Geometrical properties of the graph *p*-Laplacian spectrum

Due Giorni di Algebra Lineare Numerica e Applicazioni 2022

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A joint work with: Mario Putti, Francesco Tudisco, Nicola Segala, Martin Burger

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UNIVERSITÀ DEGLI STUDI DI PADOVA

Outline

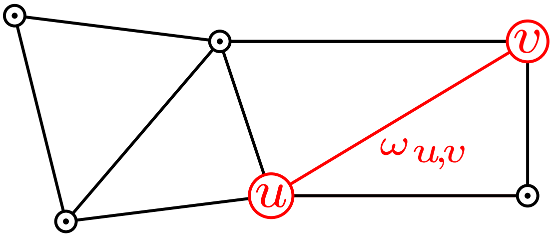
- Introduction
- Cheeger constants and $p \to 1$
- Maximal radius of balls incribed in the graph and $p \to \infty$
- Nodal Domains of the p-Laplacian eigenfunctions
- Shortest paths on the graph and $p = \infty$

p-Laplacian operator

Graph

$$\mathcal{G} = (V, E, \omega)$$

$$f : V \to \mathbb{R} \quad \Rightarrow \quad \nabla f(uv) = \omega_{uv}\Big(f(v) - f(u)\Big)$$



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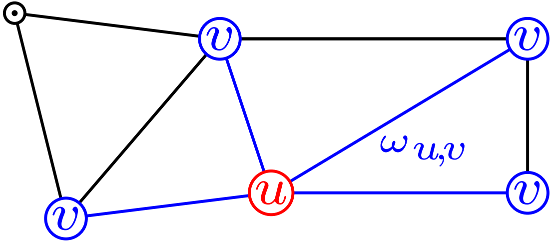
$$G : E \to \mathbb{R} \quad \Rightarrow \quad divG(u) = -\nabla^T G(u) = -\sum_{v \sim u} \omega_{uv} G(uv)$$



p-Laplacian operator

p-Laplacian

$$(\Delta_p f)(u) = -div(|\nabla f|^{p-2} \nabla f)(u) = \sum_{v \sim u} \omega_{uv}^p |\nabla f(vu)|^{p-2} \nabla f(uv)$$

p-Laplacian Eigen-equation

"Rayleigh" quotient

$$\mathcal{R}_{\Delta_p}(f) = \frac{\|\nabla f\|_p^p}{\|f\|_p^p}$$

Critical points of RQ: p-Laplacian Eigen-equation

p-Laplacian Eigen-equation

"Rayleigh" quotient

$$\mathcal{R}_{\Delta_p}(f) = \frac{\sum_{uv \in E} |\nabla f(uv)|^p}{\sum_{u \in V} |f(u)|^p}$$

Critical points of RQ: p-Laplacian Eigen-equation

$$\Delta_p(f(u)) = \lambda\, |f(u)|^{p-2} f(u)$$

p-Laplacian Eigen-equation

"Rayleigh" quotient

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Critical points of RQ: p-Laplacian Eigen-equation

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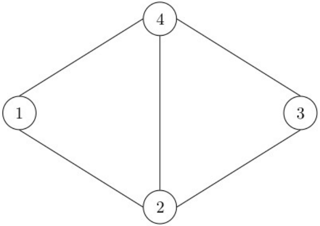
Eigenpairs > space dimension: no orthogonality

Example: 4 node graph ⇒ 5 eigenpairs

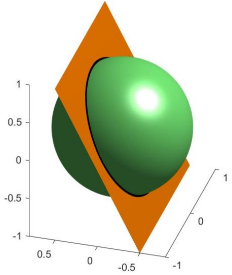
1. $\lambda_1 = 0\,,\ f_1 = (1, 1, 1, 1)$
2. $\lambda_2 = 2\,,\ f_2 = (1, 0, -1, 0)$
3. $\lambda_3 = 2 + 2^{p-1}$,
   $f_3 = (0, 1, 0, -1)$
4. $\lambda_4 = 1 + \left(1 + 2^{\frac{1}{p-1}}\right)^{p-1}$,
   $f_4 = (1, 0, 1, -2^{\frac{1}{p-1}})$
5. $\lambda_5 = 2^p,\ f_5 = (1, -1, 1, -1)$



