Relative error propagation in linear ordinary differential equations: long-time behavior of condition numbers

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Introduction

We consider

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases} \quad \text{i.e. } y(t) = \mathrm{e}^{tA} y_0$$

Analysis of perturbations in the initial value y_0.

Analysis of perturbations in the matrix A (next talk by A. Farooq).

We are interested in t large.

Errors are measured by a relative error rather than an absolute error.

Conclusions with relative error and absolute error are completely different.

We use a normwise relative error.

Relative Errors ε and $\delta(t)$

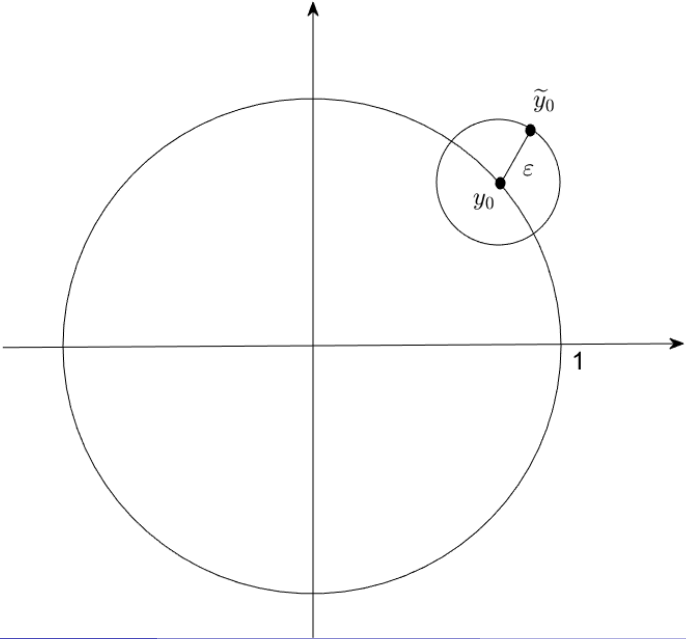
Initial value $y_0 \neq 0$ perturbed to $\widetilde{y}_0$ with relative error

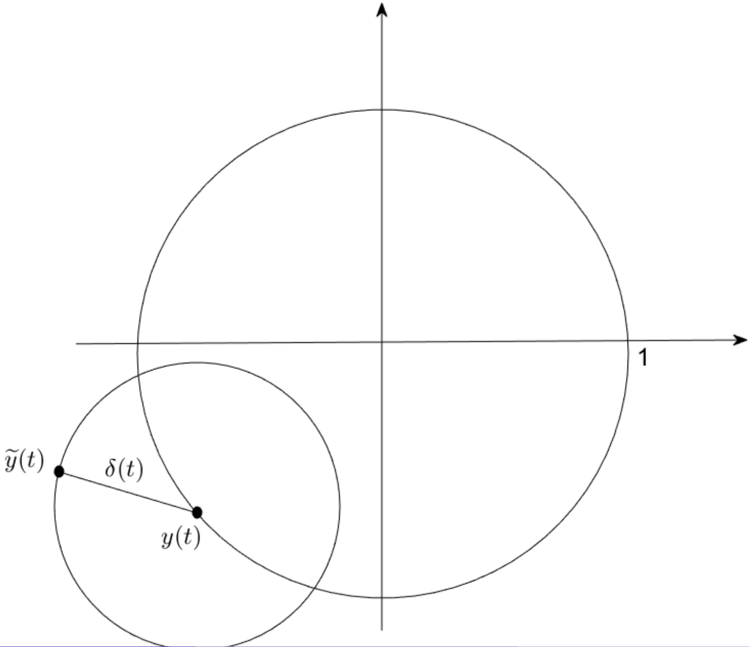
$$\varepsilon = \frac{\left\| \widetilde{y}_0 - y_0 \right\|}{\left\| y_0 \right\|}.$$

Solution $y(t)$ perturbed to $\widetilde{y}(t)$ with relative error

$$\delta(t) = \frac{\left\| \widetilde{y}(t) - y(t) \right\|}{\left\| y(t) \right\|}.$$

What is the relation between ε and $\delta(t)$?


