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ON GROUPS SATISFYING THE MAXIMAL CONDITION ON NON-NORMAL SUBGROUPS

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Abstract: The aim of this paper is the classification of non-noetherian locally graded groups satisfying the maximal condition on non-normal subgroups.

1. Introduction

In his papers [4,6,7] S.N.Černikov studied groups with the minimal condition on non-abelian subgroups, on non-normal subgroups, on abelian non-normal subgroups respectively.

This kind of researches on groups with the minimal condition on subgroups not verifying a certain property $\mathcal{P}$ is related to two different topics in infinite group theory. On one side, they are part of Černikov's investigation on groups with the minimal condition on given systems of subgroups; on the other side, they are connected with the problem of studying groups with many $\mathcal{P}$-subgroups. In particular, the minimal condition on non-$\mathcal{P}$-subgroups may be regarded as a generalization of the property "every infinite subgroup has $\mathcal{P}$" (considered for instance in [5] and [16]) and is clearly related to the property "every proper subgroup has $\mathcal{P}$" as well.

In this spirit Phillips and Wilson [17] (see also [2,3,12]) and Kurdachenko and Pylaev [13] have proved for a number of properties $\mathcal{P}$ (the property of being either serial or locally nilpotent and other stronger properties in [17]; the property of having finitely many conjugates in [13]) that a group G satisfying the minimal condition on non-$\mathcal{P}$-subgroups and some extra hypoteses is either a Černikov group or a group all proper subgroups of which have the property $\mathcal{P}$. In particular, generalizing a Černikov's result, Phillips and Wilson have shown that a locally graded group with the minimal condition on non-normal subgroups is either a Dedekind group or a Černikov group. Recall that a group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite quotient. Every locally (soluble-by-finite) group is locally graded.

Aim of this paper is the study of the dual condition, the maximal condition on non-normal subgroups, which we will denote by Max-n^-. In contrast to the above-quoted result, it turns out that there exist non-noetherian non-Dedekind groups which satisfy Max-n^-. In fact, we obtain the following:

Theorem. *A locally graded group G satisfies Max-n^- if and only if it is of one of the following types:*
(a) *G is a noetherian group*
(b) *G is a Dedekind group*
(c) *G is a central extension of $\mathbb{Z}(p^\infty)$ by a finitely generated Dedekind group*
(d) *G is the direct product of $\mathbb{Q}_2$ and a finite hamiltonian group.*

In particular a non-noetherian locally graded group with Max-n^- is nilpotent of class 2 (Corollary 2.5). By studying groups with Max-n^- the class of those groups whose non-normal subgroups are finitely generated arises in a natural way. We will denote this class by $\mathcal{D}$. In §2 some properties of groups in Max-n^- or in $\mathcal{D}$ are stated, the Theorem is proved and some of its consequences are pointed out. In §3 nilpotent $\mathcal{D}$-groups not in Max-n^- are discussed. It is also shown that a nilpotent group with the maximal condition on abelian non-normal subgroups satisfies Max-n^-. Finally we mention that a very special case of our problem has been recentely considered by Hekster and Lenstra [11].

Our notation is mostly standard. In particular we refer to [18] and [19].

2. The class Max-n^-

Lemma 2.1. *Let G be a group satisfying the maximal condition on non-normal subgroups. Then:*
(a) *G is a $\mathcal{D}$-group.*
(b) *The commutator subgroup G' of G satisfies the maximal condition. In particular G is locally noetherian.*
(c) *The group G either is soluble or satisfies the maximal condition on abelian subgroups.*

Proof — (a) Let H be a non-normal subgroup of G. Suppose, by contradiction, that H is not finitely generated. Let K be a subgroup of H which is maximal with respect to the condition of being finitely generated and non-normal in G. Since H is not finitely generated, $K < H$. For each $x \in H \setminus K$, the subgroup $\langle K,x \rangle$ is normal in G. Hence $H = \bigcup_{x \in H \setminus K} \langle K,x \rangle$ is normal in G. By this contradiction H is finitely generated.

