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Matrices associated to two conservative discretizations of Riesz fractional operators and related multigrid solvers

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Introduction

The fractional calculus finds applications in biophysics, quantum mechanics, wave theory, polymers, Lie theory, field theory, spectroscopy, analysis of signals and images, financial mathematics, cardiac electrophysiology, ...

In this work we will study 2D-space conservative Fractional Diffusion Equations (FDEs), which are useful to model *anomalous* diffusion processes in complex media.

The space fractional diffusion relates to a *non-local* diffusion with a decay in space which depends on the orders of the fractional derivatives.

We will consider two Finite Volume-type discretizations (FV and FVE).

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Statement of the problem

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We consider the following conservative FDE:

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(K_x(x,y)\dfrac{\partial^{1-\alpha}u(x,y)}{\partial|x|^{1-\alpha}}\right)-\dfrac{\partial}{\partial y}\left(K_y(x,y)\dfrac{\partial^{1-\beta}u(x,y)}{\partial|y|^{1-\beta}}\right)=v(x,y), \\ \qquad\qquad (x,y)\in\Omega, \\ u(x,y)=0, \qquad (x,y)\in\left(\mathbb{R}^2\setminus\Omega\right), \end{cases}$$

where $\frac{\partial^{1-\alpha}u(x,y)}{\partial|x|^{1-\alpha}}$, $\frac{\partial^{1-\beta}u(x,y)}{\partial|y|^{1-\beta}}$ are the Riesz fractional derivative operators with respect to x- and y-variables, respectively, $\Omega=(a_1,b_1)\times(a_2,b_2)$ is the spatial domain, $K_x(x,y), K_y(x,y)$ are the nonnegative bounded diffusion coefficients, $v(x,y)$ is the forcing term.

The Riesz fractional operator in the x-variable is defined as

$$\frac{\partial^{1-\alpha} u(x,y)}{\partial |x|^{1-\alpha}} = \eta(\alpha) \left[\frac{\partial^{1-\alpha} u(x,y)}{\partial_+^{\mathrm{RL}} x^{1-\alpha}} + \frac{\partial^{1-\alpha} u(x,y)}{\partial_-^{\mathrm{RL}} x^{1-\alpha}} \right],$$

where $\eta(\alpha) = -\frac{1}{2\cos\left(\frac{(1-\alpha)\pi}{2}\right)}$ and the left $(+)$ and right (-) derivatives are given in the Riemann-Liouville (RL) form, that is,

$$\frac{\partial^{1-\alpha} u(x,y)}{\partial_+^{\mathrm{RL}} x^{1-\alpha}} = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial x} \int_{a_1}^{x} u(\xi,y)(x-\xi)^{\alpha-1} \mathrm{d}\xi,$$

$$\frac{\partial^{1-\alpha} u(x,y)}{\partial_-^{\mathrm{RL}} x^{1-\alpha}} = -\frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial x} \int_{x}^{b_1} u(\xi,y)(\xi-x)^{\alpha-1} \mathrm{d}\xi,$$

with $\Gamma(\cdot)$ being the gamma function. Similarly one can define the Riesz fractional operator in the y-variable.

Weighted Shifted Grünwald Difference

Let $h > 0$, then

GD: $$\frac{\mathrm{d}^\alpha f}{\mathrm{d}_- x^\alpha}(x) = \frac{1}{h^\alpha}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j} f\,(x - jh) + \mathrm{O}(h)$$

⬇

SGD: $${}_L G_{h,p}^{\alpha} u(x) = \frac{1}{h^\alpha}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j} f\,(x - (j-p)h)\,, \quad p \in \mathbb{Z}$$

⬇

WSGD: $${}_L D_{h,p,q}^{\alpha} u(x) = \frac{\alpha - 2q}{2(p-q)}\ {}_L G_{h,p}^{\alpha} u(x) + \frac{2p-\alpha}{2(p-q)}\ {}_L G_{h,q}^{\alpha} u(x), \quad p \neq q$$

⬇

Let $p \neq q$ then, $${}_L D_{h,p,q}^{\alpha} u(x) = \frac{\mathrm{d}^\alpha f}{\mathrm{d}_- x^\alpha}(x) + \mathrm{O}(h^2).$$

Remark

This leads directly to a finite difference scheme.

