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A NOTE ON CENTRAL AUTOMORPHISMS OF GROUPS

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Abstract: A characterization of central automorphisms of groups is given. As an appication, we obtain a new proof of the centrality of power automorphisms.

A central automorphism of a group G is an automorphism acting trivially on the central factor group $G/Z(G)$. Equivalentely an automorphism of G is central if and only if it centralizes (in the full automorphism group $\operatorname{Aut} G$ of G) the group $\operatorname{Inn} G$ of inner automorphisms. Aim of this paper is to prove the following characterization of the central automorphisms.

Theorem. *Let G be a group and let θ be an automorphism of G. Then the following conditions are equivalent:*
(a) *θ is a central automorphism of G.*
(b) *The following hold:*
  (i) *$[g, g^\theta] = 1$ for all $g \in G$;*
  (ii) *$\langle\theta\rangle$ is normalized by $\operatorname{Inn} G$.*
(c) *The following hold:*
  (i) *$[g, g^\theta] = 1$ for all $g \in G$;*
  (ii) *$\langle[g,\theta]\rangle \triangleleft G$ for all $g \in G$.*

Throughout the paper, if g is an element of the group G, $\bar{g}$ will denote the inner automorphism of G determined by g.

Proof of the Theorem

Since (b) and (c) are obvious consequences of (a), we have only to prove the sufficiency of both conditions to ensure the centrality of θ.

Lemma 1. *Let θ be an automorphism of a group G and assume that $[g, g^\theta] = 1$ for all $g \in G$. Then:*
(i) *$[g, g^{\theta^n}] = 1$ for all $g \in G$ and $n \in \mathbb{Z}$.*
(ii) *$[g, \theta, h]^{-1} = [h^{-1}, \theta, g^{-1}]$ for all $g,\, h \in G$.*

Proof — (i) Let $g \in G$. Since $[g, g^{\theta^{-1}}] = [g^\theta, g]^{\theta^{-1}} = 1$, the statement holds for $n = -1$ and it is enough to prove that it holds for all positive n. Apply induction on n and assume the property is verified for all $m \le n$. Then

$$1 = [gg^\theta, (gg^\theta)^{\theta^n}] = [gg^\theta, g^{\theta^n} g^{\theta^{n+1}}] = [g, g^{\theta^{n+1}}]$$

so that (i) is proved.

(ii) Since $[x, x^\theta] = 1$ for $x = g,\, h^{-1},\, gh$, we get:

$$[g, \theta, h]\,[h^{-1}, \theta, g^{-1}] = gg^{-\theta}h^{-1}g^{-1}g^\theta hh^{-1}h^\theta ghh^{-\theta}g^{-1} = 1.$$

□

We are now in position to prove the first part of the Theorem.

Sufficiency of (b)

Assume first θ has infinite order and is not central. Let g, h be elements of G such that $[g,\theta,h] \neq 1$. Then clearly $[\bar{g},\theta] \neq 1$ and, by Lemma 1 (ii), $[\bar{h},\theta] \neq 1$. Since $\langle\theta\rangle$ is normalized by $\operatorname{Inn} G$, this implies $[\bar{g},\theta] = [\bar{h},\theta] = \theta^2$. Hence $[g,\theta,h] = [\bar{g},\theta,h] = [\theta^2,h] = [h,\theta,h] = 1$, a contradiction.

Suppose now θ is periodic. By Lemma 1 (i) we may assume without loss of generality that the order of θ is a power of a prime, say p^t. Let $t=1$ and assume θ is not central. Then there exists $g \in G$ such that $[\bar{g},\theta] \neq 1$. Hence $\langle[\bar{g},\theta]\rangle = \langle\theta\rangle$. But $[\bar{g},\theta,g] = [g,\theta,g] = 1$, so that $[\theta,g]=1$, again a contradiction. Thus we may assume $t>1$. Let g be an element of G such that $[g,\theta^{p^{t-1}}] \neq 1$ and let $[\bar{g},\theta] = \theta^n$. Since $[g,\theta,g]=1$, then θ^n fixes g and so $\theta^n = 1$. Thus $[\bar{g},\theta]$ is the trivial automorphism and $[g,\theta] \in Z(G)$. Therefore G is the union of the subgroups $C = C_G(\theta^{p^{t-1}})$ and $D = \{\, x \in G \mid [x,\theta] \in Z(G) \,\}$. Since C is a proper subgroup, $G = D$, which means that θ is central.

We have proved that (b) implies (a). To complete the proof of the Theorem we need another Lemma.

Lemma 2. *Let the automorphism θ verify the condition (c) of the Theorem. Then*

$$[g,\theta,h,h] = 1 \qquad \text{and} \qquad [g,\theta,h]^{-1} = [h,\theta,g]$$

for all g, $h \in G$.

Proof — By hypothesis and by Lemma 1 (ii), $[g,\theta,h] \in \langle[h,\theta]\rangle$; hence $[g,\theta,h,h]=1$. It follows, again by Lemma 1 (ii), $[g,\theta,h]^{-1} = [h,\theta,g]$. □

Sufficiency of (c)

Let g be an element of G. We shall prove that $[g,\theta] \in Z(G)$. Suppose first $[g,\theta]$ is not periodic. Then, for any $h \in G$, it holds $[g,\theta,h]^2 = [g,\theta,h^2] = 1$, as $\langle[g,\theta]\rangle \triangleleft G$. But $[g,\theta,h] \in \langle[g,\theta]\rangle$, hence $[g,\theta,h]=1$. Thus $[g,\theta] \in Z(G)$ in this case.

