Balancing Domain Decomposition by Constraints (BDDC) preconditioners for divergence free virtual element discretizations of the Stokes equations

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1. Stokes Problem
2. Virtual Element Method
3. Domain Decomposition
4. BDDC preconditioner
5. Numerical Results
6. Tridimensional Case

1. Stokes Problem:

Let $\Omega \subseteq \mathbb{R}^2$, we consider the Stokes problem with homogeneous Dirichet boundary condition:

$$\begin{cases} \text{Find } (\mathbf{u}, p) \text{ such that} & \\ -\nu \boldsymbol{\Delta} \mathbf{u} - \nabla p = \mathbf{f} & \text{in } \Omega \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

where $\mathbf{u}$ and p are the velocity and the pressure fields, respectively. Let us consider the spaces:

$$\mathbf{V} := [H_0^1(\Omega)]^2, \qquad Q := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \text{ s.t. } \int_\Omega q d\Omega = 0 \right\}$$

with norms

$$\|\mathbf{v}\|_1 := \|\mathbf{v}\|_{[H^1(\Omega)]^2}, \quad \|q\|_Q := \|q\|_{L^2(\Omega)}.$$

We assume $\mathbf{f} \in [H^{-1}(\Omega)]^2$, and $\nu \in L^\infty(\Omega)$ uniformly positive in Ω. Let the bilinear forms $a(\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $b : \mathbf{V} \times Q \to \mathbb{R}$ be defined as:

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega \qquad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{V}$$

$$b(\mathbf{v},q) := \int_\Omega \operatorname{div} \mathbf{v} q d\Omega \qquad \text{for all } \mathbf{u} \in \mathbf{V}, q \in Q.$$

Then a standard variational formulation of problem (3) reads:

$$\begin{cases} \text{find } (\mathbf{u},p) \in \mathbf{V} \times Q \text{ such that} \\ a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) = 0 & \text{for all } q \in Q, \end{cases} \tag{1}$$

where

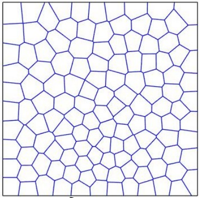
$$(\mathbf{f},\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} d\Omega.$$

Oss

(1) has unique solution $(\mathbf{u},p) \in \mathbf{V} \times Q$ [Boffi-Brezzi-Fortin, Mixed finite element methods and applications, 2013].

2. Virtual Element Method (VEM)

Let $\{T_h\}_h$ be a sequence of decompositions of Ω into general polygonal elements K with



$$h_K := \text{diameter}(K), \quad h := \sup_{K \in T_h} h_K.$$

For $k \in \mathbb{N}$, let us define the spaces:

- $\mathbb{P}_k(K)$ the set of polynomials on K of degree $\leq k$,
- $\mathbb{B}_k(\partial K) := \{v \in C^0(\partial K) \text{ s.t. } v_{|e} \in \mathbb{P}_k(e) \quad \forall \text{ edge } e \in \partial K\}$.

Classical Local Spaces[1]

On each element $K \in T_h$, for $k \geq 2$:

$$\widetilde{\mathbf{V}}_h^K := \{\mathbf{v} \in [H^1(K)]^2 \text{ s.t. } \mathbf{v}_{|\partial K} \in [\mathbb{B}_k(\partial K)]^2, \quad \nu \boldsymbol{\Delta} \mathbf{v} \in [\mathbb{P}_{k-2}(K)]^2\}$$

and

$$Q_h^K := \mathbb{P}_{k-1}(K).$$

[1] L. Beirão da Veiga et al. Basic principles of virtual element methods, 2013

2.1 Different Approach for Local Velocity Spaces

Defining:

- $G_k(K) := \nabla(\mathbb{P}_{k+1}(K)) \subseteq [\mathbb{P}_k(K)]^2$,
- $G_k(K)^\perp \subseteq [\mathbb{P}_k(K)]^2$ the L^2-orthogonal complement to $G_k(K)$.