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Matrix representation of the discretized FD

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Let $U = [u_1, u_2, ..., u_N]^{\mathrm{T}}$, with $u_j = u(x_j)$, the left SGD ${}_LG^{\alpha}_{h,p}u(x_i)$ can be rewritten as

$$ {}_LG^{\alpha}_{h,p}u(x_i) \approx \frac{1}{h^{\alpha}} \left(G_N^{\alpha,p}\right)_i U, $$

where $\left(G_N^{\alpha,p}\right)_i$ is the i-th row of the Toeplitz matrix $G_N^{\alpha,p}$:

$$ G_N^{\alpha,p} = \begin{pmatrix} g_0^{(\alpha)} & & & & \\ g_1^{(\alpha)} & g_0^{(\alpha)} & & & \\ \vdots & \ddots & \ddots & & \\ g_{N-2}^{(\alpha)} & & \ddots & \ddots & \\ g_{N-1}^{(\alpha)} & g_{N-2}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} \end{pmatrix}_{N\times N}. $$

$p > 0$ $\quad$ $p < 0$

The left WSGD can be rewritten as ${}_LD^{\alpha}_{h,p,q}u(x_i) \approx \frac{1}{h^{\alpha}} \left(A_N^{\alpha,p,q}\right)_i U$,

$$ A_N^{\alpha,p,q} = \frac{\alpha - 2q}{2(p-q)} G_N^{\alpha,p} + \frac{2p-\alpha}{2(p-q)} G_N^{\alpha,q}. $$

Toeplitz matrices and generating functions

Definition

A Toeplitz matrix $T_N \in \mathbb{C}^{N\times N}$ has the form

$$T_N = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-N+2} & t_{-N+1} \\ t_1 & t_0 & t_{-1} & & t_{-N+2} \\ \vdots & t_1 & \ddots & \ddots & \vdots \\ t_{N-2} & & \ddots & \ddots & t_{-1} \\ t_{N-1} & t_{N-2} & \cdots & t_1 & t_0 \end{pmatrix}.$$

If t_k is the k-th Fourier coefficients of $f \in L^1$ then

$$f(\theta) = \sum_{k=-N+1}^{N-1} t_k e^{ik\theta}$$

is called the *generating function* of T_N, and we denote such a matrix by $T_N(f)$.

Remark

The generating function of a symmetric Toeplitz allows to estimate its eigenvalue distribution.

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Discretization of the FDE

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We decompose the domain $\Omega = [a_1, b_1] \times [a_2, b_2]$ with the rectangles $Q_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, where

$$\begin{aligned} x_i &= a_1 + ih_x,\ \ i = 1, ..., N_x, \quad & h_x &= \tfrac{b_1 - a_1}{N_x + 1}, \\ y_j &= a_2 + jh_y,\ \ j = 1, ..., N_y, \quad & h_y &= \tfrac{b_2 - a_2}{N_y + 1}. \end{aligned}$$

and we integrate the FDE over Q_{ij} obtaining $S_1 + S_2 = S_3$, where e.g.

$$S_1 = -\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x} \left(K_x(x, y) \frac{\partial^{1-\alpha} u(x, y)}{\partial |x|^{1-\alpha}} \right) \mathrm{d}x\mathrm{d}y.$$

FVE: We replace the unknown with the piecewise linear approximation and compute exactly the integral, obtaining the linear system

$$A_{\mathrm{FVE}} x = b.$$

- $\rightarrow$ second order accurate scheme,
- $\rightarrow$ positive definite A_{FVE} when $K_x, K_y \in \mathbb{R}^+$ with spectral properties similar to the Laplacian.

Discretization of the FDE

We decompose the domain $\Omega = [a_1, b_1] \times [a_2, b_2]$ with the rectangles $Q_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, where

$$x_i = a_1 + ih_x,\ i = 1, \ldots, N_x, \quad h_x = \tfrac{b_1 - a_1}{N_x + 1},$$
$$y_j = a_2 + jh_y,\ j = 1, \ldots, N_y, \quad h_y = \tfrac{b_2 - a_2}{N_y + 1}.$$

and we integrate the FDE over Q_{ij} obtaining $S_1 + S_2 = S_3$, where e.g.

$$S_1 = -\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x}\left(K_x(x,y)\frac{\partial^{1-\alpha} u(x,y)}{\partial |x|^{1-\alpha}}\right) \mathrm{d}x\mathrm{d}y.$$

FV: We approximate the fractional operator through finite differences obtaining the linear system

$$A_{\mathrm{FV}} x = b.$$

$\rightarrow$ second order accurate scheme through a non integer shift,
$\rightarrow$ spectral properties?

