Mediterr. j. math. 2 (2005), 125–135
1660-5446/020125-11, DOI 10.1007/s00009-005-0034-x

Mediterranean Journal of Mathematics

Noetherian Automorphisms of Groups

Fausto De Mari and Francesco de Giovanni

Abstract. An automorphism α of a group G is called a *noetherian automorphism* if for each ascending chain

$$X_1 < X_2 < \ldots < X_n < X_{n+1} < \ldots$$

of subgroups of G there is a positive integer m such that $X_n^\alpha = X_n$ for all $n \geq m$. The structure of the group of all noetherian automorphisms of a group is investigated in this paper.

Mathematics Subject Classification (2000). Primary 20E15; Secondary 20F28.

Keywords. Power automorphism, maximal condition.

1. Introduction

A *power automorphism* of a group G is an automorphism mapping every subgroup of G onto itself, and the set $PAut\,G$ of all power automorphisms of G is an abelian normal subgroup of the full automorphism group $Aut\,G$ of G. The structure of the group $PAut\,G$ has been widely investigated by Cooper [2]. Power automorphisms play a relevant role in many questions, but when the attention is especially focused on infinite groups, it could be necessary to know the behaviour of automorphisms leaving invariant all subgroups which are large in some suitable sense. This point of view was in particular adopted in [3], and more recently in [1], where the group $IAut\,G$ consisting of all automorphisms of a group G fixing every infinite subgroup is studied. It turns out that in many cases the group $IAut\,G$ coincides with $PAut\,G$; moreover, $IAut\,G$ is always metabelian, provided that G contains an infinite abelian subgroup.

We shall say that an automorphism α of a group G is a *noetherian automorphism* if for each ascending chain

$$X_1 < X_2 < \ldots < X_n < X_{n+1} < \ldots$$

of subgroups of G there is a positive integer m such that $X_n^\alpha = X_n$ for all $n \geq m$. Clearly, the set $NAut\,G$ of all noetherian automorphisms of G is a normal subgroup of $Aut\,G$, and $NAut\,G$ contains the group $PAut\,G$; note also that if G is any group satisfying the maximal condition on subgroups, then $NAut\,G = Aut\,G$.

The aim of this article is to describe the structure of $NAut\,G$. In particular, it is proved in Section 2 that $NAut\,G = IAut\,G$ for any infinite locally finite group G. The study of noetherian automorphisms of abelian groups and locally nilpotent groups is the subject of Section 3 and Section 4, respectively, while the final Section 5 deals with the derived length of the group of noetherian automorphisms of a soluble group.

Most of our notation is standard and can for instance be found in [10]. In particular, $\underset{i\in I}{\mathrm{Dr}}\, G_i$ denotes the direct product of the groups G_i's.

2. Noetherian automorphisms of locally finite groups

We start with an easy remark, that will be very useful in our consideration.

Lemma 2.1. *Let G be a group and let α be a noetherian automorphism of G. If X is a subgroup of G which is not finitely generated, then $X^\alpha = X$.*

Proof. Let x be any element of X. As X is not finitely generated, there exist infinitely many elements $x_1, x_2, \dots, x_n, \dots$ of X such that

$$\langle x\rangle < \langle x, x_1\rangle < \langle x, x_1, x_2\rangle < \dots < \langle x, x_1, x_2, \dots, x_n\rangle < \dots$$

Thus

$$\langle x, x_1, x_2, \dots, x_m\rangle^\alpha = \langle x, x_1, x_2, \dots, x_m\rangle$$

for some positive integer m, so that $x^\alpha \in X$ and hence X^α is contained in X. The same argument applied to the noetherian automorphism α^{-1} shows that $X \leq X^\alpha$, and hence $X^\alpha = X$. □

Recall that a group is called *radical* if it has an ascending (normal) series with locally nilpotent factors. It was proved in [3] that if G is a locally finite non-Černikov group, then $IAut\,G = PAut\,G$ provided that G contains a locally radical subgroup of finite index. Our next lemma shows that this latter hypothesis can be dropped out; the proof here depends ultimately on the classification of finite simple groups.