Search for n eigenpairs: Variational Eigenvalues

Linear case:

$$\lambda_k = \min_{dim(A)\geq k} \max_{f\in A} \mathcal{R}_{\Delta_2}(f)\,.$$



Nonlinear case:

$$\lambda_k = \min_{A\in\mathcal{F}_k} \max_{f\in A} \mathcal{R}_{\Delta_p}(f)\,.$$

$$\mathcal{F}_k(S_p) := \left\{ \begin{array}{l} A = -\bar{A} \subseteq S_p\,,\ s.t.\ \exists m \geq k \\ \exists\psi \in C(A, S^m),\ \psi(x) = -\psi(-x) \end{array} \right\}$$

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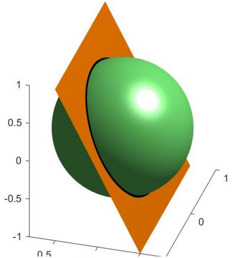

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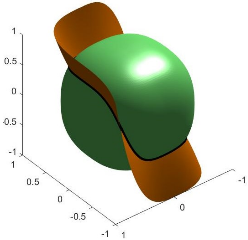


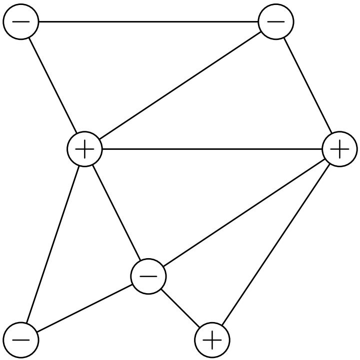
Nodal Domains

Nodal Domain

$$f : V \to \mathbb{R}.$$

Nodal domain of f = maximal connected subgraph of $\{f(u) > 0\}$ and $\{f(u) < 0\}$

$\mathcal{N}(f)$ = number of nodal domains



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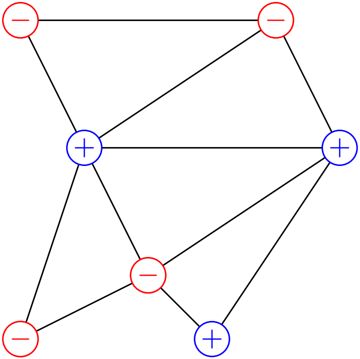

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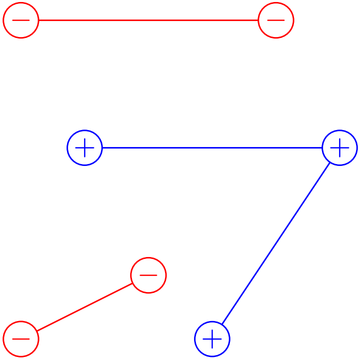


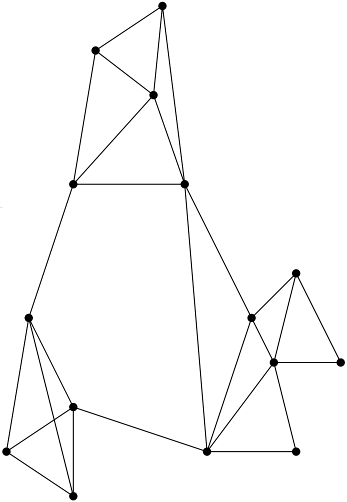
Cheeger constants

$$cut(A) = \frac{\omega(E(A \to A^c))}{\#(A)}$$

$$\mathcal{D}_k(\mathcal{G}) = \{A_1, \dots, A_k \neq \emptyset \mid A_i \cap A_j = \emptyset\}$$

k-th Cheeger Constant of $\mathcal{G}$

$$h_k(\mathcal{G}) := \min_{\mathcal{P} \in \mathcal{D}_k(\mathcal{G})} \max_{A \in P} cut(A)$$



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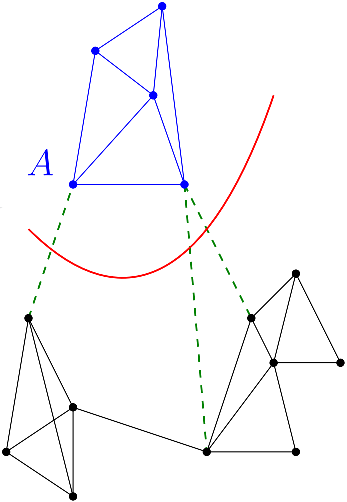