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Choice of the shifts

If $K_x, K_y \in \mathbb{R}^+$, then $A_{\mathrm{FV}} = T_N(\mathcal{F}(x,y))$ with $N = N_x N_y$ and

$$\mathcal{F}(x,y) = K_x \frac{f_{\alpha,p_1,q_1}(x) + \overline{f}_{\alpha,\tilde{p}_1,\tilde{q}_1}(x)}{h_x^{\alpha}} + K_y \frac{f_{\beta,p_2,q_2}(y) + \overline{f}_{\beta,\tilde{p}_2,\tilde{q}_2}(y)}{h_y^{\beta}}$$

where $f_{\alpha,p,q}(x)$ is the generating function of the left WSGD operator.

Constraints on the shifts:

Choice of the shifts

If $K_x, K_y \in \mathbb{R}^+$, then $A_{\mathrm{FV}} = T_N(\mathcal{F}(x,y))$ with $N = N_x N_y$ and

$$\mathcal{F}(x,y) = K_x \frac{f_{\alpha,p_1,q_1}(x) + \overline{f}_{\alpha,\tilde{p}_1,\tilde{q}_1}(x)}{h_x^\alpha} + K_y \frac{f_{\beta,p_2,q_2}(y) + \overline{f}_{\beta,\tilde{p}_2,\tilde{q}_2}(y)}{h_y^\beta}$$

where $f_{\alpha,p,q}(x)$ is the generating function of the left WSGD operator.

Constraints on the shifts:

- $\mathcal{F}(x,y)$ real $\rightarrow p_1 = \tilde{p}_1, q_1 = \tilde{q}_1, \quad p_2 = \tilde{p}_2, q_2 = \tilde{q}_2$

$$\mathcal{F}(x,y) = K_x \frac{f_{\alpha,p_1,q_1}(x) + \overline{f}_{\alpha,p_1,q_1}(x)}{h_x^\alpha} + K_y \frac{f_{\beta,p_2,q_2}(y) + \overline{f}_{\beta,p_2,q_2}(y)}{h_y^\beta}$$

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Choice of the shifts

If $K_x, K_y \in \mathbb{R}^+$, then $A_{\mathrm{FV}} = T_N(\mathcal{F}(x,y))$ with $N = N_x N_y$ and

$$\mathcal{F}(x,y) = K_x \frac{f_{\alpha,p_1,q_1}(x) + \overline{f}_{\alpha,\tilde{p}_1,\tilde{q}_1}(x)}{h_x^\alpha} + K_y \frac{f_{\beta,p_2,q_2}(y) + \overline{f}_{\beta,\tilde{p}_2,\tilde{q}_2}(y)}{h_y^\beta}$$

where $f_{\alpha,p,q}(x)$ is the generating function of the left WSGD operator.

Constraints on the shifts:

- $\mathcal{F}(x,y)$ real $\rightarrow p_1 = \tilde{p}_1, q_1 = \tilde{q}_1, \quad p_2 = \tilde{p}_2, q_2 = \tilde{q}_2$

$$\mathcal{F}(x,y) = K_x \frac{f_{\alpha,p_1,q_1}(x) + \overline{f}_{\alpha,p_1,q_1}(x)}{h_x^\alpha} + K_y \frac{f_{\beta,p_2,q_2}(y) + \overline{f}_{\beta,p_2,q_2}(y)}{h_y^\beta}$$

- Same shift in each dimension $\rightarrow p_1 = p_2, q_1 = q_2,$

$$\mathcal{F}(x,y) = K_x \frac{f_{\alpha,p_1,q_1}(x) + \overline{f}_{\alpha,p_1,q_1}(x)}{h_x^\alpha} + K_y \frac{f_{\beta,p_1,q_1}(y) + \overline{f}_{\beta,p_1,q_1}(y)}{h_y^\beta}$$