(b) Let K be a cyclic subgroup of G. The interval $[K^G/K]$ of the subgroup lattice of G satisfies the maximal condition. It follows by a standard argument that K^G satisfies the maximal condition on normal subgroups, and hence K^G satisfies the maximal condition on subgroups. If G is a Dedekind group, then G' is finite. If not, let H be a subgroup of G which is maximal with respect to the condition of being non-normal in G. Then H is finitely generated and, by the above, H^G satisfies the maximal condition. Moreover, by the choice of H, the quotient G/H^G is a Dedekind group, and so $G'/(G' \cap H^G) \simeq G'H^G/H^G$ is finite. Therefore G' satisfies the maximal condition.

(c) Suppose that G contains an abelian subgroup A which is not finitely generated. Then A is normal in G and G/A is a Dedekind group by (a) and (b). Hence G is soluble. □

The proof of (a) in the previous lemma shows that a group with the maximal condition on finitely generated non-normal subgroups is a $\mathcal{D}$-group. Therefore Max-n^- is equivalent to the maximal condition on finitely generated non-normal subgroups.

The next two lemmas establish some restrictions for the centre of a non-Dedekind $\mathcal{D}$-group.

Lemma 2.2. *Let G be a non-Dedekind $\mathcal{D}$-group. Then $Z(G) = T \times K$, where T is either finite or direct product of a finite group and a Prüfer p-group (p prime) and K is torsion-free of finite rank. Furthermore:*
(i) *if T is infinite, K is finitely generated and every p'-subgroup of G is normal in G;*
(ii) *if G has a finite non-normal subgroup, then K is finitely generated.*

Proof — Let H be a cyclic non-normal subgroup of G. Suppose that $Z(G)$ has two subgroups A and B which are not finitely generated and such that $A \cap B = H \cap Z(G)$. Then AH and BH are normal in G, since they are not finitely generated. Hence $H = HA \cap HB$ is normal in G, which is a contradiction. The first part of the lemma follows now readily. Moreover, if $T = \operatorname{tor} Z(G)$ is infinite, then K is finitely generated. Let H be a cyclic non-normal q-subgroup of G (q prime). If L is a subgroup of $Z(G)$ with no element of order q, then H is the q-component of HL, hence HL is not normal in G and L is finitely generated. By using this observation, the proof can be easily completed. □

Lemma 2.3. *Let G be a $\mathcal{D}$-group. Suppose that $Z(G)$ contains a torsion-free subgroup A such that G/A is not periodic and A is not finitely generated. Then G is abelian.*

Proof — Let H/A be a cyclic subgroup of G/A. Since H is abelian and A is not finitely generated, H is not finitely generated and so it is normal in G. Hence G/A is a Dedekind group. Since G/A is non-periodic, it is abelian. We proceed now by induction on the rank r of A (which may be assumed finite by Lemma 2.2) to prove that G is abelian. Let $r = 1$ and assume that G is not abelian. Since G is generated by its elements which have infinite order modulo A, there exists $a \in G \setminus Z(G)$ of infinite order modulo A and $b \in G$ such that $c = [a, b] \neq 1$. Clearly $c \in A$, and the p'-component of $A/\langle c \rangle$ is infinite for a suitable prime p. Let $H = \langle c^p \rangle$ and let P/H be the p'-component of A/H. Since $P\langle a \rangle$ is not finitely generated it is normal in G, so that $c \in P\langle a \rangle \cap A = P(\langle a \rangle \cap A) = P$. Since c has order p modulo H, this is impossible. Hence G is abelian if $r = 1$. Suppose now $r > 1$ and let H_1, H_2 be two nontrivial cyclic subgroups of A such that $H_1 \cap H_2 = 1$. For $i = 1, 2$ let T_i/H_i be the torsion subgroup of A/H_i. Then $T_1 \cap T_2 = 1$. If one of the subgroups T_1 and T_2 is not finitely generated, then, by the case $r = 1$, G is abelian. If T_1 and T_2 are both finitely generated, then A/T_1 and A/T_2 are not finitely generated, so that, by induction, G/T_1 and G/T_2 are abelian and $G' \leq T_1 \cap T_2 = 1$. □