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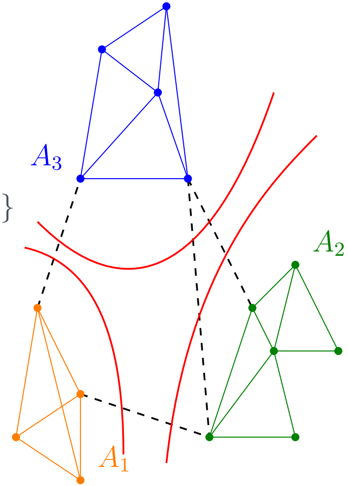

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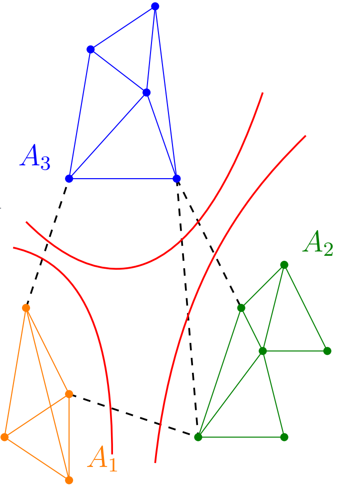

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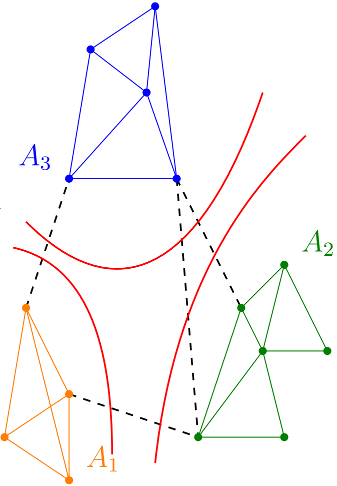
$$p \to 1$$



Dirichlet Problem

Dirichlet p-Laplacian Eigen-equation

$B \subset V$ BOUNDARY

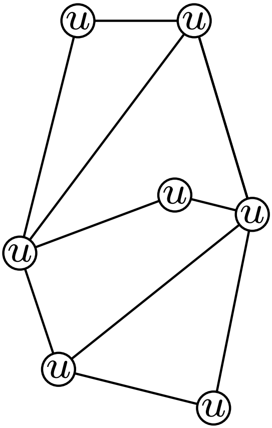
$$\begin{cases} \Delta_p f(u) = \lambda |f(u)|^{p-2} f(u) & u \in V \setminus B \\ f(u) = 0 & u \in B \end{cases}$$

Distance

$$d(u,v) = \min_{\substack{u=u_1 \\ v=u_n}} \sum \omega_{u_i u_{i+1}}^{-1}$$

Boundary Distance

$$d_B(u) = \min_{v \in B} d(u,v)$$



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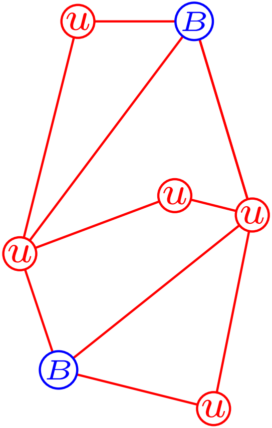

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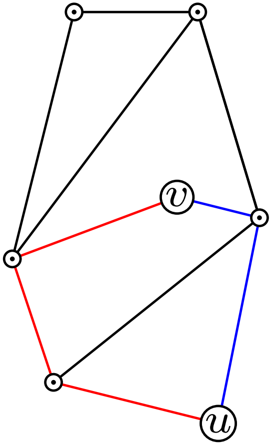

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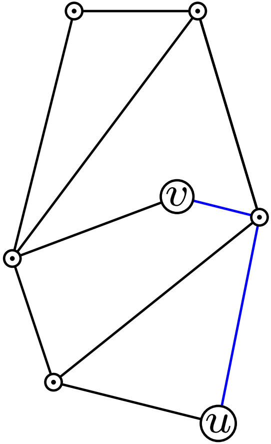

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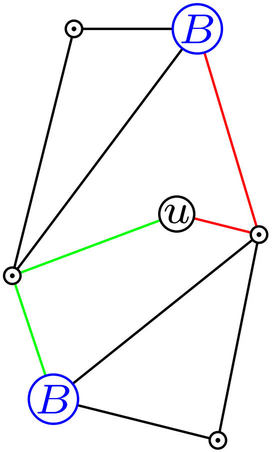