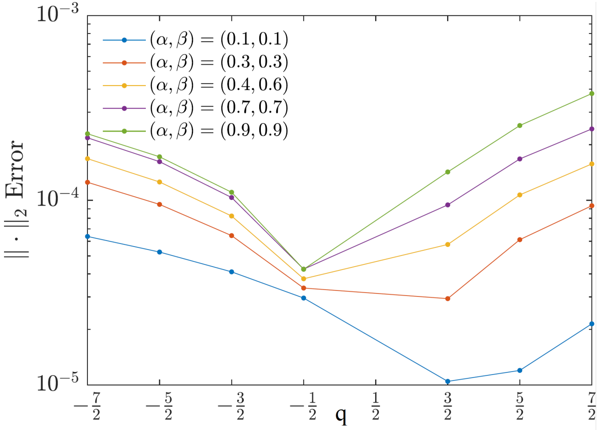

Figure: Relative error varying α, β and q, with fixed $p = \frac{1}{2}$.

Remark

The shift $\left(\frac{1}{2}, -\frac{1}{2}\right)$ *leads to an increasing non-negative function* $\mathcal{F}(x, y)$ *s.t.* $\mathcal{F}(0, 0) = 0$.

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MGM for Toeplitz matrices

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V-Cycle is known to converge when *smoothing property* and *approximation property* hold.

In the case of a positive definite Toeplitz matrix generated by $f(\boldsymbol{\theta})$:

- *Smoothing property*: holds whenever weighted Jacobi converges (when $f(0,0) = 0$);
- *Approximation property*: Let $f(\boldsymbol{\theta}) > 0$ that vanishes only at $\boldsymbol{\theta}_0$, then the projectors have to be generated by $p(\boldsymbol{\theta})$ s.t.

$$\limsup_{\boldsymbol{\theta}\to\boldsymbol{\theta}_0} \frac{p(\hat{\boldsymbol{\theta}})}{f(\boldsymbol{\theta})} = \mathfrak{c} < +\infty, \qquad \forall \hat{\boldsymbol{\theta}} \in \mathscr{M}(\boldsymbol{\theta}),$$

where $\mathscr{M}(\boldsymbol{\theta}) = \{(\theta_1, \pi - \theta_2), (\pi - \theta_1, \theta_2), (\pi - \theta_1, \pi - \theta_2)\}$ is the set of the "mirror points" of $\boldsymbol{\theta}$.

Convergence of MGM

Consider constant diffusion coefficients.

Smoothing property

Approximation property

Convergence of MGM

Consider constant diffusion coefficients.

Smoothing property $\rightsquigarrow$ holds whenever weighted Jacobi converges.

- the choice of a fixed weight ω could lead to a slow algorithm, hence numerical estimates are needed.

Approximation property $\rightsquigarrow$ holds with the same projectors as the standard diffusion equation due to the generating functions of the two discretized equations having similar properties, i.e., a zero at $(x, y) = (0, 0)$ of order upper bounded by 2.

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Estimate of a suitable weight ω

$$A_N = \frac{1}{h_x^\alpha(N)} A_{x,N} + \frac{1}{h_y^\beta(N)} A_{y,N} \quad \to \quad \tilde{A}_{\tilde{N}} = \frac{1}{h_x^\alpha(N)} A_{x,\tilde{N}} + \frac{1}{h_y^\beta(N)} A_{y,\tilde{N}},$$

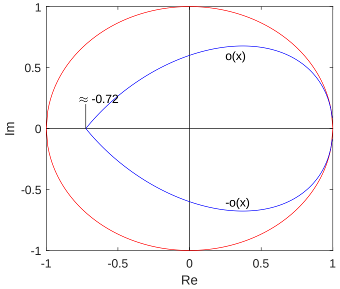
with $\tilde{N} \leq N$.