Assume then $[g,\theta]$ is periodic of minimal order subject to $[g,\theta] \notin Z(G)$. It is clear that $[g,\theta]$ has order a power of a prime p. By Lemma 2 and the previous paragraph, there exists an element $h \in G$ such that $c = [g,\theta,h] \neq 1$ and $[h,\theta]$ has prime-power order. Since $c \in \langle[g,\theta]\rangle \cap \langle[h,\theta]\rangle$, then $[h,\theta]$ is a p-element. Without loss of generality, we may assume $c = [h,\theta]^{p^t} = [g,\theta]^{p^k}$. By the choice of g, it holds $k \leq t$. Moreover, it follows from the hypothesis that the group $\langle[g,\theta],[h,\theta]\rangle$ is nilpotent of class at most 2. Therefore we get:

$$\begin{aligned}
[h^{-p^{t-k}}g,\theta]^{p^k} &= \left(\left([h,\theta]^{-p^{t-k}}\right)^g [g,\theta]\right)^{p^k} = \left(\left([h,\theta]^{-p^{t-k}}[g,\theta]\right)^{p^k}\right)^g \\
&= \left([h,\theta]^{-p^t}[g,\theta]^{p^k}\,[\,[g,\theta],[h,\theta]^{-p^{t-k}}\,]^{\frac{p^k(p^k-1)}{2}}\right)^g \\
&= \left(c^{-1}c[\,[g,\theta],[h,\theta]\,]^{\frac{p^t(1-p^k)}{2}}\right)^g = [\,[g,\theta],[h,\theta]\,]^{\frac{p^t(1-p^k)}{2}}.
\end{aligned}$$

By Lemma 2, $[h^{-p^{t-k}}g,\theta,h] = c \neq 1$, so that $[h^{-p^{t-k}}g,\theta] \notin Z(G)$. By the minimality of the order of $[g,\theta]$, it follows $[\,[g,\theta],[h,\theta]\,]^{\frac{p^t(1-p^k)}{2}} \neq 1$. By the same reason, $[g,\theta]^p \in Z(G)$ and $c^p = 1$. Therefore $p=2$, $t=k=1$ and $[g,\theta]$ has order 4. Since $\langle[g,\theta]\rangle$ is normal in G, it holds $[g,\theta]^h = [g,\theta]^{[h,\theta]} = [g,\theta]^{-1}$, which implies $[g,\theta,h^\theta]=1$. We have proved that any element of G not belonging to $C = C_G([g,\theta])$ is mapped by θ in C. Hence $G = C \cup C^{\theta^{-1}}$, a contradiction since C is a proper subgroup of G. □

Let G be the infinite dihedral group. If $\langle x \rangle$ is an infinite cyclic subgroup of G, and $\theta = \bar{x}$, then $\langle \theta \rangle \lhd \operatorname{Aut} G$ and $\langle [g,\theta] \rangle \lhd G$ for all $g \in G$ but θ is not central.

As an application of the above Theorem, we give an alternative proof of the well-known theorem by Cooper stating that power automorphisms (i.e. automorphisms fixing all subgroups) of a group are central (see [1]). This follows from the implication $(c) \Rightarrow (a)$ of the Theorem and the following lemma.

Lemma 3. *Let G be a group and let θ be an automorphism of G such that $[x, x^\theta] = 1$ for all $x \in G$. Let $g \in G$. If θ fixes all conjugates of $\langle g \rangle$ in G, then $\langle [g,\theta] \rangle \lhd G$.*

Proof — Let h be any element of G. Since the group of the automorphisms of G fixing all conjugates of $\langle g \rangle$ in G is normalized by $\operatorname{Inn} G$, it holds $[g, \theta, h] = [h^{-1}, \theta, g^{-1}]^{-1} \in \langle g \rangle$. Then $[g, \theta]^h \in \langle g \rangle$, as $[g, \theta] \in \langle g \rangle$. We conclude that $\langle [g,\theta] \rangle$ is contained in the normal core of $\langle g \rangle$ and so is normal. $\square$

As a matter of fact the above proof of Cooper's theorem may be actually shortened, since the group of power automorphisms is always abelian. In fact, if Γ is an abelian group of automorphisms of a group G normalized by $\operatorname{Inn} G$ and such that $[g, g^\theta] = 1$ for all $g \in G$ and $\theta \in \Gamma$, then $[G, \Gamma]$ is abelian. This information clearly simplifies the proof of the implication $(c) \Rightarrow (a)$ of the Theorem. To prove that $[G, \Gamma]$ is abelian (where Γ is defined as above), let g, $h \in G$ and α, $\beta \in \Gamma$. Then $\overline{[g, \alpha, \beta]} = 1$, as $\overline{[g, \alpha]} \in \Gamma$, hence $[g, \alpha, \beta, h] = 1$ and, by Lemma 1, $[\,[h, \beta]\,, [g, \alpha]\,] = 1$, as we wanted to show.

As a final remark, we point out that there exist groups G and abelian normal subgroups A of $\operatorname{Aut} G$ such that $[g, g^\alpha] = 1$ for all $g \in G$ and $\alpha \in A$ but not all elements of A are central. An example is given by the group A of quasi-power automorphisms of certain infinite groups (see [2]).

References

[1] C.D.H.Cooper, Power automorphisms of a group, *Math. Z.* **107** (1968) 335–356.

[2] G.Cutolo, Quasi-power automorphisms of infinite groups, to appear.

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