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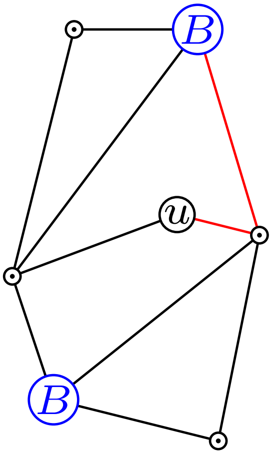

Limit p-Laplacian eigenequation $p \to \infty$

$$R_k = \sup \left\{ \begin{matrix} \mathbf{r} \;\; s.t. \;\; \exists v_1, \ldots, v_k \\ d(v_i, v_j) \geq 2\mathbf{r}\,, \;\; d(v_i, B) \geq \mathbf{r} \end{matrix} \right\}$$

$$\begin{cases} \left(f_{k,p}, \lambda_{k,p}^{\frac{1}{p}}\right) \xrightarrow[p\to\infty]{} (f_k, \Lambda_k) \\ \Delta_p f_{k,p} = \lambda_{k,p} |f_{k,p}|^{p-2} f_{k,p} \end{cases}$$

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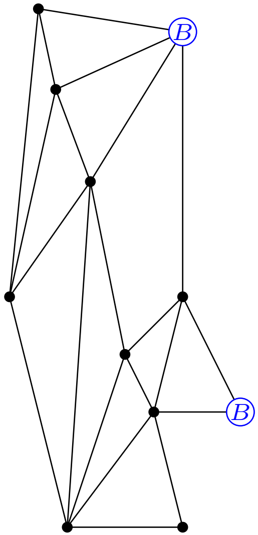

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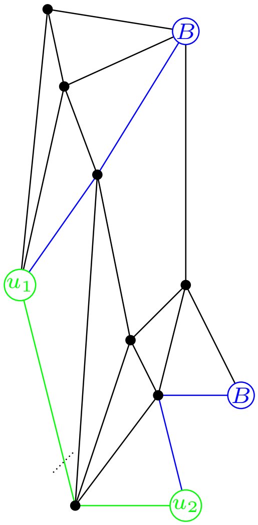

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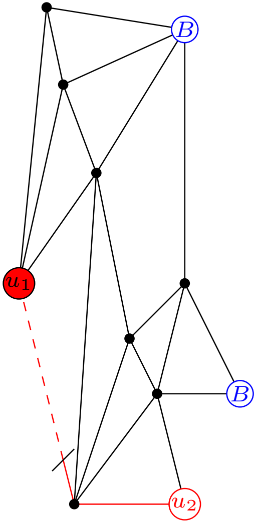

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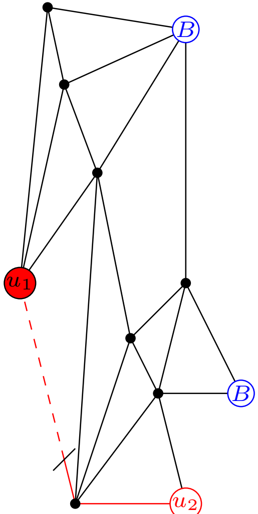

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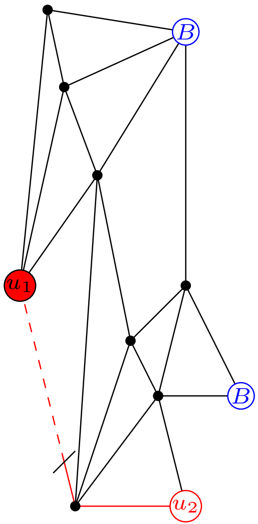

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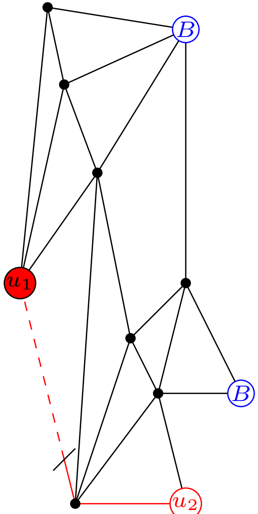

Nodal domains count on a general graph

Theorem

$\mathcal{G}$ connected.