Div Free Approach[2]:

$$\widehat{\mathbf{V}}_h^K := \left\{ \mathbf{v} \in [H^1(K)]^2 \text{ s.t. } \mathbf{v}_{|\partial K} \in [\mathbb{B}_k(\partial K)]^2, \begin{cases} -\nu \boldsymbol{\Delta} \mathbf{v} - \nabla s \in G_{k-2}(K)^\perp, \\ \text{div } \mathbf{v} \in \mathbb{P}_{k-1}(K), \end{cases} \text{for some} s \in L^2(K) \right\}$$

- $\text{div}\widehat{\mathbf{V}}_h \subseteq Q_h$, this leads to an exactly divergence free discrete velocity;

[2] L. Beirão da Veiga, C. Lovadina, and G. Vacca. Divergence free virtual elements for the stokes problem on polygonal meshes, 2017

Given a function $\mathbf{v} \in \widehat{\mathbf{V}}_h^K$ we take the following linear operators $\mathbf{D}_{\widehat{\mathbf{V}}}$, split into four subsets:

- $\mathbf{D}_{\widehat{\mathbf{V}}}\mathbf{1}$: the values of $\mathbf{v}$ at the vertices of the polygon K,
- $\mathbf{D}_{\widehat{\mathbf{V}}}\mathbf{2}$: the values of $\mathbf{v}$ at $k-1$ distinct points of every edge $e \in \partial K$ ($k-1$ internal points of the $(k+1)$-Gauss-Lobatto quadrature rule),
- $\mathbf{D}_{\widehat{\mathbf{V}}}\mathbf{3}$: the moments of the values of $\mathbf{v}$

$$\int_K \mathbf{v} \cdot \mathbf{g}_{k-2}^{\perp} dK \qquad \text{for all } \mathbf{g}_{k-2}^{\perp} \in \mathbf{G}_{k-2}(K)^{\perp}$$

- $\mathbf{D}_{\widehat{\mathbf{V}}}\mathbf{4}$: the moments up to order $k-1$ and greater than zero of div$\mathbf{v}$ in K, *i.e.*

$$\int_K (\text{div } \mathbf{v}) q_{k-1} dK \qquad \text{for all } q_{k-1} \in \mathbb{P}_{k-1}(K)/\mathbb{R}.$$

Furthermore, for the local pressure, given $q \in Q_h^K$, we consider the linear operators $\mathbf{D_Q}$:

- $\mathbf{D_Q}$: the moments up to order $k-1$ of q, *i.e.*

$$\int_K q \; p_{k-1} dK \qquad \text{for all } p_{k-1} \in \mathbb{P}_{k-1}(K).$$

We are able to compute $\forall \mathbf{u} \in \widehat{\mathbf{V}}_h^K, q \in Q_h^K$:

$$b_h^K(\mathbf{v}, q) := b^K(\mathbf{v}, q) = \int_K \mathrm{div}\mathbf{v} q dK,$$

and, because we can write $\nu \boldsymbol{\Delta} \mathbf{q}_k = \nabla q_{k-1} + \mathbf{g}_{k-2}^{\perp}$:

$$a^K(\mathbf{q}_k, \mathbf{v}) = \int_K \nu \nabla \mathbf{q}_k : \nabla \mathbf{v} dK = \int_K \nu \boldsymbol{\Delta} \mathbf{q}_k \cdot \mathbf{v} dK + \int_{\partial K} (\nu \nabla \mathbf{q}_k \mathbf{n}) \cdot \mathbf{v} ds.$$

We define:

$$a_h^K(\mathbf{w}_k, \mathbf{v}_h) := a^K(\Pi_k^{\nabla,K} \mathbf{w}_k, \Pi_k^{\nabla,K} \mathbf{v}_h) + S^K((I - \Pi_k^{\nabla,K})\mathbf{w}_h, (I - \Pi_k^{\nabla,K})\mathbf{v}_h)$$

with $\Pi_k^{\nabla,K} : \mathbf{V}_h^K \to [\mathbb{P}_k(K)]^2$ and $S^K : \mathbf{V}_h^K \times \mathbf{V}_h^K \to \mathbb{R}$.