Lemma 2.2. *Let G be a locally finite group. If G is not a Černikov group, then $IAut\,G = PAut\,G$.*

Proof. Suppose first that G contains an element x of prime power order such that the centralizer $C_G(x)$ is a Černikov group. Then G is a finite extension of a locally soluble group (see [6], Theorem 1), and hence $IAut\,G = PAut\,G$ (see [3], Theorem A). Therefore it can be assumed that for each element y of G with prime power order the subgroup $C_G(y)$ is not a Černikov group, so that $C_G(y)$ contains an abelian subgroup A which does not satisfy the minimal condition on subgroups (see [11]). The socle S of A is infinite, and so $B = \langle y, S\rangle$ is an infinite abelian residually finite subgroup of G; thus $IAut\,B = PAut\,B$ (see [3], Lemma 2.1) and hence $\langle y\rangle^\alpha = \langle y\rangle$ for any automorphism α in $IAut\,G$. Therefore $IAut\,G = PAut\,G$. □

It follows in particular from Lemma 2.1 that any noetherian automorphism of a locally finite group G fixes all infinite subgroups of G, so that in this case $NAut\,G$ is a subgroup of $IAut\,G$. We shall prove that these groups of automorphisms coincide for all locally finite groups.

Lemma 2.3. *If G is a divisible abelian group, then $IAut\,G = PAut\,G$.*

Proof. Let x be any element of G. Then G contains a divisible subgroup X of rank 1 such that $x \in X$ (see [4], p.107). Since all subgroups of X are characteristic, it follows that $\langle x\rangle$ is fixed by any element of $IAut\,G$. Therefore $IAut\,G = PAut\,G$. □

Theorem 2.4. *If G is a locally finite group, then $NAut\,G = IAut\,G$.*

Proof. If G is not a Černikov group, it follows from Lemma 2.2 that

$$NAut\,G = IAut\,G = PAut\,G.$$

Suppose that G is a Černikov group, and assume by contradiction that there exists an automorphism $\alpha \in IAut\,G$ which is not noetherian. Then G contains infinitely many subgroups $X_1, X_2, \dots, X_n, \dots$ such that

$$X_1 < X_2 < \dots < X_n < \dots$$

and $X_n^\alpha \neq X_n$ for all n. Obviously, each X_n must be finite, while

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

is an infinite subgroup of G and so $X^\alpha = X$. Let D be the largest divisible abelian subgroup of X. As X/D is finite, there is a positive integer m such that $X_kD = X_mD$ for all $k \geq m$. Moreover, X_m^α is a finite subgroup of X and hence $X_m^\alpha \leq X_t$ for some $t \geq m$. As

$$X_t = X_mD \cap X_t = X_m(D \cap X_t)$$

and the subgroup $D \cap X_t$ is fixed by α by Lemma 2.3, it follows that

$$X_t^\alpha = X_m^\alpha(D \cap X_t) \leq X_t.$$

Therefore $X_t^\alpha = X_t$, and this contradiction proves that $NAut\,G = IAut\,G$. □

As an obvious consequence of Theorem 2.4 and Lemma 2.2, we have the following result.

Corollary 2.5. *Let G be a locally finite group. If G is not a Černikov group, then $NAut\,G = PAut\,G$.*

Corollary 2.6. *If G is an infinite locally finite group, then the group $NAut\,G$ is metabelian.*

Proof. It follows from Theorem 2.4 and from Corollary 2.4 of [3]. □

3. Noetherian automorphisms of abelian groups

Let p be a prime number, and consider the direct product $G = P \times \langle a \rangle$, where P is a group of type p^∞ and $\langle a \rangle$ is an infinite cyclic group. It is easy to see that the automorphism α of G defined by $x^\alpha = x$ for all $x \in P$ and $a^\alpha = ay$, where y is a fixed non-trivial element of P, is noetherian, but obviously α is not a power automorphism. The main result of this section shows that for abelian groups this is one of the few situations in which $PAut\,G$ is properly contained in $NAut\,G$.