$\tilde{y}_0$
ε
y_0
1


1
$\widetilde{y}(t)$
$\delta(t)$
$y(t)$

Condition Numbers

Let

$$\widetilde{y}_0 = y_0 + \varepsilon \|y_0\| \widehat{z}_0.$$

We have, with $\widehat{y}_0 = \frac{y_0}{\|y_0\|}$,

$$\delta(t) = K\left(t, A, y_0, \widehat{z}_0\right) \varepsilon, \quad K\left(t, A, y_0, \widehat{z}_0\right) := \frac{\left\| \mathrm{e}^{tA} \widehat{z}_0 \right\|}{\left\| \mathrm{e}^{tA} \widehat{y}_0 \right\|}.$$

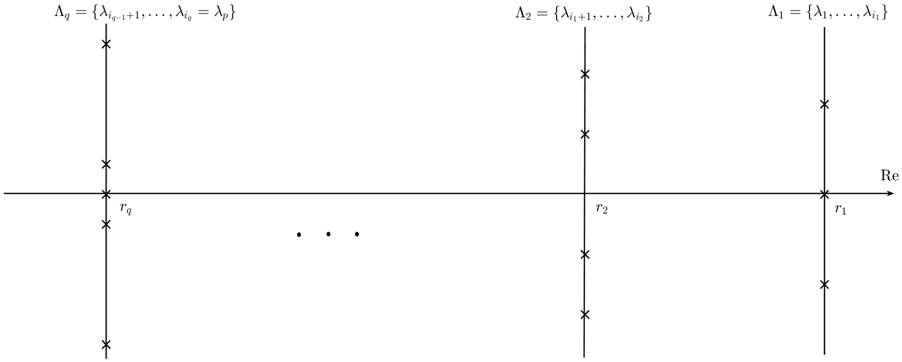
$$K(t, A, y_0) := \max_{\substack{\widehat{z}_0 \in \mathbb{C}^n \\ \left\|\widehat{z}_0\right\| = 1}} K\left(t, A, y_0, \widehat{z}_0\right) = \frac{\left\| \mathrm{e}^{tA} \right\|}{\left\| \mathrm{e}^{tA} \widehat{y}_0 \right\|}.$$

$$K(t, A) := \max_{y_0 \in \mathbb{C}^n \setminus \{0\}} K(t, A, y_0) = \left\| \mathrm{e}^{tA} \right\| \cdot \left\| \mathrm{e}^{-tA} \right\|.$$

We study the asymptotic behavior of $K\left(t, A, y_0, \widehat{z}_0\right)$ and $K(t, A, y_0)$.

Partition of the spectrum

The spectrum $\{\lambda_1, \ldots, \lambda_p\}$ of A is partitioned by decreasing real part:



The real parts of the eigenvalues are $r_1 > r_2 > \cdots > r_q$.

Formula for the matrix exponential

Formula for a function f of matrix based on the JCF of A:

$$f(A) = \sum_{i=1}^{p} \sum_{l=0}^{m_i - 1} \frac{f^{(l)}(\lambda_i)}{l!} P_{il},$$

where m_i is the index of λ_i and P_{il} are suitable matrices: P_{i0} is the projection on the eigenspace of λ_i.

For the matrix exponential case, we have

$$\mathrm{e}^{tA} = \sum_{j=1}^{q} \mathrm{e}^{r_j t} \sum_{l=0}^{L_j} \frac{t^l}{l!} Q_{jl}(t)$$

where $L_j := \max_{\lambda_i \in \Lambda_j} m_i - 1$ and

$$Q_{jl}(t) := \sum_{\substack{\lambda_i \in \Lambda_j \\ m_i \geq l+1}} \mathrm{e}^{\sqrt{-1}\,\omega_i t} P_{il}$$

with ω_i the imaginary part of λ_i.