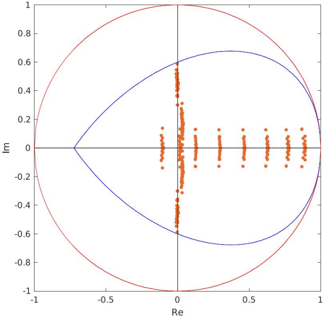
It can be proven that $\rho(\tilde{B}^\omega_{\tilde{N}}) \approx \rho(B^\omega_N)$, where

$$B^\omega_N = I_N - \omega D^{-1} A_N \quad \text{and} \quad \tilde{B}^\omega_{\tilde{N}} = I_{\tilde{N}} - \omega D^{-1} \tilde{A}_{\tilde{N}}$$

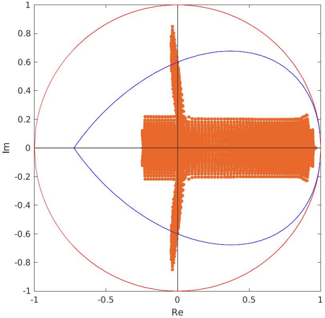
are the respective iteration matrices of weighted Jacobi.

We choose the biggest ω s.t. $\Lambda(\tilde{B}^\omega_{\tilde{N}}) \subset \{x : -o(x) \leq x \leq o(x)\}$, where $o(x) = \sqrt{1-x^2} + 0.4x - 0.4$.





Eigenvalue distribution of $\tilde{B}^{0.8}_{2^8}$



Eigenvalue distribution of $B^{0.8}_{2^{14}}$

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Consider the FDE with

- $u(x,y) = x^2y^2(1-x)^2(1-y)^2, \ (x,y) \in \Omega = [0,1] \times [0,1]$,
- $K_x(x,y) = K_y(x,y) = \mathrm{e}^{4x+4y}$
- $v(x,y,t)$ is built from the exact solution $u(x,y)$,

discretized over a square mesh with $N_x = N_y \in \mathbb{N}$ grid points.
We consider the following solvers

- **V**, V-Cycle applied to A_N with automatic estimate of ω;
- **V5**, which is V applied to the pentadiagonal block-band approximation of A_N,
- **V($\tilde{\omega}$)**, which is V-Cycle with weight $\tilde{\omega} = 0.75 + \frac{\sqrt{\min(\alpha,\beta)}}{4}$;
- **VL(gal)**, which is V-Cycle with Galerkin approach applied to the 2D Laplacian and $\omega = 0.75$;
- **VL(geo)**, which is **VL(gal)** implemented through the geometric approach;
- $\mathcal{P}_\star$, denotes the use of GMRES as main solver with $\star$ as preconditioner.