Lemma 3.1. *Let G be a torsion-free abelian group which is not finitely generated. Then $NAut\,G = PAut\,G$.*

Proof. Assume by contradiction that G admits a noetherian automorphism α that is not a power automorphism, so that a subgroup X of G can be chosen which is maximal with respect to the condition $X^\alpha \neq X$. Then X is finitely generated by Lemma 2.1, and so G/X is infinite. Since any subgroup of G properly containing X is fixed by α, the group G/X cannot contain non-trivial subgroups with trivial intersection. Thus G/X is a group of type p^∞ for some prime number p. Let $X = \langle x_1 \rangle \times \ldots \times \langle x_t \rangle$, and put

$$X_{i,q} = \langle x_i \rangle \times \Big(\operatorname*{Dr}_{j \neq i} \langle x_j^q \rangle \Big),$$

for $i = 1, \ldots, t$ and for all primes $q \neq p$. Then

$$G/X_{i,q} = X/X_{i,q} \times B_{i,q}/X_{i,q},$$

where $B_{i,q}/X_{i,q}$ is a group of type p^∞. Clearly $B_{i,q}$ is not finitely generated, so that $B_{i,q}^\alpha = B_{i,q}$ and hence also

$$B_i = \bigcap_{q \neq p} B_{i,q}$$

is a subgroup of G fixed by α. Moreover,

$$B_i \cap X = \bigcap_{q \neq p} (B_{i,q} \cap X) = \bigcap_{q \neq p} X_{i,q} = \langle x_i \rangle,$$

so that B_i has rank 1. Since α is noetherian, there exists a finite subgroup $C_i/\langle x_i \rangle$ of $B_i/\langle x_i \rangle$ such that $C_i^\alpha = C_i$; clearly, C_i is cyclic and hence also $\langle x_i \rangle$ is fixed by α. Therefore $X^\alpha = X$ and this contradiction proves the lemma. $\square$

Theorem 3.2. *Let G be an abelian group such that $NAut\,G \neq PAut\,G$. Then either G is finitely generated or G is the direct product of a finitely generated group and a group of type p^∞ for some prime number p.*

Proof. Let α be a noetherian automorphism of G that is not a power automorphism, and consider the subgroup T of G consisting of all elements of finite order. Suppose first that T is not a Černikov group, so that in particular $NAut\,T = PAut\,T$ by Corollary 2.5. The socle S of T is infinite, and so $S = S_1 \times S_2$

where both S_1 and S_2 are infinite. If a is any element of infinite order of G, the subgroups $\langle a, S_1\rangle$ and $\langle a, S_2\rangle$ are fixed by α by Lemma 2.1; as $\langle a, S_1\rangle \cap \langle a, S_2\rangle = \langle a\rangle$, it follows that $\langle a\rangle^\alpha = \langle a\rangle$, a contradiction because α is not a power automorphism.

Therefore T is a Černikov group, and $G = T \times A$, where A is a torsion-free subgroup. Assume that T is not a finite extension of a group of type p^∞, so that T contains Prüfer subgroups P_1 and P_2 such that $\langle P_1, P_2\rangle = P_1 \times P_2$. In this case

$$NAut\,T = IAut\,T = PAut\,T,$$

(see [3], Proposition 2.5) and the argument used above can be repeated to show that α fixes all infinite cyclic subgroups of G, so that α is a power automorphism. This contradiction shows that T is a finite extension of a group of type p^∞.

Assume finally that A is not finitely generated, so that in particular $A^\alpha = A$ and $NAut\,A = PAut\,A$ by Lemma 3.1. If x is any element of T, there exist infinitely many elements $a_1, \ldots, a_n, \ldots$ of A such that

$$\langle x, a_1\rangle < \langle x, a_1, a_2\rangle < \ldots < \langle x, a_1, a_2, \ldots, a_n\rangle < \ldots,$$

and so $\langle x, a_1, \ldots, a_t\rangle^\alpha = \langle x, a_1, \ldots, a_t\rangle$ for a suitable positive integer t. As $\langle x\rangle$ is a characteristic subgroup of $\langle x, a_1, \ldots, a_t\rangle$, it follows that $\langle x\rangle^\alpha = \langle x\rangle$ and α induces a power automorphism on T. Therefore G contains an element of infinite order u such that $\langle u\rangle^\alpha \neq \langle u\rangle$. If A has infinite rank, there exist subgroups B_1 and B_2 of A that are not finitely generated and such that $B_1 \cap B_2 = \langle B_1, B_2\rangle \cap \langle u\rangle = \{1\}$; then $\langle u\rangle = \langle u, B_1\rangle \cap \langle u, B_2\rangle$ is fixed by α by Lemma 2.1, a contradiction. Thus A must have finite rank r. Let G be so that r is smallest possible with respect to the above properties. Consider a positive integer k such that $u^k \in A$, so that $\langle u^k\rangle^\alpha = \langle u^k\rangle$ and α induces on $\bar{G} = G/\langle u^k\rangle$ a noetherian automorphism which is not a power automorphism. In particular $NAut\,\bar{G} \neq PAut\,\bar{G}$ and the group $\bar{G}$ is not a finite extension of a Prüfer group; on the other hand, the torsion-free rank of $\bar{G}$ is $r-1$, and the minimal choice of r produces the final contradiction. □