Notations

For scalar functions $f(t)$ and $g(t)$ of t

- $f(t) \sim g(t)$, $t \to +\infty$, means

$$f(t) = g(t)(1 + e(t)), \ \lim_{t \to +\infty} e(t) = 0.$$

- $f(t) \approx g(t)$ with precision ϵ means

$$f(t) = g(t)(1 + e(t)), \ |e(t)| \leq \epsilon.$$

Asymptotic condition numbers

Theorem

In a generic situation for y_0 and $\widehat{z}_0$, we have, as $t \to +\infty$,

$$K\left(t, A, y_0, \widehat{z}_0\right) \sim K_\infty\left(t, A, y_0, \widehat{z}_0\right), \quad K\left(t, A, y_0\right) \sim K_\infty\left(t, A, y_0\right).$$

where

$$K_\infty\left(t, A, y_0, \widehat{z}_0\right) := \frac{\left\| Q_{1L_1}(t)\widehat{z}_0 \right\|}{\left\| Q_{1L_1}(t)\widehat{y}_0 \right\|}, \quad K_\infty\left(t, A, y_0\right) := \frac{\left\| Q_{1L_1}(t) \right\|}{\left\| Q_{1L_1}(t)\widehat{y}_0 \right\|}$$

with

$$Q_{1L_1}(t) := \sum_{\substack{\lambda_i \in \Lambda_1 \\ m_i = L_1 + 1}} \mathrm{e}^{\sqrt{-1}\,\omega_i t} P_{iL_1}$$

As functions of t, $K_\infty\left(t, A, y_0, \widehat{z}_0\right)$ and $K_\infty\left(t, A, y_0\right)$ are bounded and away from zero.

Onset of the asymptotic behavior

$$K\left(t,A,y_0,\widehat{z}_0\right)\approx K_\infty\left(t,A,y_0,\widehat{z}_0\right),\ \text{with precision}\ \frac{\epsilon(t,A,\widehat{z}_0)+\epsilon(t,A,\widehat{y}_0)}{1-\epsilon(t,A,\widehat{y}_0)}$$

$$K\left(t,A,y_0\right)\approx K_\infty\left(t,A,y_0\right),\ \text{with precision}\ \frac{\epsilon(t,A)+\epsilon(t,A,\widehat{y}_0)}{1-\epsilon(t,A,\widehat{y}_0)}$$

where

$$\epsilon(t,A):=\sum_{l=0}^{L_1-1}\frac{L_1!}{l!}t^{l-L_1}\frac{\|Q_{1l}(t)\|}{\|Q_{1L_1}(t)\|}+\sum_{j=2}^{q}\mathrm{e}^{(r_j-r_1)t}\sum_{l=0}^{L_j}\frac{L_1!}{l!}t^{l-L_1}\frac{\|Q_{jl}(t)\|}{\|Q_{1L_1}(t)\|}$$

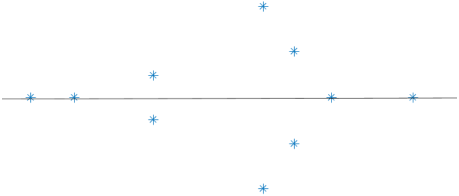
$$\epsilon(t,A,u):=\sum_{l=0}^{L_1-1}\frac{L_1!}{l!}t^{l-L_1}\frac{\|Q_{1l}(t)u\|}{\|Q_{1L_1}(t)\|}+\sum_{j=2}^{q}\mathrm{e}^{(r_j-r_1)t}\sum_{l=0}^{L_j}\frac{L_1!}{l!}t^{l-L_1}\frac{\|Q_{jl}(t)u\|}{\|Q_{1L_1}(t)\|}$$

$u=\widehat{y}_0$ or $u=\widehat{z}_0$

$\epsilon(t,A)\to 0$ and $\epsilon(t,A,u)\to 0$, $t\to+\infty$.

A generic real matrix A

Suppose A real in the generic situation: any Λ_j, $j = 1, \ldots, q$, is constituted by a unique simple real eigenvalue, or by a unique pair of simple complex conjugate eigenvalues.



