independent set in $K \cap Z$ (here $a_i \neq a_j$ if $i \neq j$). Let $A = \langle a_i \mid i \in \mathbb{N}\rangle$, let $B = \{b_i \mid i \in \mathbb{N}\}$ be a countable abelian subgroup of T and $S := \langle a_i b_i \mid i \in \mathbb{N}\rangle$. Then S is torsion-free abelian, implying that S^θ is finite, as is A^θ. Now $B^\theta \le (AS)^\theta = A^\theta S^\theta$, hence B^θ is finite. We can use Lemma 4.8 again to conclude that T^θ is finite. Therefore $F := G^\theta = T^\theta K^\theta$ is finite and (a) holds.

Conversely, it is obvious that G satisfies $\mathcal{P}^\sharp$ boundedly in case (a). To complete the proof we only need to show that the same conclusion can be drawn from (b). Assume that (b) holds. As G' and $\operatorname{tor} Z$ are finite both G/Z and T have finite exponent. If $H \le G$ then H splits over H', because $\exp(\operatorname{tor} H)$ is finite, hence $H = (H \cap T)V$ for some $V \le H$ such that $V \cap T = H'$. Of course $H^\sharp = (H \cap T)^\sharp V^\sharp$. Now, $F := J \cap T$ is finite, because J/Z is finite, and $(H \cap T)F = (H \cap T)J \cap T$ char G by hypothesis, hence $|(H \cap T)^\sharp : H \cap T| \le |F|$. Thus we need only consider V. By hypothesis $VT \cap VJ$ char G, hence $V^\sharp \le VT \cap VJ = V(T \cap VJ)$. But $\operatorname{rk}(V) \le \operatorname{rk}(G')\operatorname{rk}(G/T)$ and $\exp(G/J)$ is finite, hence $|VJ/J| \le n$ for some integer n independent of H. Thus $|V^\sharp : V| \le |T \cap VJ| \le n|F|$. It follows that the index of H in $H^\sharp = (H \cap T)^\sharp V^\sharp$ is finite and bounded above by a number independent of H. Thus G satisfies $\mathcal{P}^\sharp$ boundedly and the proof is complete. □

Together with Theorem 4.5 this result proves the Theorem in the Introduction: $\mathcal{P}^\sharp$ and $\mathcal{P}^\sharp_{\mathrm{a}}$, and also the property of satisfying $\mathcal{P}^\sharp$ boundedly, are equivalent.

Our final remarks provide further examples of $\mathcal{P}^\sharp$-groups. If T is any $\mathcal{P}^\sharp$-group of finite exponent and U is an (abelian) torsion-free group of finite rank such that $|\operatorname{Aut} U| = 2$ then $G := T \times U \in \mathcal{P}^\sharp$. For $Z(T)$ is finite by Lemma 4.2, Theorem 4.5 (see the remark following it) implies that T has a finite subgroup J_0 such that all subgroups of T containing it are characteristic in T, and it follows that $UZ(T)J_0$ has the property required for J in condition (b) of Theorem 4.9. If T^2 is infinite (which is the case, for instance, if T is isomorphic to one of the groups constructed in Theorem 3.1 when the prime p is odd) then no finite subgroup of G has the property required for F in Theorem 4.9(a). In fact, suppose that $F \leq T$ and all subgroups of G containing F are characteristic. Let $1 \neq u \in U$ and $t \in T$, then $tu^{-1} \in F\langle tu\rangle$ because $F\langle tu\rangle$ char G and G has an automorphism centralising T and inverting every element of U. Hence $tu^{-1} = xt^n u^n$ for some $x \in F$ and $n \in \mathbb{Z}$. As $x \in T$ we have $t = xt^n$ and $u^{-1} = u^n$. Therefore $n = -1$ and $t^2 = x$. Hence $T^2 \leq F$, so F is infinite. Thus we have more examples of groups satisfying $\mathcal{P}^\sharp$ for a non-trivial reason, which shows that condition (b) in Theorem 4.9 cannot be dismissed. By comparison, if we modify this last example by letting U have infinite rank then certainly G does not satisfy $\mathcal{P}^\sharp$ unless T^2 is finite, this also follows from Theorem 4.9 and the above argument. A further variation is the following. It provides examples of groups of type (a) in Theorem 4.9 with possibly infinite torsion-free rank, in which $T = \operatorname{tor} G$ is not a factor in a central decomposition of G. It also shows that, even in this case, it is not necessary that $\operatorname{Aut} G$ acts trivially on G/T: the groups obtained for $p = 2$ have an automorphism centralising T and inverting every element of U.

Example 4.10. *Let p be a prime and let $G = T \rtimes U$ where T is a p-group isomorphic to one of the groups in Theorem 3.1 and U is an abelian torsion-free group such that $|\operatorname{Aut} U| = 2$. If $TC_G(T) < G$ then G satisfies $\mathcal{P}^\sharp$.*

Proof. It is clear that $T = \operatorname{tor} G$ char G. As we know from Section 3, $\operatorname{Aut} T$ is abelian; this implies that $[G, \operatorname{Aut} G] \leq C_G(T)$. If, furthermore, $[G, \operatorname{Aut} G] \leq T$ then $[G, \operatorname{Aut} G] \leq C_G(T) \cap T = Z(T)$; as $Z(T)$ is finite the result follows in this case. The only nontrivial automorphism of G/T is the inversion mapping. Now, $\operatorname{Aut} T$ is a p-group, hence $G/TC_G(T)$ is a nontrivial (by hypothesis) p-group centralised by $\operatorname{Aut} G$. If $p > 2$ it follows that $[G, \operatorname{Aut} G] \leq T$. Thus we may assume that $p = 2$. Since $(\operatorname{Aut} T)^2 = 1$ we have $G^2 \leq C_G(T)$. Then every automorphism of G inducing the identity (respectively the inversion) on G/T also induces the identity (respectively the inversion) on $G/C_G(T)$, and therefore on $G/Z(T)$. The result follows. □

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