Theorem

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the solution of problem* (1) *and* $(\mathbf{u}_h, p_h) \in \widehat{\mathbf{V}}_h \times Q_h$ *be the solution of the discrete problem. Then it holds*

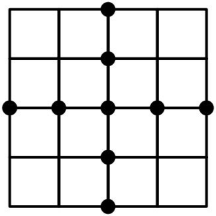
$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leq Ch^k(|\mathbf{f}|_{k-1} + |\mathbf{u}|_{k+1})$$

and

$$\|\mathbf{p} - \mathbf{p}_h\|_1 \leq Ch^k(|\mathbf{f}|_{k-1} + |\mathbf{u}|_{k+1} + |p|_k).$$

Note

- The error on the velocity is not affected by pressure;
- Possibility to use a "Reduced Space" ($\mathbb{P}_0(K)$ for pressure).



We write the discrete problem in variational form:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}. \tag{2}$$

We decompose the discrete velocity and pressure space $\widehat{\mathbf{V}}$ and Q (instead of $\widehat{\mathbf{V}}_h \times Q_h$) into:

$$\widehat{\mathbf{V}} = \mathbf{V}_I \bigoplus \widehat{\mathbf{V}}_\Gamma, \quad Q = Q_I \bigoplus Q_0. \tag{3}$$

$\mathbf{V}_I$ and Q_I are direct sums of subdomain interior velocity spaces $\mathbf{V}_I^{(i)}$, and subdomain interior pressure spaces $Q_I^{(i)}$, respectively, i.e.,

$$\mathbf{V}_I = \bigoplus_{i=1}^{N} \mathbf{V}_I^{(i)}, \quad Q_I = \bigoplus_{i=1}^{N} Q_I^{(i)}. \tag{4}$$

Whit this decomposition of the solution space as in (3), the global saddle point problem (2) can be written as: find $(\mathbf{u}_I, p_I, \mathbf{u}_\Gamma, p_0) \in (\mathbf{V}_I, Q_I, \widehat{\mathbf{V}}_\Gamma, Q_0)$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & \widehat{A}_{\Gamma I}^T & 0 \\ B_{II} & 0 & \widehat{B}_{I\Gamma} & 0 \\ \widehat{A}_{\Gamma I} & \widehat{B}_{I\Gamma}^T & \widehat{A}_{\Gamma\Gamma} & \widehat{B}_{0\Gamma}^T \\ 0 & 0 & \widehat{B}_{0\Gamma}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Gamma \\ 0 \end{bmatrix}. \tag{5}$$

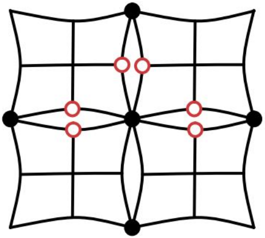
Eliminating the indipendent subdomain variables $(\mathbf{u}_I, p_I)$ from the global problem (5), we have the global interface problem

$$\widehat{S}\ \widehat{u} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix} = \widehat{\mathbf{g}}, \tag{6}$$

4. BDDC Preconditioner

We introduce a partially assembled interface velocity space $\widetilde{\mathbf{V}}_\Gamma$:

$$\widetilde{\mathbf{V}}_\Gamma = \widehat{\mathbf{V}}_\Pi \bigoplus \mathbf{V}_\Delta = \widehat{\mathbf{V}}_\Pi \bigoplus \Big(\prod_{i=1}^{N} \mathbf{V}_\Delta^{(i)} \Big).$$



Introducing restriction and scaling operators we obtain the BDDC preconditioned problem: find $(\mathbf{u}_\Gamma, p_0) \in \widehat{\mathbf{V}}_\Gamma \times Q_0$, such that

$$M^{-1}\widehat{S}\begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = M^{-1}\begin{bmatrix} \mathbf{g}_\Gamma \\ 0 \end{bmatrix}, \tag{7}$$

with $M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D$.