λ_j denotes the real eigenvalue and λ_j and $\overline{\lambda}_j$ denote the complex conjugate pair. Let $v^{(j)}$ be an eigenvector of λ_j and let $w^{(j)}$ be a left eigenvector of λ_j. Let

$$\widehat{v}^{(j)} := \frac{v^{(j)}}{\left\| v^{(j)} \right\|}, \ \widehat{w}^{(j)} := \frac{w^{(j)}}{\left\| w^{(j)} \right\|}, \ f_j := \left\| w^{(j)} \right\| \cdot \left\| v^{(j)} \right\|.$$

Theorem

Suppose that Λ_1 *is constituted by a unique simple real eigenvalue. For* $w^{(1)} y_0 \neq 0$ *and* $w^{(1)} \widehat{z}_0 \neq 0$*,*

$$K_\infty\left(t, A, y_0, \widehat{z}_0\right) = \frac{\left|\widehat{w}^{(1)} \widehat{z}_0\right|}{\left|\widehat{w}^{(1)} \widehat{y}_0\right|}, \quad K_\infty\left(t, A, y_0\right) = \frac{1}{\left|\widehat{w}^{(1)} \widehat{y}_0\right|}.$$

Onset of the asymptotic behavior:

$$\epsilon(t, A) \ll 1, \quad \epsilon(t, A, u) \ll 1 \text{ for } u = \widehat{y}_0 \text{ or } u = \widehat{z}_0$$

$$\epsilon(t, A) \leq C \sum_{j=2}^{q} \mathrm{e}^{(r_j - r_1)t} \frac{f_j}{f_1}, \quad \epsilon(t, A, u) \leq C \sum_{j=2}^{q} \mathrm{e}^{(r_j - r_1)t} \frac{f_j}{f_1} \cdot \frac{\left|\widehat{w}^{(j)} u\right|}{\left|\widehat{w}^{(1)} u\right|}$$

C depends only on the dimension of A ($C = 2$ for a p-norm)

Theorem

Suppose that Λ_1 is constituted by a unique complex conjugate pair of simple eigenvalues of imaginary part ω_1. For $w^{(1)}y_0 \neq 0$ and $w^{(1)}\widehat{z}_0 \neq 0$,

$$K_\infty\left(t, A, y_0, \widehat{z}_0\right) = \frac{\left|\widehat{w}^{(1)}\widehat{z}_0\right|}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|} \cdot G_\infty\left(t, A, y_0, \widehat{z}_0\right)$$

$$K_\infty\left(t, A, y_0\right) = \frac{1}{\left|w^{(1)}\widehat{y}_0\right|} \cdot G_\infty\left(t, A, y_0\right),$$

where $G_\infty\left(t, A, y_0, \widehat{z}_0\right)$ and $G_\infty\left(t, A, y_0\right)$ are periodic functions of t with period $\frac{2\pi}{\omega_1}$. We have

$$c\sqrt{1 - V_1} \leq G_\infty\left(t, A, y_0, \widehat{z}_0\right) \leq \frac{C}{\sqrt{1 - V_1}}$$

$$d\sqrt{1 - V_1} \leq G_\infty\left(t, A, y_0\right) \leq \frac{D}{\sqrt{1 - V_1}},$$

where c, C, d, D depend only on the dimension of A (for the euclidean norm, $c = \frac{\sqrt{2}}{2}$, $C = \sqrt{2}$, $d = \frac{1}{2}$, $D = \sqrt{2}$) and $V_1 = |(\widehat{v}^{(1)})^T\widehat{v}^{(1)}| \in [0, 1)$.

In the asymptotic condition numbers

$$K_\infty\left(t,A,y_0,\widehat{z}_0\right)=\frac{\left|\widehat{w}^{(1)}\widehat{z}_0\right|}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}\cdot G_\infty\left(t,A,y_0,\widehat{z}_0\right)$$

$$K_\infty\left(t,A,y_0\right)=\frac{1}{\left|w^{(1)}\widehat{y}_0\right|}\cdot G_\infty\left(t,A,y_0\right)$$

the factors

$$G_\infty\left(t,A,y_0,\widehat{z}_0\right),\ G_\infty\left(t,A,y_0\right)$$

are called oscillating factors and the factors

$$\frac{\left|\widehat{w}^{(1)}\widehat{z}_0\right|}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|},\ \frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}$$

are called oscillation scale factors.
Onset of the asymptotic behavior:

$$\epsilon(t,A)\ll 1,\ \ \epsilon(t,A,u)\ll 1 \text{ for } u=\widehat{y}_0 \text{ or } u=\widehat{z}_0$$

$$\epsilon(t,A)\leq\frac{C}{\sqrt{1-V_1}}\sum_{j=2}^{q}\mathrm{e}^{(r_j-r_1)t}\frac{f_j}{f_1},\ \ \epsilon(t,A,u)\leq\frac{C}{\sqrt{1-V_1}}\sum_{j=2}^{q}\mathrm{e}^{(r_j-r_1)t}\frac{f_j}{f_1}\cdot\frac{\left|\widehat{w}^{(j)}u\right|}{\left|\widehat{w}^{(1)}u\right|}$$

C depends only on the dimension of A ($C=\sqrt{2}$ for the euclidean norm)

Numerical test I

Matrix A from the MATLAB gallery test: $A = gallery('lesp', n)$ with dimension $n = 10$.

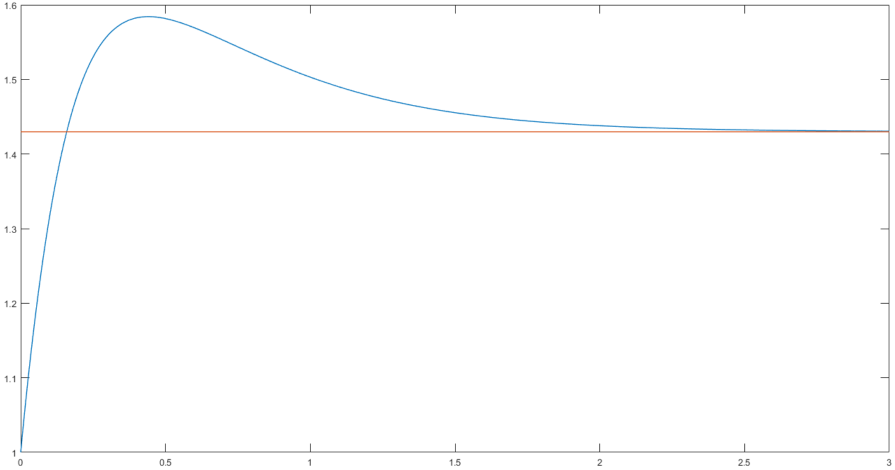
The matrix has 10 distinct real eigenvalues: the rightmost two are -4.5491 and -6.9531.

We use the euclidean norm.

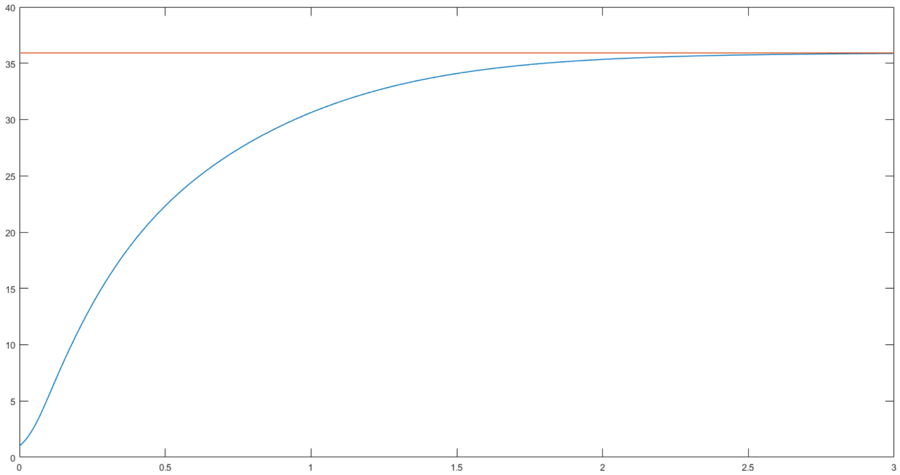
$K(t, A, y_0)$ blue line.

$K_\infty(t, A, y_0) = \frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}$ red line.

$y_0 = (1, \ldots, 1)$



$y_0 = ((-1.2)^l)_{l=1,\ldots,10}$



Numerical test II

Matrix A from the MATLAB gallery test: $A = -gallery('parter', n)$ with dimension $n = 10$.