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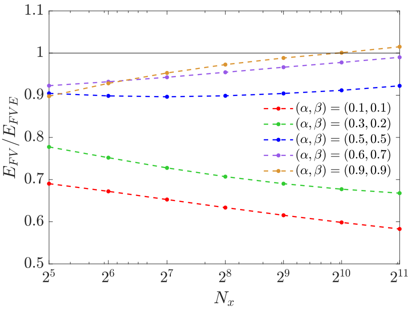
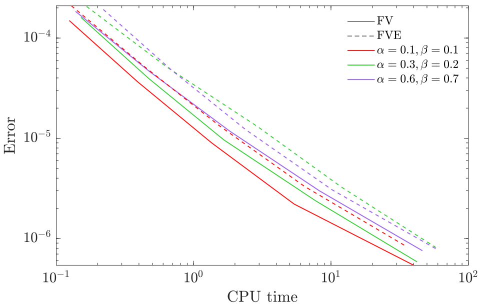
$\begin{pmatrix}\alpha\\\beta\end{pmatrix}$	N_x	V-cycle				Preconditioned GMRES									
		V		**V**($\tilde{\omega}$)		$\mathcal{P}_{\mathbf{V5}}$		$\mathcal{P}_{\mathbf{VL}(geo)}$		$\mathcal{P}_{\mathbf{VL}(gal)}$		$\mathcal{P}_{\mathbf{V}}$		$\mathcal{P}_{\mathbf{V}(\tilde{\omega})}$	
		FVE	FV	FVE	FV	FVE	FV	FVE	FV	FVE	FV	FVE	FV	FVE	FV
$\begin{pmatrix}0.1\\0.1\end{pmatrix}$	2^6	10	16	11	15	**7**	10	9	12	9	12	7	10	7	8
	2^7	10	15	11	16	8	**10**	11	12	11	12	7	10	8	9
	2^8	10	15	12	16	9	**10**	12	14	11	14	7	9	8	9
	2^9	11	15	12	17	10	**10**	14	**13**	12	16	8	9	8	9
	2^{10}	11	16	13	17	10	**10**	13	17	14	17	8	10	8	12
	2^{11}	**11**	16	13	18	10	13	16	18	15	18	8	10	8	11
$\begin{pmatrix}0.3\\0.2\end{pmatrix}$	2^6	24	22	18	25	**11**	11	20	22	17	22	11	11	10	12
	2^7	20	24	20	27	12	**13**	22	23	22	24	10	11	12	12
	2^8	23	26	22	29	12	**13**	26	29	23	29	13	13	12	14
	2^9	25	28	24	32	14	**14**	33	36	28	34	15	13	12	14
	2^{10}	25	31	26	34	14	**15**	36	37	34	37	14	15	13	16
	2^{11}	27	33	28	37	15	**16**	37	40	35	44	15	16	14	16
$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	2^6	8	11	9	11	9	**9**	19	21	20	23	6	8	6	7
	2^7	**9**	**11**	10	12	**11**	**12**	26	29	26	29	6	7	6	8
	2^8	9	11	10	13	14	**12**	30	31	30	34	6	8	7	8
	2^9	**10**	12	11	13	14	15	40	39	40	42	7	8	7	8
	2^{10}	**10**	12	11	14	18	16	43	46	44	56	7	9	7	8
	2^{11}	**11**	13	12	14	20	18	54	57	54	63	7	9	8	9
$\begin{pmatrix}0.6\\0.7\end{pmatrix}$	2^6	13	16	13	18	12	**12**	31	34	31	35	8	9	9	10
	2^7	14	18	14	19	14	**14**	36	45	36	49	9	9	8	10
	2^8	16	19	16	21	16	**16**	50	52	51	54	10	11	9	10
	2^9	17	21	17	22	19	**18**	60	74	61	67	10	11	9	11
	2^{10}	18	22	18	24	23	**21**	92	88	94	90	12	11	10	12
	2^{11}	19	24	20	26	30	**24**	93	103	106	115	12	12	11	14
$\begin{pmatrix}0.9\\0.9\end{pmatrix}$	2^6	7	9	7	9	12	**12**	32	33	33	36	5	6	5	6
	2^7	**7**	9	7	9	16	15	40	50	41	55	5	6	5	6
	2^8	**7**	10	8	10	21	18	69	59	63	77	5	6	5	6
	2^9	**8**	10	**8**	10	27	23	84	87	86	92	5	7	5	7
	2^{10}	**8**	10	**8**	10	36	30	103	109	102	110	5	7	6	7
	2^{11}	**8**	11	9	11	48	38	147	154	145	156	6	7	6	7