Note

(7) is symmetric definite positive, use CG method.

Assumption 1

For any $\mathbf{v}_\Delta \in \mathbf{V}_\Delta$, $\int_{\partial\Omega_i} \mathbf{v}_\Delta^{(i)} \cdot \mathbf{n} = 0$ and $\int_{\partial\Omega_i} (E_{D,\Delta}\mathbf{v}_\Delta)^{(i)} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the outward normal of $\partial\Omega_i$. We can equivalently write $B_{0\Delta}^{(i)}\mathbf{v}_\Delta^{(i)} = 0$ and $B_{0\Delta}^{(i)}(E_{D,\Delta}\mathbf{v}_\Delta)^{(i)} = 0$

Assumption 2

There exist a positive constant C, which is indipendent of H, h and the number of subdomains, such that

$$|\bar{R}_\Gamma(E_{D,\Gamma}\mathbf{v}_\Gamma)|_{1/2,\Gamma} \leq C\left(1 + log\frac{Hk^2}{h}\right)|\bar{R}_\Gamma\mathbf{v}_\Gamma|_{1/2,\Gamma}, \quad \forall \mathbf{v}_\Gamma \in \mathbf{V}_\Gamma.$$

Satifying the assumption

- (1) $\int_{\Gamma_{ij}} \mathbf{v}_\Gamma^{(i)} \cdot \mathbf{n}_{ij} = \int_{\Gamma_{ij}} \mathbf{v}_\Gamma^{(j)} \cdot \mathbf{n}_{ij}$;
- (2) Bertoluzza,Pennacchio,Prada. BDDC and FETI-DP for the virtual element method 2017.

Theorem 1

Let Assumption 1 and 2 hold. The preconditioned operator $M^{-1}\widehat{S}$ is then symmetric, positive definite with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\widehat{S}}$ on the benign space $\widehat{\mathbf{V}}_{\Gamma,B} \times Q_0$. Its minimum eigenvalue is 1 and its maximum eigenvalue is bounded by

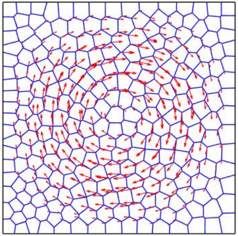
$$C \frac{1}{\beta^2} \left(1 + log \frac{Hk^2}{h} \right)^2.$$

Here, C is a constant which is indipendent of H, h and the number of subdomains and β is the inf-sup stability constant.

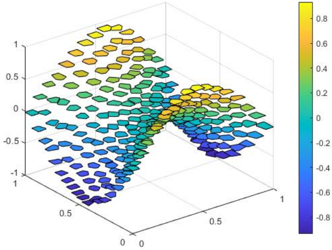
$\Omega = [0,1] \times [0,1]$, homogeneous boundary conditions on the whole $\partial\Omega$ and $\mathbf{f}$ such that the analytical solution is

$$\mathbf{u}(x,y) = \begin{pmatrix} -sin(\pi x)sin(\pi x)sin(2\pi y) \\ sin(\pi y)sin(\pi y)sin(2\pi x) \end{pmatrix}, \quad p(x,y) = sin(\pi x) - sin(\pi y).$$

Test performed in Matlab (serial code).



(a) Velocity field.



(b) Pressure.

5.1 Square meshes

(a) Schur (GMRES) complement.

1/H \ 1/h	8 it	16 it	32 it	64 it	128 it
2	7	13	18	27	40
4	x	40	57	79	99
8	x	x	140	203	282
16	x	x	x	295	437
32	x	x	x	x	605

(b) BDDC with only vertices.