The matrix has five distinct complex conjugate pairs of eigenvalues: the rightmost two pairs are $-0.9066 \pm \sqrt{-1} \cdot 2.7709$ and $-1.601 \pm \sqrt{-1} \cdot 2.366$.

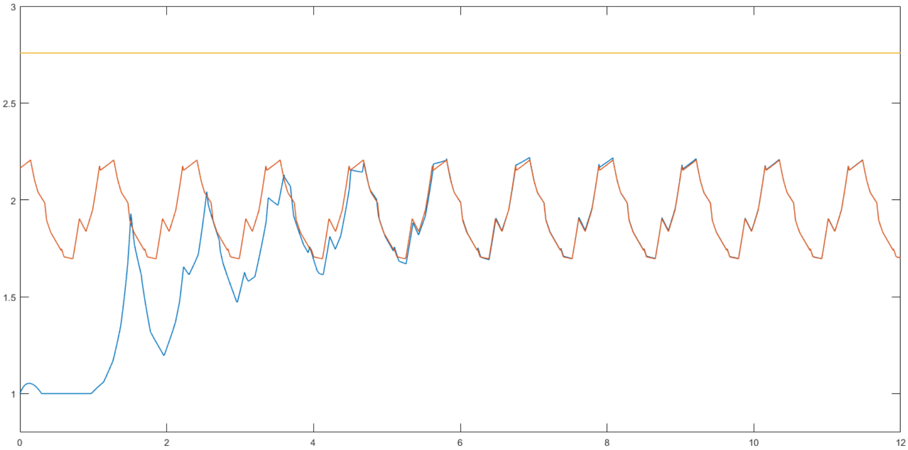
We use the ∞-norm.

$K(t, A, y_0)$ blue line.

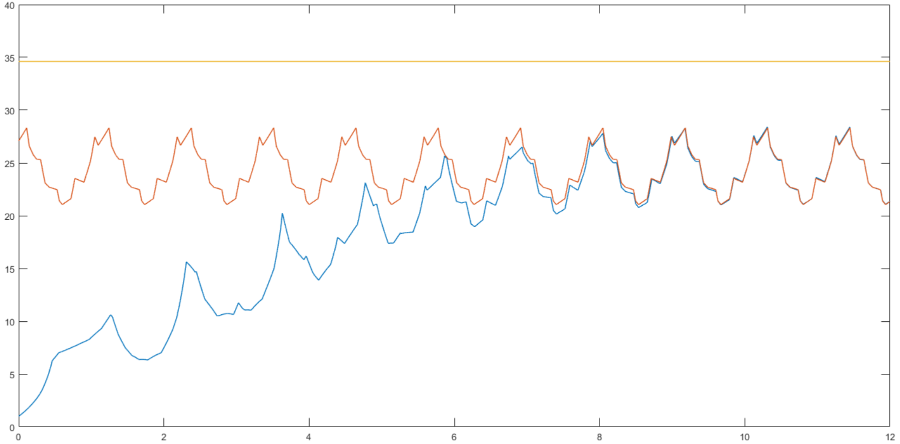
$K_\infty(t, A, y_0) = \frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|} \cdot G_\infty(t, A, \widehat{y}_0)$ red line.

$\frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}$ yellow line.

$y_0 = (1, \ldots, 1)$



$$y_0 = (0.9, -1.4, 0.2, 0.2, -0.2, 0.9, -0.4, -0.8, 0.3, 0.5)$$



Non-normal matrices

Normal matrices have

$$V_1 = |(\widehat{v}^{(1)})^T \widehat{v}^{(1)}| = 0$$
$$f_j = \left\| w^{(j)} \right\| \cdot \left\| v^{(j)} \right\| = 1,\ j = 1, \ldots, q,$$

in the euclidean norm.

Non-normal matrices have

V_1 arbitrarily close to 1

$f_j,\ j = 1, \ldots, q$, arbitrarily large.

What is the impact of an high non-normality on the asymptotic condition numbers?