$$\{\lambda_i\}_{i=1}^N \quad \text{Variational Eigenvalues}$$

$$(f,\lambda) \to l(f) \text{ constant sign loops},$$
$$\to z(f) \text{ zeros},$$
$$\to \beta \text{ edges that differentiate } \mathcal{G} \text{ from a tree}$$

P1. *If (f,λ) is an eigenpair of Δ_p, s.t. $\lambda < \lambda_k$, then.*

$$\mathcal{N}(f) \leq k - 1$$

P2. *If (f,λ) is an eigenpair of Δ_p, s.t. $\lambda \geq \lambda_k$, then.*

$$\mathcal{N}(f) \geq k - \beta + l(f) - z(f)$$

∞-Eigenvalue Problem

$$\left.\begin{array}{l} f \to \|\nabla f\|_\infty \\ f \to \|f\|_\infty \end{array}\right\} \quad \text{CONVEX FUNCTIONS}$$

$$\partial\Big(x \to \|x\|_\infty\Big) := \left\{ \frac{q}{\|q\|_1} \;\middle|\; \begin{array}{l} q_i = 0 \;\; if \; |x_i| \neq \|x\|_\infty \\ |q_i|\,|x_i| = q_i x_i \end{array} \right\}$$

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$$\partial\Big(x \to \|x\|_\infty\Big) := \left\{ \frac{q}{\|q\|_1} \middle| \begin{array}{l} q_i = 0 \;\; if \; |x_i| \neq \|x\|_\infty \\ |q_i|\,|x_i| = q_i x_i \end{array} \right\}$$

(f, Λ) "critical points" of $\frac{\|\nabla f\|_\infty}{\|f\|_\infty}$

$$\exists (q_E^f, q_V^f) \in \Big(\partial\|\nabla f\|_\infty, \partial\|f\|_\infty\Big)$$

$$-\mathrm{div}(q_E^f) = \Lambda q_V^f$$

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$$\Longrightarrow \boxed{\begin{array}{c} -div(q_E^{f_1}) = \sqrt{\lambda} q_v^{f_1} \\ \left(f_1, \sqrt{\lambda_1}\right) \end{array}} \quad 1^{st} \textit{ Infinity Eigenpair}$$

Optimal Paths

Theorem

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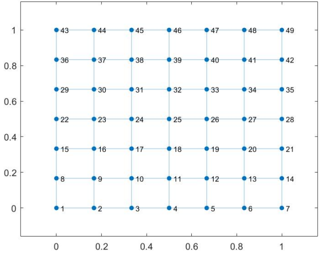

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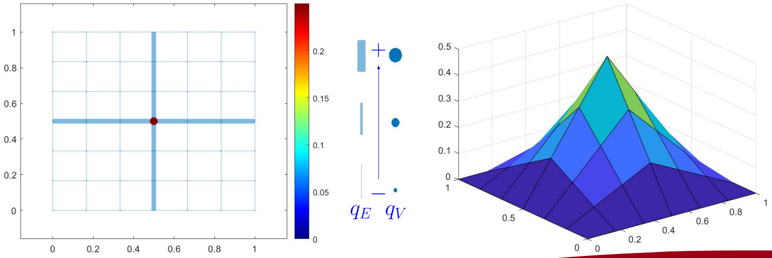

Heuristics

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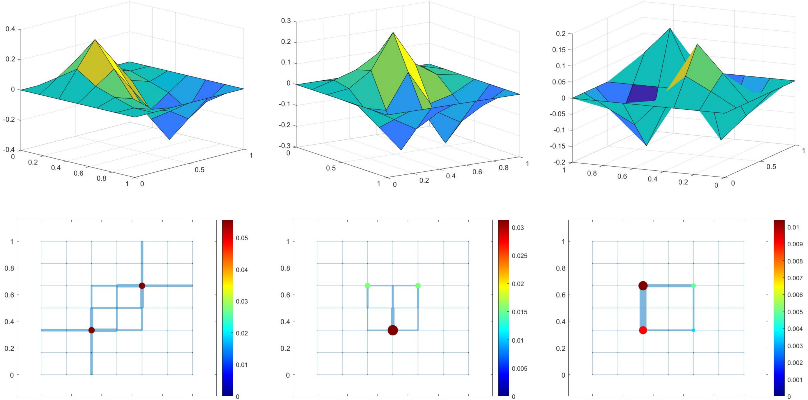

$k = 2$ $k = 3$ $k = 4$

Thank you for you attention!

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