1/H \ 1/h	8 it	16 it	32 it	64 it	128 it
2	7	7	7	7	8
4	x	12	13	15	17
8	x	x	22	23	29
16	x	x	x	22	26
32	x	x	x	x	22

(c) BDDC with vertices and one basis function per edge.

1/H \ 1/h	8		16		32		64		128	
	k2	it	k2	it	k2	it	k2	it	k2	it
2	1,82	7	2,11	7	2,37	7	2,80	7	3,24	8
4	x	x	2,80	9	5,75	10	7,20	11	8,76	12
8	x	x	x	x	5,78	13	7,81	15	10,09	16
16	x	x	x	x	x	x	6,16	16	8,46	19
32	x	x	x	x	x	x	x	x	6,29	16

Table: Iteration count and condition number (where possible) to solve the interface problem for different mesh elements size and different number of subdomains.

5.2 Voronoi Meshes

(a) Schur (GMRES) complement.

1/h \ 1/H	8	16	32	64	128
	it	it	it	it	it
2	31	44	55	79	101
4	x	89	104	134	172
8	x	x	267	307	336
16	x	x	x	885	924
32	x	x	x	x	2522

(b) BDDC with only vertices.

1/h \ 1/H	8	16	32	64	128
	it	it	it	it	it
2	16	17	17	17	17
4	x	26	30	32	33
8	x	x	36	41	41
16	x	x	x	50	50
32	x	x	x	x	51

(c) BDDC with vertices and one basis function per edge.

1/h \ 1/H		8		16		32		64		128
	k2	it	k2	it	k2	it	k2	it	k2	it
2	5,82	14	6,97	15	8,16	16	9,32	16	10,23	16
4	x	x	10,20	20	13,87	21	15,98	22	17,22	23
8	x	x	x	x	22,24	27	21,43	28	23,13	28
16	x	x	x	x	x	x	30,12	34	28,34	33
32	x	x	x	x	x	x	x	x	30,28	35

Table: Iteration count and condition number (where possible) to solve the interface problem for different mesh elements size and different number of subdomains.

5.3 BDDC vertexes and two basis function per edge

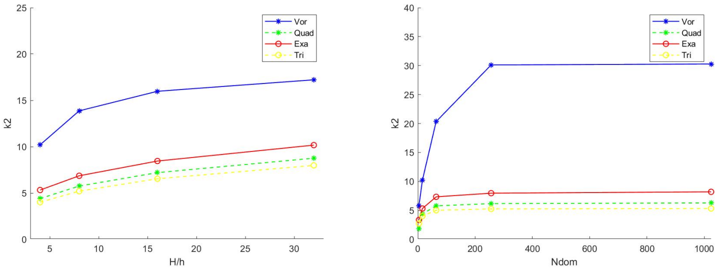
(a) Square Meshes.

1/H \ 1/h	8 k2	8 it	16 k2	16 it	32 k2	32 it	64 k2	64 it	128 k2	128 it
2	1,48	7	1,68	7	1,90	7	2,18	8	2,51	7
4	x	x	2,80	9	3,73	10	4,78	10	5,93	11
8	x	x	x	x	2,99	10	4,05	11	5,20	13
16	x	x	x	x	x	x	2,72	9	3,67	10
32	x	x	x	x	x	x	x	x	2,64	8

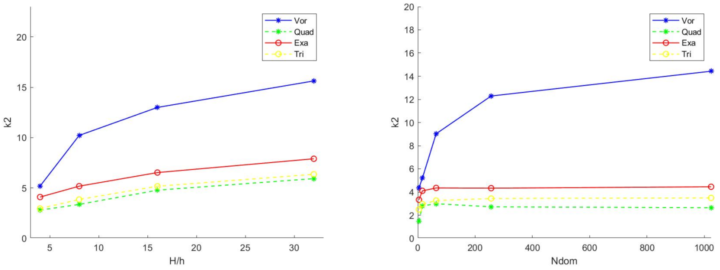
(b) Voronoi meshes.