4. Noetherian automorphisms of locally nilpotent groups

Let $\mathfrak{X}$ be a class of groups which is closed with respect to forming subgroups and extensions. We shall say that $\mathfrak{X}$ is a *BZ-class* if every soluble group which is not in $\mathfrak{X}$ contains an abelian non-$\mathfrak{X}$ subgroup. An easy example of BZ-class is provided by the class consisting of all finite groups. Moreover, two relevant theorems by Mal'cev and by Baer and Zaicev prove that the class of groups with the maximal condition on subgroups and that of minimax groups are BZ-classes (see for instance [10] Part 1, Theorem 3.31 and Part 2, Theorem 10.35).

Lemma 4.1. *Let $\mathfrak{X}$ be a BZ-class of groups and let G be a nilpotent group which does not belong to $\mathfrak{X}$. Then for each element x of G the centralizer $C_G(x)$ is not an $\mathfrak{X}$-group.*

Proof. It can obviously be assumed that $Z(G)$ belongs to $\mathfrak{X}$, so that $G/Z(G)$ cannot be an $\mathfrak{X}$-group. By induction on the nilpotency class of G, we have that also the centralizer $C/Z(G) = C_{G/Z(G)}(xZ(G))$ is not in $\mathfrak{X}$ and so it contains an abelian non-$\mathfrak{X}$ subgroup $A/Z(G)$ since $\mathfrak{X}$ is a BZ-class. As the map

$$\varphi : a \in A \longmapsto [a, x] \in Z(G)$$

is a homomorphism with $ker\varphi = C_A(x)$, the group $A/C_A(x)$ belongs to $\mathfrak{X}$, and so $C_A(x)$ is not in $\mathfrak{X}$. Therefore $C_G(x)$ is not an $\mathfrak{X}$-group. □

Our next result extends Lemma 3.1 to the case of nilpotent groups.

Proposition 4.2. *Let G be a torsion-free nilpotent group which is not finitely generated. Then $NAut\,G = PAut\,G$.*

Proof. Let x be any element of G. As groups satisfying the maximal condition on subgroups form a BZ-class, it follows from Lemma 4.1 that the centralizer $C_G(x)$ is not finitely generated, and so it contains an abelian subgroup A that also is not finitely generated. By Lemma 3.1 we have that $NAut\,\langle x, A\rangle = PAut\,\langle x, A\rangle$, so that in particular $\langle x\rangle^\alpha = \langle x\rangle$ for any noetherian automorphism α of G. Thus $NAut\,G = PAut\,G$. □

Let G be a group and let Γ be a group of automorphisms of G. Recall that Γ *stabilizes* a finite normal series

$$\{1\} = X_0 < X_1 < \ldots < X_t = G$$

if $[X_i, \Gamma] \leq X_{i-1}$ for each $i = 1, \ldots, t$; in this case a well known result by L.A. Kalužnin proves that Γ is nilpotent with class at most $t - 1$ (see for instance [7], Theorem 1.C.1). For our pourposes we need certain informations on radical groups of automorphisms of soluble minimax groups that were already known for hyperabelian groups of automorphisms.