Λ_1 is constituted by the real eigenvalue:

- no impact on the asymptotic condition number $\frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}$;
- impact on the onset of the asymptotic behavior only when the ratios $\frac{f_j}{f_1}$, $j = 2, \ldots, q$, are large; possible delay on the onset: recall

$$\epsilon(t, A) \leq C \sum_{j=2}^{q} \mathrm{e}^{(r_j - r_1)t} \frac{f_j}{f_1}, \quad \epsilon(t, A, u) \leq C \sum_{j=2}^{q} \mathrm{e}^{(r_j - r_1)t} \frac{f_j}{f_1} \cdot \frac{\left|\widehat{w}^{(j)} u\right|}{\left|\widehat{w}^{(1)} u\right|}$$

Λ_1 is constituted by the complex conjugate pair:

- impact on the oscillation factors $G_\infty(t, A, y_0, \widehat{z}_0)$ and $G_\infty(t, A, y_0)$ only when V_1 is close to 1; possible wide oscillations:

$$G_\infty\left(t, A, y_0, \widehat{z}_0\right) \leq \frac{C}{\sqrt{1 - V_1}}, \; G_\infty\left(t, A, y_0\right) \leq \frac{D}{\sqrt{1 - V_1}}$$

- impact on the onset of the asymptotic behavior only when V_1 is close to 1 or the ratios $\frac{f_j}{f_1}$, $j = 2, \ldots, q$, are large; possible delay on the onset:

$$\epsilon(t, A) \leq \frac{C}{\sqrt{1 - V_1}} \sum_{j=2}^{q} \mathrm{e}^{(r_j - r_1)t} \frac{f_j}{f_1}, \quad \epsilon(t, A, u) \leq \frac{C}{\sqrt{1 - V_1}} \sum_{j=2}^{q} \mathrm{e}^{(r_j - r_1)t} \frac{f_j}{f_1} \cdot \frac{\left|\widehat{w}^{(j)} u\right|}{\left|\widehat{w}^{(1)} u\right|}$$

Numerical test III

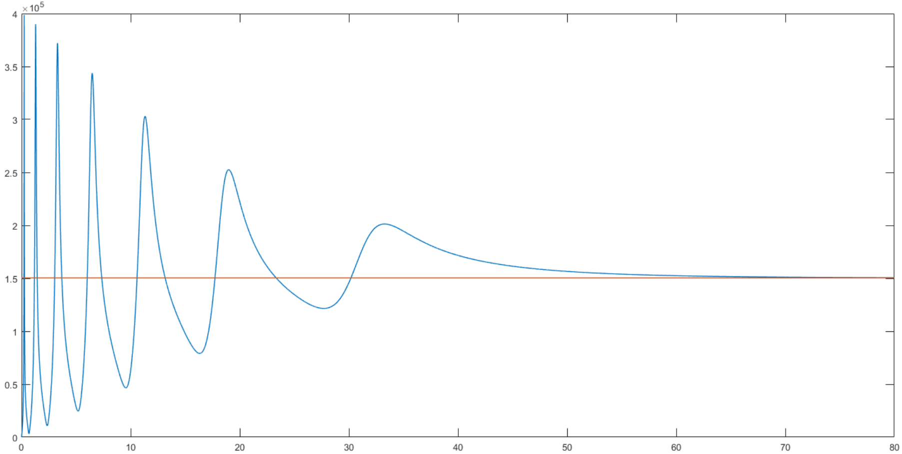
$A = VDV^{-1}$, where V is the Hilbert matrix of order 8 and $D = \mathrm{diag}(0.1, 0, -0.1, \ldots, -0.6)$.

We use the euclidean norm.