Proposition 2.4. *A torsion-free nilpotent $\mathcal{D}$-group G is either finitely generated or abelian.*

Proof — Assume that G is not abelian. It follows from Lemma 2.3 that $Z(G)$ is finitely generated. By induction on the nilpotency class of G, we obtain that $G/Z(G)$ is abelian and, in particular, G' is finitely generated. For any subgroup H of G, the quotient H/H_G satisfies the maximal condition, so that $G/Z(G)$, and hence G, is finitely generated (see [8], Theorem 5.9). □

Let p be a prime and let $\mathbb{Q}_p$ be the additive group of all rational numbers whose denominator is a power of p. Let α be the automorphism of $\mathbb{Q}_p$ defined by $x^\alpha = px$. Then $\mathbb{Q}_p \rtimes \langle \alpha \rangle$ is an example of a non-abelian finitely generated torsion-free metabelian $\mathcal{D}$-group.

Proof of the Theorem

Let G be a group in Max-n^- which is neither noetherian nor a Dedekind group. Suppose first that G is soluble-by-finite. For each subgroup H of G, the quotient H/H_G has the maximal condition. Since G' is poycyclic-by-finite (see Lemma 2.1(b)), this implies that $G/Z(G)$ is polycyclic-by-finite by Theorem 5.9 of [8]. In particular $Z(G)$ is not finitely generated. By Lemma 2.1 it follows that $G/Z(G)$ is a Dedekind group, so that G is nilpotent. By Proposition 2.4, either $G' \leq \operatorname{tor} G$ or $G/\operatorname{tor} G$ is finitely generated. In the latter case $\operatorname{tor} G$ is not finitely generated, thus $G/\operatorname{tor} G$ is abelian. Then G' is torsion and so finite in any case. Hence $G/Z(G)$ is periodic and so finite.

Let $T = \operatorname{tor} Z(G)$. Suppose first that G/T is finitely generated. Then, by Lemma 2.2, it follows that T contains a subgroup $A \simeq \mathbb{Z}(p^\infty)$ such that G/A is finitely generated. We conclude that G/A is a Dedekind group and G is of type (c) in our statement. Assume now that G/T is not finitely generated. Since $G/Z(G)$ is finite, it follows by Lemma 2.2 that T is finite and that every finite subgroup of G is normal. Then G contains a non-normal infinite cyclic subgroup H. As $H \cap G' = 1$, we have $H_G = H \cap Z(G) \neq 1$ and H/H_G is a finite non-normal subgroup of G/H_G. By the above part of the proof, G/H_G contains a central subgroup $P/H_G \simeq \mathbb{Z}(p^\infty)$ such that G/P is a finitely generated Dedekind group. Then $P \leq Z(G)$ and $P = P_0 \times P_1$, where P_0 is finite and $P_1 \simeq \mathbb{Q}_p$. The quotient G/P_1 is a Dedekind group and, since $G' \cap P_1 = 1$, it is hamiltonian and so finite. Hence $|G'| = 2$ and $G/Z(G)$ has exponent 2. By Lemma 2.2, every p'-subgroup of G/H_G is normal, so that H/H_G is a p-group and p divides $|G/Z(G)|$. Hence $p = 2$ and $P_1 \simeq \mathbb{Q}_2$. Let V/P_1 be the $2'$-component of G/P_1. Then V is abelian, since $V' \leq G' \cap P_1 = 1$. Hence $V = V_0 \times V_1$, where V_0 is finite and $V_1 \simeq \mathbb{Q}_2$. Also G/V_1 is a finite hamiltonian group. Let U/V_1 be the Sylow 2-subgroup of G/V_1. Then $G' \leq U$ and $V_1G'/G' \simeq V_1 \simeq \mathbb{Q}_2$, so that there exists a subgroup B of U containing G' such that $U = (V_1G')B$ and $V_1G' \cap B = G'$ (see [9], vol.I, p.223). Hence $U = V_1 \times B$ and B is finite. Therefore $G = UV = V(V_1 \times B) = V_1 \times (BV_0)$ and $BV_0 \simeq G/V_1$ is hamiltonian. Then G is of type (d) and the theorem holds for soluble-by-finite groups.