Lemma 4.3. *Let G be a torsion-free abelian minimax group and let Γ be a radical group of automorphisms of G. Then Γ is a soluble minimax group.*

Proof. Each abelian subgroup of Γ is minimax (see [10] Part 2, Corollary to Lemma 10.37) and hence Γ itself is a soluble minimax group (see [10] Part 2, Theorem 10.35). □

Lemma 4.4. *Let G be a soluble residually finite minimax group and let Γ be a radical group of automorphisms of G. Then Γ is likewise a soluble residually finite minimax group.*

Proof. The group G has a finite characteristic series

$$\{1\} = K_0 < K_1 < \ldots < K_t \leq K_{t+1} = G$$

such that G/K_t is finite and K_i/K_{i-1} is torsion-free abelian for each $i = 1, \ldots, t$ (see [10] Part 2, Theorem 3.39.3). Consider in Γ the normal subgroup

$$\Lambda = \bigcap_{i=1}^{t+1} C_\Gamma(K_i/K_{i-1}).$$

As by Lemma 4.3 the group $\Gamma/C_\Gamma(K_i/K_{i-1})$ is soluble for $i = 1, \dots, t$ and $\Gamma/C_\Gamma(G/K_t)$ is finite, it follows that also the factor group Γ/Λ is soluble. On the other hand, Λ stabilizes a series of finite length of G, so that Λ is nilpotent and Γ is a soluble group. Thus Γ is also residually finite and minimax (see [8], Theorem 3). □

We will also need the following lemma, which is a special case of a result by L.A. Kurdachenko and H. Smith (see [9], Lemma 3.2).

Lemma 4.5. *Let G be a metabelian group whose derived subgroup G' satisfies the minimal condition on subgroups. If there exists an element $x \in G \setminus G'$ such that the centralizer $C_G(x)$ is minimax, then also G is minimax.*

Recall that the *Baer radical* of a group G is the subgroup generated by all abelian subnormal subgroups of G, and that G is a *Baer group* if it coincides with its Baer radical. Clearly, all Baer groups are locally nilpotent.

Lemma 4.6. *Let G be a radical group and let x be any element of the Baer radical of G. If G is not minimax, then either $\langle x\rangle^G$ is a periodic divisible abelian group or $C_G(x)$ contains an abelian non-minimax subgroup.*

Proof. If $\langle x\rangle$ is a normal subgroup of G, the group $G/C_G(x)$ is finite, so that $C_G(x)$ cannot be minimax and hence it contains an abelian non-minimax subgroup (see [10] Part 2, Theorem 10.35). Assume that $\langle x\rangle$ is not normal in G and suppose first that the normal closure $\langle x\rangle^G$ is minimax. Without loss of generality, it can be assumed that $\langle x\rangle^G$ properly contains its finite residual J. Put $\bar{G} = G/J$. Then $\bar{G}/C_{\bar{G}}(\langle\bar{x}\rangle^{\bar{G}})$ is minimax by Lemma 4.4, so that $C_{\bar{G}}(\langle\bar{x}\rangle^{\bar{G}})$ is not minimax and hence contains an abelian non-minimax subgroup $\bar{A}$. As the subgroup $\bar{B} = \langle\bar{x}, \bar{A}\rangle$ is abelian, the derived subgroup B' of B is contained in J and so it is an abelian group satisfying the minimal condition on subgroups. Moreover, x lies in $B \setminus B'$ and so $C_B(x)$ contains an abelian non-minimax subgroup by Lemma 4.5. Suppose now that $\langle x\rangle^G$ is not a minimax group. Then by induction on the defect of $\langle x\rangle$ in G, we have that either $C_G(\langle x\rangle^G)$ contains an abelian non-minimax subgroup or $\langle x\rangle^{G,2} = \langle x\rangle^{\langle x\rangle^G}$ is a periodic divisible abelian subgroup; in this latter case also $\langle x\rangle^G = (\langle x\rangle^{G,2})^G$ is periodic divisible abelian (see [10] Part 1, Lemma 4.46). The lemma is proved. □

We can now prove the following result, that should be seen in relation to Theorem 3.2.

Theorem 4.7. *Let G a Baer group which is not minimax. Then $NAut\,G = PAut\,G$.*

Proof. Let α be any noetherian automorphism of G. Consider an element x of G, and assume first that the normal closure $\langle x\rangle^G$ is a periodic divisible abelian group. Then α fixes $\langle x\rangle^G$ by Lemma 2.1 and it follows from Lemma 2.3 and Theorem 2.4 that $NAut\,(\langle x\rangle^G) = PAut\,(\langle x\rangle^G)$; in particular, $\langle x\rangle^\alpha = \langle x\rangle$. Suppose now that $\langle x\rangle^G$ is not a periodic divisible abelian group, so that by Lemma 4.6 the centralizer $C_G(x)$ contains an abelian non-minimax subgroup A. Then the abelian subgroup $\langle x, A\rangle$ is not finitely generated and so it is fixed by α by Lemma 2.1; moreover, it follows from Theorem 3.2 that $NAut\,\langle x, A\rangle = PAut\,\langle x, A\rangle$ and hence $\langle x\rangle^\alpha = \langle x\rangle$. Therefore α is a power automorphism of G and $NAut\,G = PAut\,G$. $\square$

In the last part of this section we prove that the above theorem also holds for locally nilpotent groups, provided that they are assumed to have finite abelian section rank.