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$	N_x	V-cycle				Preconditioned GMRES									
		V		$\mathbf{V}(\tilde{\omega})$		$\mathcal{P}_{\mathbf{V5}}$		$\mathcal{P}_{\mathbf{VL}(geo)}$		$\mathcal{P}_{\mathbf{VL}(gal)}$		$\mathcal{P}_{\mathbf{V}}$		$\mathcal{P}_{\mathbf{V}(\tilde{\omega})}$	
		FVE	FV	FVE	FV	FVE	FV	FVE	FV	FVE	FV	FVE	FV	FVE	FV
$\begin{pmatrix}0.1\\ 0.1\end{pmatrix}$	2^6	5.9e-2	9.3e-2	6.6e-2	8.7e-2	**2.5e-2**	3.7e-2	3.0e-2	4.0e-2	3.0e-2	4.1e-2	9.2e-2	1.1e-1	7.0e-2	7.6e-2
	2^7	1.3e-1	1.8e-1	1.7e-1	2.4e-1	1.3e-1	**1.2e-1**	1.4e-1	1.3e-1	1.3e-1	1.3e-1	2.2e-1	3.1e-1	2.5e-1	2.6e-1
	2^8	4.3e-1	6.2e-1	5.1e-1	6.6e-1	6.2e-1	**3.9e-1**	4.3e-1	4.7e-1	4.2e-1	4.9e-1	6.9e-1	7.7e-1	8.5e-1	8.7e-1
	2^9	1.6e+0	2.0e+0	1.7e+0	2.3e+0	2.4e+0	**1.4e+0**	1.9e+0	**1.4e+0**	1.6e+0	2.0e+0	3.0e+0	2.6e+0	2.8e+0	2.9e+0
	2^{10}	6.4e+0	8.9e+0	7.4e+0	9.3e+0	9.9e+0	**5.4e+0**	6.4e+0	8.0e+0	8.4e+0	8.3e+0	1.3e+1	1.4e+1	1.2e+1	1.7e+1
	2^{11}	**3.6e+1**	4.9e+1	4.3e+1	5.5e+1	4.6e+1	4.0e+1	4.6e+1	4.2e+1	4.4e+1	4.6e+1	7.0e+1	7.4e+1	6.4e+1	9.0e+1
$\begin{pmatrix}0.3\\ 0.2\end{pmatrix}$	2^6	1.4e-1	1.2e-1	1.1e-1	1.2e-1	**3.2e-2**	3.4e-2	5.8e-2	6.1e-2	4.6e-2	6.1e-2	1.2e-1	1.2e-1	1.2e-1	1.3e-1
	2^7	2.6e-1	3.2e-1	2.9e-1	4.1e-1	1.7e-1	**1.5e-1**	2.2e-1	2.0e-1	2.1e-1	2.1e-1	2.8e-1	2.8e-1	4.2e-1	3.5e-1
	2^8	9.8e-1	1.1e+0	9.4e-1	1.2e+0	6.3e-1	**4.7e-1**	8.6e-1	8.2e-1	7.3e-1	8.5e-1	1.4e+0	1.2e+0	1.1e+0	1.2e+0
	2^9	3.5e+0	3.8e+0	3.3e+0	4.3e+0	2.9e+0	**1.7e+0**	3.7e+0	3.4e+0	3.0e+0	3.3e+0	5.5e+0	3.8e+0	3.8e+0	4.0e+0
	2^{10}	1.4e+1	1.7e+1	1.5e+1	1.9e+1	1.2e+1	**7.8e+0**	1.5e+1	1.4e+1	1.5e+1	1.4e+1	1.9e+1	2.0e+1	1.7e+1	2.1e+1
	2^{11}	8.6e+1	9.9e+1	8.8e+1	1.1e+2	5.8e+1	**4.2e+1**	7.8e+1	7.3e+1	7.6e+1	8.4e+1	1.1e+2	1.1e+2	1.1e+2	1.1e+2
$\begin{pmatrix}0.5\\ 0.5\end{pmatrix}$	2^6	4.8e-2	6.4e-2	5.4e-2	6.4e-2	3.3e-2	**2.9e-2**	5.6e-2	5.7e-2	5.9e-2	6.3e-2	6.2e-2	9.7e-2	6.2e-2	6.9e-2
	2^7	**1.4e-1**	**1.4e-1**	1.6e-1	1.8e-1	**1.4e-1**	**1.4e-1**	2.6e-1	2.6e-1	2.6e-1	2.7e-1	1.6e-1	2.1e-1	1.7e-1	2.7e-1
	2^8	3.9e-1	4.4e-1	4.2e-1	5.4e-1	7.1e-1	**3.7e-1**	9.5e-1	9.0e-1	9.5e-1	9.4e-1	4.7e-1	8.7e-1	6.9e-1	7.2e-1
	2^9	**1.4e+0**	1.6e+0	1.6e+0	1.7e+0	2.3e+0	1.7e+0	4.2e+0	3.5e+0	4.3e+0	3.9e+0	2.3e+0	2.9e+0	2.3e+0	2.4e+0
	2^{10}	**5.7e+0**	6.7e+0	6.3e+0	7.7e+0	1.3e+1	7.4e+0	1.8e+1	1.6e+1	1.8e+1	2.0e+1	9.4e+0	1.3e+1	9.4e+0	1.1e+1
	2^{11}	**3.4e+1**	4.0e+1	3.7e+1	4.4e+1	6.4e+1	4.0e+1	1.1e+2	1.0e+2	1.1e+2	1.1e+2	5.1e+1	6.9e+1	5.6e+1	7.1e+1