Let now G be locally graded. Let $N = G''$ be the second term of the derived series of G and let H be a subgroup of finite index in N. It follows from Lemma 2.1(b) that H_G has finite index in N. By the first part of the proof, the non-noetherian group G/H_G is metabelian, so that $H_G = N$ and N has no proper subgroups of finite index. By the definition of locally graded group, $N = 1$ and G is soluble. The necessity of the condition is proved.

Conversely, a group of type (a) or (b) satisfies trivially Max-n^-. Let G be a group of type (c). Then $G = NA$, where $A \simeq \mathbb{Z}(p^\infty)$ is contained in $Z(G)$, the quotient G/A is a Dedekind group, N is polycyclic (and obviously normal). Assume that G has an infinite, strictly ascending sequence of non-normal subgroups

$$K_1 < K_2 < \cdots < K_n < \cdots$$

and let K be the union of the K_i. Since G/A is finitely generated, $K/(A \cap K)$ has Max, so that $A \cap K \notin$ Max and $A \cap K = A$. Then $A \leq K$. Since G/A is a Dedekind group, $K \triangleleft G$. Since N satisfies the maximal condition, there exists an integer n such that $K \cap N = K_n \cap N$, and so $K \cap N \leq K_n < K \triangleleft G$. But $K/(K \cap N) \simeq KN/N \simeq \mathbb{Z}(p^\infty)$, so that K_n must be normal in G. This contradiction shows that G satisfies Max-n^-. Let finally $G = Q \times F$ be a group of type (d), where $Q \simeq \mathbb{Q}_2$ and F is a finite hamiltonian group. Let $K_1 \leq K_2 \leq \cdots \leq K_n \leq \cdots$ be an ascending chain of non-normal subgroups of G and let K be its union. It is clear that $N = K_1 \cap Q \neq 1$; thus $Q/N = A \times D$, where $A \simeq \mathbb{Z}(2^\infty)$ and D is a finite abelian group of odd order. Hence $G/N = A \times D \times (FN/N)$ is a group of type (c), and so it belongs to Max-n^-. It follows that $K = K_n$ for some integer n. Therefore $G \in$ Max-n^-. □

Remark. The proof of the Theorem also shows that *a PC-group which is a $\mathcal{D}$-group satisfies Max-n^-*. Here a *PC*-group is a group G such that $G/C_G(x^G)$ is polycyclic-by-finite for each element x of G (see [8]).

Corollary 2.5. *Let G be a non-noetherian locally graded group in Max-n^-. Then G is a nilpotent central-by-finite group of class at most 2.*

Proof — We have only to prove that a group G of type (c) (see Theorem) has class 2. We may assume that G is a 2-group and that G/A is a finite hamiltonian group, where the subgroup A is isomorphic with $\mathbb{Z}(2^\infty)$. Clearly the Schur multiplicator $M(G/Z(G))$ of $G/Z(G)$ has exponent 2, so that $G' \cap Z(G)$ has exponent 2 and $|A \cap G'| \leq 2$. It follows that $|G'| \leq 4$. Hence, for each $x \in G$, we have $|G : C_G(x)| \leq 4$. But $A \leq C_G(x)$ and each subgroup of G/A of index ≤ 4 contains $G'A/A$, so that $G' \leq Z(G)$ and G has class 2. □

Corollary 2.6. *Every non-Dedekind group G in Max-n^- is countable.*

Proof — By Lemma 2.1(b), the union of any chain of soluble subgroups of G is still soluble. Hence we can apply Zorn's Lemma to obtain a maximal soluble subgroup H of G. We may assume that $H < G$. Then H is polycyclic by Lemma 2.1(c) and H^G satisfies the maximal condition on subgroups. Hence H^G has only countably many subgroups, so that $|G : N_G(H)|$ is countable. But $N_G(H) = H$, so the corollary is proved. □

3. Other results and counterexamples

Examples of soluble non-noetherian $\mathcal{D}$-groups which are not nilpotent (and so do not satisfy the maximal condition on non-normal subgroups) are easily obtained (for instance the example given in §2, or any non-central extension of a Prüfer group by a finitely generated Dedekind group). Even in the nilpotent case the property $\mathcal{D}$ does not imply Max-n^-, as the following example shows.