Proposition 4.8. *Let G be a group and let H be the Hirsch-Plotkin radical of G. If $NAut\,G \neq PAut\,G$, then the set of primes $\pi(H)$ is finite.*

Proof. Assume by contradiction that the set $\pi = \pi(H)$ is infinite, so that in particular the subgroup T consisting of all elements of finite order of H is not a Černikov group. By hypothesis there exist a noetherian automorphism α and an element x of G such that $\langle x\rangle^\alpha \neq \langle x\rangle$. Clearly the subgroup $\langle x, T\rangle$ is not finitely generated, so that it is fixed by α by Lemma 2.1 and it follows from Corollary 2.5 that x must have infinite order. Let $\pi = \rho \cup \sigma$, where both ρ and σ are infinite and $\rho \cap \sigma = \emptyset$. Write $\rho = \{p_1, p_2, \ldots, p_n, \ldots\}$ and $\sigma = \{q_1, q_2, \ldots, q_n, \ldots\}$; moreover, for each positive integer n put

$$P_n = \underset{i\leq n}{\mathrm{Dr}}\, T_{p_i} \qquad \text{and} \qquad Q_n = \underset{i\leq n}{\mathrm{Dr}}\, T_{q_i}.$$

Clearly, $\langle x, P_n\rangle < \langle x, P_{n+1}\rangle$ and $\langle x, Q_n\rangle < \langle x, Q_{n+1}\rangle$ for all n and hence there exist positive integers r, s such that $\langle x, P_r\rangle$ and $\langle x, Q_s\rangle$ are fixed by α. Thus α leaves invariant also $\langle x\rangle = \langle x, P_r\rangle \cap \langle x, Q_s\rangle$ and this contradiction proves the lemma. $\square$

Theorem 4.9. *Let G be a locally nilpotent group with finite abelian section rank. If G is not minimax, then $NAut\,G = PAut\,G$.*

Proof. Assume by contradiction that $NAut\,G \neq PAut\,G$, so that there exist a noetherian automorphism α and an element x of G such that $\langle x\rangle^\alpha \neq \langle x\rangle$. By Proposition 4.8 the subgroup T consisting of all elements of finite order of G has finitely many Sylow subgroups; then T is a Černikov group and G/T is a torsion-free nilpotent group (see [10] Part 2, p.38). Let J be the finite residual of T; as T/J is finite, also the group G/J is nilpotent and it follows from Lemma 4.1 that the centralizer $C_{G/J}(xJ)$ contains an abelian non-minimax subgroup A/J. Put $B = \langle x, A\rangle$. Since B/J is abelian, the derived subgroup B' of B is contained in J; moreover, $NAut\,J = PAut\,J$ by Lemma 2.3 and Theorem 2.4, so that x does not belong to J. Application of Lemma 4.5 yields now that the centralizer $C_B(x)$ is not minimax, so that it also contains an abelian non-minimax subgroup C (see [10] Part 2, Theorem 10.35). Then $\langle x, C\rangle$ is an abelian non-minimax group and so $\langle x\rangle^\alpha = \langle x\rangle$ by Theorem 3.2. This contradiction completes the proof of the theorem. $\square$