$\begin{pmatrix}0.6\\ 0.7\end{pmatrix}$	2^6	7.7e-2	9.2e-2	7.7e-2	1.0e-1	4.0e-2	**3.5e-2**	8.3e-2	8.4e-2	8.2e-2	8.8e-2	9.9e-2	1.0e-1	1.1e-1	1.1e-1
	2^7	2.2e-1	2.3e-1	1.9e-1	2.9e-1	1.6e-1	**1.4e-1**	3.3e-1	3.8e-1	3.2e-1	4.1e-1	3.6e-1	2.5e-1	2.7e-1	3.0e-1
	2^8	6.8e-1	7.8e-1	6.8e-1	8.7e-1	7.0e-1	**4.7e-1**	1.5e+0	1.4e+0	1.5e+0	1.5e+0	1.0e+0	1.1e+0	8.0e-1	8.3e-1
	2^9	2.4e+0	2.8e+0	2.4e+0	2.9e+0	2.9e+0	**1.8e+0**	5.9e+0	6.2e+0	6.0e+0	5.8e+0	3.4e+0	3.5e+0	2.7e+0	3.4e+0
	2^{10}	1.0e+1	1.2e+1	1.0e+1	1.3e+1	1.4e+1	**8.2e+0**	3.5e+1	2.8e+1	3.5e+1	3.0e+1	1.8e+1	1.4e+1	1.4e+1	1.5e+1
	2^{11}	6.0e+1	7.3e+1	6.6e+1	7.9e+1	8.7e+1	**4.6e+1**	1.8e+2	1.7e+2	2.0e+2	1.9e+2	1.1e+2	8.4e+1	8.5e+1	1.0e+2
$\begin{pmatrix}0.9\\ 0.9\end{pmatrix}$	2^6	4.2e-2	5.2e-2	4.2e-2	5.3e-2	3.3e-2	**3.0e-2**	8.5e-2	8.2e-2	8.7e-2	8.9e-2	5.4e-2	6.1e-2	5.5e-2	6.1e-2
	2^7	**9.4e-2**	1.3e-1	1.1e-1	1.4e-1	1.5e-1	1.2e-1	3.4e-1	4.1e-1	3.5e-1	4.6e-1	1.4e-1	1.6e-1	1.5e-1	1.7e-1
	2^8	**3.1e-1**	4.1e-1	3.4e-1	4.2e-1	7.9e-1	4.9e-1	2.1e+0	1.6e+0	1.8e+0	2.0e+0	4.0e-1	4.4e-1	4.1e-1	4.6e-1
	2^9	**1.1e+0**	1.4e+0	**1.1e+0**	1.4e+0	3.7e+0	2.1e+0	8.0e+0	7.2e+0	8.4e+0	7.8e+0	1.4e+0	2.2e+0	1.4e+0	2.2e+0
	2^{10}	**4.6e+0**	5.6e+0	**4.6e+0**	5.6e+0	2.0e+1	1.1e+1	3.8e+1	3.4e+1	3.7e+1	3.6e+1	5.5e+0	9.3e+0	8.7e+0	9.2e+0
	2^{11}	**2.5e+1**	3.4e+1	2.8e+1	3.4e+1	1.2e+2	7.1e+1	2.7e+2	2.5e+2	2.7e+2	2.6e+2	4.6e+1	5.1e+1	4.8e+1	4.9e+1

Comparison between FV and FVE



(a) Ratio $\frac{E_{FV}}{E_{FVE}}$ as N_x increases.



(b) 2-norm error versus the CPU time.

- When $\alpha, \beta \approx 0$ FV allows same error with lower CPU time with respect to FVE.
- For large values of N, when $\alpha, \beta \approx 1$ FVE seems to be more suitable than FV.

Conclusion

- A second order FV discretization for FDEs in Riesz form;
- Spectral study of the coefficient matrix;
- Numerical comparison between FV and FVE approaches.

Further works

- The use of two different shifts between first and second spatial dimension;
- The addition of time dependency and related spectral study;
- The use of non-uniform meshes and related ad-hoc multigrid solver.

Introduction
Discretization
Multigrid methods for Toeplitz matrices
Numerical Results

- M. Donatelli, R. Krause, M. Mazza, M. Semplice, K. Trotti, *Matrices associated to two conservative discretizations of Riesz fractional operators and related multigrid solvers*, under revision.

Thank you

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