Example 3.1. Let p be a prime and A be a torsion-free abelian group of rank $n > 1$ with no infinite cyclic quotients containing a finitely generated subgroup B such that $A/B \simeq \mathbb{Z}(p^\infty)$ (for the existence of such groups, see [10] or [9], vol.II, p.128). Since A is p-minimax, the Schur multiplicator $M(A)$ of A is also p-minimax. It is well-known that $M(A)$ has torsion-free rank $n(n-1)/2 > 0$. On the other hand, $\mathrm{Hom}(M(A)\,,\,\mathbb{Z}) \rightarrowtail \mathrm{Hom}(A \otimes A\,,\,\mathbb{Z}) = 0$, so that $M(A)$ has no infinite cyclic quotients and hence it has a quotient isomorphic with $\mathbb{Z}(p^\infty)$. Consider $C \simeq \mathbb{Z}(p^\infty)$ as a trivial A-module and let $\varphi : M(A) \twoheadrightarrow C$ be an epimorphism. Then φ determines a central extension:

$$C \rightarrowtail G \twoheadrightarrow A$$

where $G' = C$. Assume that there exists a subgroup H of G which is not finitely generated and does not contain G'. Then $H \cap G'$ is finite and HG'/G' is not finitely generated. Let K/G' be a finitely generated subgroup of G/G' such that $G/K \simeq \mathbb{Z}(p^\infty)$. Then $HK/K \simeq HG'/((H \cap K)G')$ is not finitely generated, and so $G = HK$. Then G/HG' is finitely generated, hence finite, since $G/G' \simeq A$ has no infinite cyclic quotient. Therefore H is a near-complement of G' in G. This is impossible, since the cohomology class of $C \rightarrowtail G \twoheadrightarrow A$ has infinite order (see [20]). This contradiction proves that every subgroup of G which is not finitely generated contains G' and so is normal. Thus G is a nilpotent $\mathcal{D}$-group. On the other hand, G does not satisfy Max-n^-, since G' is infinite. □

It is well-known that a soluble group satisfies the maximal condition on subgroups if and only if its abelian subgroups have the same property, and a similar result holds for soluble groups satisfying the minimal condition on subgroups (see [18]). On the other hand there exist soluble groups which are not Min-by-Max whose abelian subgroups are Min-by-Max. The structure of groups of this type has been investigated by Newell [14,15]. Our next result gives a description of those nilpotent $\mathcal{D}$-groups which do not satisfy Max-n^- as groups with the property considered above.

Proposition 3.2. *Let G be a nilpotent group and let T be the torsion subgroup of G. Then G is a $\mathcal{D}$-group not in Max-n^- if and only if it satisfies the following conditions:*
(i) *$G' \simeq \mathbb{Z}(p^\infty)$ and T/G' is finite;*
(ii) *every abelian subgroup of G is Min-by-Max but G is not Min-by-Max.*