In contrast to Theorem 3.2, it can be observed that there exists a locally nilpotent Černikov group G which is not a finite extension of a Prüfer subgroup and such that $NAut\,G$ is not abelian (and so in particular $NAut\,G \neq PAut\,G$). To see this, let

$$P = \langle x_0, x_1, \ldots, x_n, \ldots \mid x_0 = 1, x_{n+1}^3 = x_n \rangle$$

and

$$Q = \langle y_0, y_1, \ldots, y_n, \ldots \mid y_0 = 1, y_{n+1}^3 = y_n \rangle$$

be two groups of type 3^∞, and put $H = P \times Q$. Consider the semidirect product $G = \langle z \rangle \ltimes H$, where $z^3 = 1$, $x_n^z = y_n$ and $y_n^z = x_n^{-1} y_n^{-1}$ for all n, and let α be the automorphism of G defined by $z^\alpha = z x_1 y_1^{-1}$ and $h^\alpha = h$ for all $h \in H$. Since G does not contain normal subgroups of type 3^∞, every infinite subgroup of G either contains or is contained in H; it follows that α fixes all infinite subgroups of G, so that in particular α is a noetherian automorphism of G which is not in $PAut\,G$. Moreover, α does not commute with the noetherian automorphism β of G defined by $z^\beta = z$ and $h^\beta = h^{-1}$ for all $h \in H$.

5. Noetherian automorphisms of soluble groups

The aim of this section is to prove that (with the obvious exceptions) the group of all noetherian automorphisms of a soluble group is likewise soluble, and that it is possible to bound the derived length of such group.

Lemma 5.1. *Let G be a group whose chief factors are abelian. If G contains a divisible abelian non-trivial subgroup D, then the derived subgroup of $NAut\,G$ acts trivially on D^G and G/D^G. In particular, the group $NAut\,G$ is metabelian.*

Proof. If g is any element of G, the subgroup D^g is fixed by all noetherian automorphisms of G by Lemma 2.1, and it follows from Theorem 3.2 that $\Gamma = NAut\,G$ acts on D^g as a group of power automorphisms. Thus the derived subgroup Γ' of Γ acts trivially on the normal closure D^G of D. Let x be any element of G and assume by contradiction that the subgroup $H = \langle x, D^G \rangle$ is finitely generated. Then D^G is the normal closure in G of a finite subset, and hence it contains a maximal proper G-invariant subgroup K. As the chief factor D^G/K is abelian, the group H/K is metabelian and so also residually finite (see [10] Part 2, Theorem 9.51). This is a contradiction since D^G has no proper subgroups of finite index. It follows that H cannot be finitely generated, so that H is fixed by all noetherian automorphsms of G by Lemma 2.1 and hence Γ induces a group of power automorphisms on G/D^G. Therefore Γ' acts trivially also on G/D^G. In particular, the group Γ is metabelian. □

Observe that, under the assumptions of Lemma 5.1, it cannot be proved that $NAut\,G$ is abelian, even when G is abelian. In fact, if the group $G = P \times A$ is the direct product of a group P of type p^∞ and a group A of order p (where p is an odd prime), then $NAut\,G$ is isomorphic to $Aut\,P \times Hol\,A$ and in particular it is not abelian (see [1]).

Lemma 5.2. *Let G be a group and let J be the finite residual of G. If Γ is the group of all automorphisms of G fixing every subgroup of finite index, then $[G, \Gamma'] \leq J$.*

Proof. Let N be any normal subgroup of finite index of G. Then N is fixed by Γ and Γ induces on G/N a group of power automorphisms. In particular, Γ' acts trivially on G/N and hence $[G, \Gamma'] \leq J$. □

Lemma 5.3. *Let G be a soluble group with derived length k and let Γ be the group of all noetherian automorphisms of G fixing every subgroup of finite index. If G is not polycyclic, then Γ' stabilizes a finite normal series of length at most $k + 1$.*

Proof. Suppose first that G is abelian. Clearly, it is enough to consider the case $NAut\,G \neq PAut\,G$, so that by Theorem 3.2 we have $G = P \times E$, where P is a Prüfer group and E is finitely generated. Then Γ' acts trivially on G/P by Lemma 5.2 and so it stabilizes the series $\{1\} < P < G$. Suppose now that $k > 1$. By Lemma 5.1 it can be assumed that G does not contain divisible abelian non-trivial subgroups. Let A be the smallest non-trivial term of the derived series of G. Suppose that A is finitely generated. Then $G/C_G(A)$ is polycyclic (see [10] Part 1, Theorem 3.27) and $C_G(A)$ must contain an abelian subgroup B which is not finitely generated (see [10] Part 1, Theorem 3.31). Then the abelian subgroup $H = AB$ is fixed by Γ and $NAut\,H = PAut\,H$, so that in particular Γ' acts trivially on H and hence $[A, \Gamma'] = \{1\}$. On the other hand, if A is not finitely generated, it follows from Theorem 3.2 that $\Gamma/C_\Gamma(A)$ is abelian and so $[A, \Gamma'] = \{1\}$ also in this case. If G/A is polycyclic, it follows from Lemma 5.2 that Γ' acts trivially on G/A. Assume finally that G/A is not polycyclic. By induction on k we have that Γ' stabilizes a finite normal series of G/A with length at most k, and hence Γ' also stabilizes a finite normal series of G with length at most $k + 1$. □

We can now prove the main result of this section.