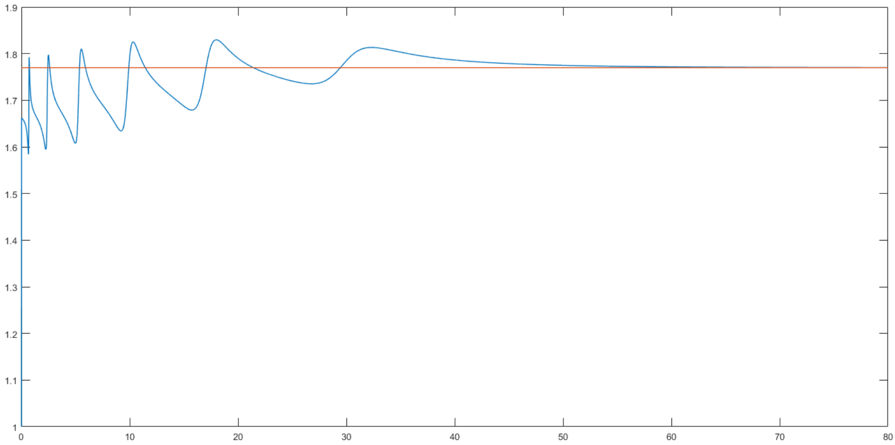
$K(t, A, y_0)$ blu line.

$K_\infty(t, A, y_0) = \frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}$ red line.

$y_0 = (1, \ldots, 1)$



$$y_0 = (1, 1, 1, 1, -1, -1, -1, -1)$$



Numerical test IV

$A = V\mathrm{diag}(\sqrt{-1}, -\sqrt{-1}, -1)V^{-1}$ with

$$V = \begin{bmatrix} 1+\sqrt{-1} & 1-\sqrt{-1} & 1 \\ 1+\sqrt{-1} & 1-\sqrt{-1} & 0 \\ 1+0.9\sqrt{-1} & 1-0.9\sqrt{-1} & 0 \end{bmatrix}.$$

Since $\mathrm{Re}(v^{(1)})$ and $\mathrm{Im}(v^{(1)})$ are close to be linearly dependent, $V_1 = 0.999$ is close to one.

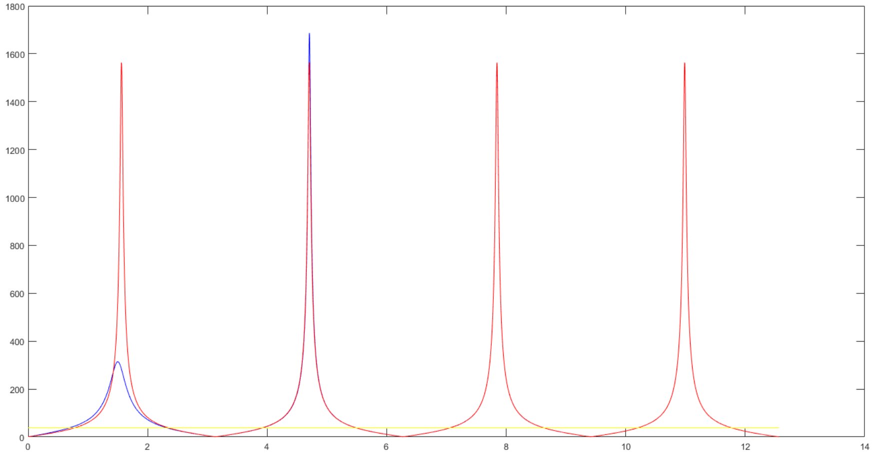
We use the euclidean norm.

$K(t, A, y_0)$ blue line.

$K_\infty(t, A, y_0) = \frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|} \cdot G_\infty(t, A, y_0, \widehat{z}_0)$ red line.

$\frac{1}{\left|\widehat{w}^{(1)}\widehat{y}_0\right|}$ yellow line.

$$y_0 = (0, -0.72, -0.69)$$



Conclusions

For a linear ODE, we have analyzed the long-time propagation of perturbations in the initial value by looking to a normwise relative error.

In the generic situation, the initial error is asymptotically magnified in the worst case by $\frac{1}{|\widehat{w}^1 \widehat{y}_0|}$. This term is multiplied by an oscillation term when there is a complex conjugate pair as rightmost eigenvalues.

Non-normality has an impact only when:

- V_1 is close to one: the eigenvectors of the complex conjugate pair are close to be linearly dependent;
- ratios $\frac{f_j}{f_1}$ large: the inverse of the eigenvectors matrix has non-rightmost rows much larger than the rightmost rows (by assuming unit eigenvectors).

Talk by A. Farooq will deal with perturbations in the matrix, rather than perturbations in the initial value.