Proof — Let G be a nilpotent $\mathcal{D}$-group not in Max-n^-. The remark following the Theorem shows in particular that G' is infinite. Assume that T contains two infinite subgroups A and B such that $A \cap B = 1$. Then A and B are normal in G and the factor groups G/A and G/B are Dedekind groups. Hence $G'/(A \cap G') \simeq AG'/A$ and $G'/(B \cap G') \simeq BG'/B$ are finite, so that G' is finite. By this contradiction, it follows that T does not contain such a pair A, B. It follows that each abelian subgroup of T is either finite or direct product of a finite group by a Prüfer group. Hence T is a Černikov group. By the Theorem, G/T is not finitely generated, so that G is not Min-by-Max and $G' \leq T$ by Proposition 2.4. Then T is infinite and contains a subgroup of finite index $P \simeq \mathbb{Z}(p^\infty)$. Now G/P is a Dedekind group and so is abelian, as $T < G$. Thus $G' = P$ and (i) holds. Let A be a maximal abelian subgroup of G. Since $G' \leq Z(G)$, then $G' \leq A$ and $A = G' \times B$ for a suitable subgroup B of A. Since $G' \cap B = 1$ and B is not contained in $Z(G)$, it follows that B is not normal in G and so it is finitely generated. Whence A is Min-by-Max and also (ii) is proved.

Conversely, let G satisfy (i) and (ii) and let H be a subgroup of G which is not finitely generated. Then H contains an abelian subgroup A which is not finitely generated. Since A is Min-by-Max, the torsion subgroup of A must be infinite, so that A contains G' and $G' \leq H$. Hence $H \triangleleft G$ and G is a nilpotent $\mathcal{D}$-group. Since G' is infinite, G does not belong to Max-n^-. □

Corollary 3.3. *Let G be a nilpotent $\mathcal{D}$-group with torsion-free rank 1. Then G satisfies the maximal condition on non-normal subgroups.*

Proof — Assume that G does not satisfy Max-n^-. Then it follows from Proposition 3.2 that $G' \leq Z(G)$ and G/G' has a locally cyclic torsion-free subgroup of finite index A/G'. Clearly A is abelian and G is abelian-by-finite, which is impossible by Proposition 3.2. □

By Proposition 3.2 and a result of Baer [1], nilpotent $\mathcal{D}$-groups not in Max-n^- are minimax groups. More precisely, we have:

Corollary 3.4. *Let G be a nilpotent $\mathcal{D}$-group which is not in Max-n^-. If $A = G' \times B$ is any maximal abelian subgroup of G, then G/A is an infinite p-group with the minimal condition and $A = B^G$.*

Proof — The quotient G/A is isomorphic with a group of automorphisms of A which centralizes $G' \simeq \mathbb{Z}(p^\infty)$ and $A/G' \simeq B$, so that G/A embeds in $\mathrm{Hom}\,(B, \mathbb{Z}(p^\infty))$, which is a p-group with the minimal condition, since B is finitely generated by Proposition 3.2. Since G is not Min-by-Max, G/A is infinite. Finally $C_G(B^G) = C_G(A) = A$, so that B^G is not finitely generated and $A = B^G$.□

It follows easily from Corollary 3.4 that every nilpotent $\mathcal{D}$-groups which is not in Max-n^- has a normal subgroup of the type described in Example 3.1. The corollary above also has the following consequence:

Corollary 3.5. *Let the nilpotent group G satisfy the maximal condition on abelian non-normal subgroups. Then G satisfies the maximal condition on non-normal subgroups.*

Proof — The same argument used in the proof of Lemma 2.1(a), shows that every abelian non-normal subgroup of G is finitely generated. Let H be any non-normal subgroup of G. Since H is generated by its maximal abelian subgroups, H must contain a maximal abelian subgroup U which is not normal in G. Hence U is finitely generated and it follows easily that also H is finitely generated. Thus G is a $\mathcal{D}$-group. If G does not satisfy Max-n^- and $A = G' \times B$ is a maximal abelian subgroup of G, then $B^G/B = A/B$ does not satisfy the maximal condition on subgroups. This contradiction proves that $G \in$ Max-n^-. □

The hypothesis of nilpotency in Corollary 3.5 cannot be weakened. In fact, if A is the direct product of a Prüfer 2-group and a cyclic group of order 4 and α is the inversion automorphism of A, the hypercentral metabelian group $G = A \rtimes \langle\alpha\rangle$ has the maximal condition on abelian non-normal subgroups, but is not a $\mathcal{D}$-group.

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