Theorem 5.4. *Let G be a soluble group with derived length k. If G is not finitely generated, then the derived subgroup of $NAut\,G$ is nilpotent of class at most k. In particular, the group $NAut\,G$ is soluble with derived length at most $k + 1$.*

Proof. By Lemma 2.1 all noetherian automorphisms of G fix every subgroup of finite index, so that the statement follows from Lemma 5.3. □

We finally note that for certain locally soluble groups the above theorem can be improved. This holds in particular for all locally polycyclic groups.

Theorem 5.5. *Let G be a group whose finitely generated subgroups are soluble of finite rank. If G is not finitely generated, then the group $NAut\,G$ is metabelian.*

Proof. By Lemma 5.1 and Lemma 5.2 it can be assumed without loss of generality that the group G is not minimax. Suppose first that G has finite rank, so that in particular it is hyperabelian (see [10] Part 2, p.178) and hence it contains an abelian non-minimax subgroup A (see [10] Part 2, Theorem 10.35). It follows from Lemma 2.1 and Theorem 3.2 that $NAut\,G$ acts as a group of power automorphisms on A^g for each element g of G. In particular, the derived subgroup of $\Gamma = NAut\, G$ acts trivially on the normal closure $N = A^G$ of A. If x is any element of G, the subgroup $\langle x, N\rangle$ cannot be finitely generated (see [10] Part 2, Theorem 10.38) and hence it is fixed by each noetherian automorphism of G. Therefore $[G, \Gamma'] \leq N$, so that Γ' is abelian and the group $NAut\, G$ is metabelian. Assume now that G has infinite rank, so that it contains an abelian subgroup A of infinite rank (see [5]), and the above argument can be repeated to obtain that $NAut\, G$ is metabelian. $\square$

References

[1] J.C. Beidleman and H. Heineken, *A note on I-automorphisms*, J. Algebra **234** (2000), 694–706.

[2] C.D.H. Cooper, *Power automorphisms of a group*, Math. Z. **107** (1968), 335–356.

[3] M. Curzio, S. Franciosi and F. de Giovanni, *On automorphisms fixing infinite subgroups of groups*, Arch. Math. (Basel) **54** (1990), 4–13.

[4] L. Fuchs, *Infinite Abelian Groups, vol. I.*, Academic Press, New York (1970).

[5] Yu.M. Gorčakov, *The existence of abelian subgroups of infinite rank in locally soluble groups*, Soviet Math. Dokl. **5** (1964), 591–594.

[6] B. Hartley, *Fixed points of automorphisms of certain locally finite groups and Chevalley groups*, J. London Math. Soc. **37** (1988), 421–436.

[7] O.H. Kegel and B.A.F. Wehrfritz, *Locally Finite Groups*, North-Holland, Amsterdam, 1973.

[8] L.A. Kurdachenko and J. Otal, *Frattini properties of groups with minimax conjugacy classes*, Quaderni Mat. **8** (2001), 221–237.

[9] L.A. Kurdachenko and H. Smith, *Groups with the weak minimal condition for nonsubnormal subgroups*, Ann. Mat. Pura Appl. **173** (1997), 299–312.

[10] D.J.S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin, 1972.

[11] V.P. Šunkov, *On the minimality problem for locally finite groups*, Algebra Logic **9** (1970), 137–151.

Fausto De Mari and Francesco de Giovanni
Dipartimento di Matematica e Applicazioni
Università di Napoli Federico II
Complesso Universitario Monte S. Angelo
Via Cintia
I-80126 Napoli
Italy
e-mail: `fausto.demari@dma.unina.it`
`degiovan@unina.it`

Received: November 11, 2004
Revised: April 15, 2005
Accepted: April 29, 2005