1/H \ 1/h	8 k2	8 it	16 k2	16 it	32 k2	32 it	64 k2	64 it	128 k2	128 it
2	4,35	13	5,27	14	6,63	15	13,88	15	10,03	16
4	x	x	5,20	15	10,22	18	13,00	19	15,63	20
8	x	x	x	x	9,03	20	17,52	23	18,41	22
16	x	x	x	x	x	x	12,29	21	19,21	24
32	x	x	x	x	x	x	x	x	14,43	23

Table: BDDC algorithm iteration count to solve the interface problem for different mesh elements size and different number of subdomains with a primal coarse space spanned by corner and two edge basis function for each edge for different meshes.



(c) Optimality (4x4 procs) and weak Scaling with one basis function per edge.



(d) Optimality (4x4 procs) and weak Scaling with two basis function per edge.

Joint work with: Simone Scacchi, Franco Dassi and Stefano Zampini.

Library (Parallel Implementation):

- Vem++ (Franco Dassi);
- Petsc (BDDC by Stefano Zampini).

Test performed on the Linux Cluster INDACO (www.indaco.unimi.it) of the University of Milan.
Results for cube meshes, with BDDC preconditioner with Deluxe Scaling and an adaptive Coarse space.

6.1 Optimality Test (Cube Meshes)

(a) Refining the mesh, k = 2, procs = 32.

nEl	nDofs	tol = 2			tol = ∞		
		nPrimal	it	cond	nPrimal	it	cond
4096	124195	3099	8	1,53	80	76	198
8000	239763	3959	8	1,52	80	79	211
13824	408243	4775	8	1,53	80	81	221
21952	643387	5633	8	1,52	80	83	230
32768	954947	6381	8	1,51	80	85	238

(b) Refining the mesh, k = 3, procs = 32.

nEl	nDofs	tol = 2			tol = ∞		
		nPrimal	it	cond	nPrimal	it	cond
512	40667	2823	8	1,43	80	82	211
1728	131655	4587	8	1,40	80	86	237
4096	305587	6311	8	1,41	80	90	253
8000	589343	8055	8	1,45	80	92	264
13824	1009803	9839	8	1,48	80	95	274

(c) Increasing the polynomial degree, nEl = 4096, procs = 32.

k	nDofs	tol = 2			tol = ∞		
		nPrimal	it	cond	nPrimal	it	cond
2	124195	3099	8	1,53	80	76	198
3	305587	6311	8	1,41	80	90	253
4	579395	9867	8	1,53	80	106	310

6.2 Strong Scalability and other Solvers (Cube Meshes)

(d) Strong Scalability, k = 2, nEl = 21952, nDofs = 643387.

procs	S^{id}	Tass	S_p	tol = 2 nPrimal	it	Tsol	S_p	tol = ∞ nPrimal	it	Tsol	S_p
4		1652		447	8	2467		4	19	2316	
8	2	842	2,0	1159	8	687	3,6	12	32	625	3,7
16	4	462	3,6	2243	8	387	6,4	36	45	345	6,7
32	8	226	7,3	4775	8	151	16,3	80	80	133	17,4

(e) Strong Scalability, k = 3, nEl = 8000, nDofs = 589343.

procs	S^{id}	Tass	S_p	tol = 2 nPrimal	it	Tsol	S_p	tol = ∞ nPrimal	it	Tsol	S_p
4		3815		783	7	7982		4	23	10001	
8	2	1923	2,0	2007	7	2853	2,8	12	38	2454	4,1
16	4	1011	3,8	3843	8	1466	5,4	36	51	1264	7,9
32	8	500	7,6	8055	8	467	17,1	80	92	500	18,2

(f) Comparision with other Solvers, procs = 32.

nEl	k	nDofs	Pardiso Tsol	MUMPS Tsol	BDDC it	tol = 5 Tsol
32768	2	954947	8854	7284	20	827
13824	3	1009803	12213	7033	15	1321

Thanks for your attention