# Wreath product of graphs: topological indices and spectrum

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Motivations

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Results

## GRAPH COMPOSITIONS

$$\updownarrow$$

## MATRIX COMPOSITIONS

The correspondence is achieved by the notion of **ADJACENCY MATRIX**.

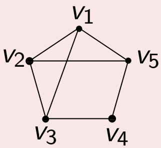
**Spectra** of **adjacency matrices** and **Laplacians** are the main object of **Spectral graph theory**:
connectivity, regularity and other graph invariants; expander graphs; random walks and rapidly mixing Markov chains; isospectrality problems; determination and characterization problem; applications to Mathematical Chemistry.

## The adjacency matrix of a graph

$G = (V_G, E_G)$ undirected simple finite graph.
The **adjacency matrix** of $G$ is the matrix $A_G = (a_{u,v})_{u,v \in V_G}$,
where $a_{u,v} = \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{if } u \not\sim v. \end{cases}$

## Example

$G$




$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

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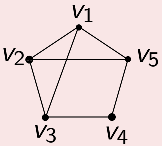



$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

**Remarks:**

♢ $G$ undirected $\Rightarrow A_G$ symmetric;

♢ $\deg u = \sum_{v \in V_G} a_{u,v} =$ number of vertices adjacent to $u$;

♢ $G$ $d$-regular $\Leftrightarrow \sum_{v \in V_G} a_{u,v} = d$ for each $u \in V_G$.

## Cartesian product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs.

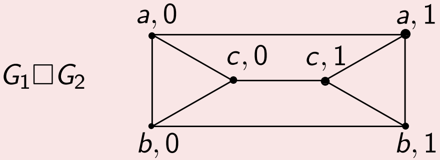
The *Cartesian product* $G_1 \square G_2$ is the graph with:

- vertex set $V_1 \times V_2$
- where $(v_1, v_2) \sim (w_1, w_2)$ if:
  1. either $v_1 = w_1$ and $v_2 \sim w_2$ in $G_2$;
  2. or $v_2 = w_2$ and $v_1 \sim w_1$ in $G_1$.

Then:

$$A_{G_1 \square G_2} = I_{G_1} \otimes A_{G_2} + A_{G_1} \otimes I_{G_2}.$$

## Example

## Direct product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs.

The *direct product* $G_1 \times G_2$ is the graph with:

- vertex set $V_1 \times V_2$
- where $(v_1, v_2) \sim (w_1, w_2)$ if

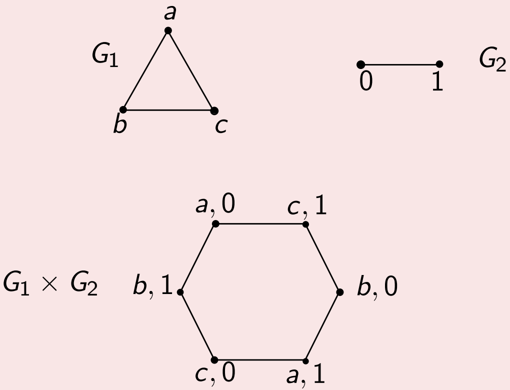
$$v_1 \sim w_1 \text{ in } G_1 \ \text{ and } v_2 \sim w_2 \text{ in } G_2.$$

Then:

$$A_{G_1 \times G_2} = A_{G_1} \otimes A_{G_2}.$$

(Kronecker product of matrices)

## Example



## Strong product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs.

The *strong product* $G_1 \boxtimes G_2$ is the graph with:

- vertex set $V_1 \times V_2$
- where $(v_1, v_2) \sim (w_1, w_2)$ if:
  1. $v_1 = w_1$ and $v_2 \sim w_2$ in $G_2$;
  2. or $v_2 = w_2$ and $v_1 \sim w_1$ in $G_1$;
  3. or $v_1 \sim w_1$ in $G_1$ and $v_2 \sim w_2$ in $G_2$.

Then:

$$A_{G_1 \boxtimes G_2} = I_{G_1} \otimes A_{G_2} + A_{G_1} \otimes I_{G_2} + A_{G_1} \otimes A_{G_2}.$$

## Lexicographic product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs.

The *lexicographic product* $G_1 \circ G_2$ is the graph with:

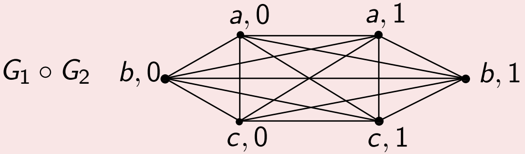
- vertex set $V_1 \times V_2$
- where $(v_1, v_2) \sim (w_1, w_2)$ if:
  1. either $v_1 \sim w_1$ in $G_1$;
  2. or $v_1 = w_1$ and $v_2 \sim w_2$ in $G_2$.

Then:

$$A_{G_1 \circ G_2} = A_{G_1} \otimes J_{G_2} + I_{G_1} \otimes A_{G_2},$$

where $J_{G_2}$ is the matrix indexed by $V_2$ whose entries are all equal to 1.

## Example

## References

[1] G. Sabidussi: The composition of graphs, *Duke Math. J.* **26** (1959), 693–696

[2] W. Imrich, H. Izbicki: Associative Products of Graphs. *Monatsh. Math.* **80** (1975), no. 4, 277–281.

[3] R. Hammack, W. Imrich, S. Klavžar, Handbook of product graphs. Second edition. *Discrete Mathematics and its Applications (Boca Raton)*. CRC Press, Boca Raton, FL, 2011.

## The NEPS construction (non-complete extended $p$-sum)

Let $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,0,\ldots,0)\}$. The NEPS of the graphs $G_1,\ldots,G_n$ with basis $\mathcal{B}$ has vertex set $V_{G_1} \times \cdots \times V_{G_n}$, where $(x_1,\ldots,x_n) \sim (y_1,\ldots,y_n)$ if there exists $b = (b_1,\ldots,b_n) \in \mathcal{B}$ s.t.

- $x_i = y_i$ whenever $b_i = 0$;
- $x_i \sim y_i$ in $G_i$ whenever $b_i = 1$.

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- $x_i = y_i$ whenever $b_i = 0$;
- $x_i \sim y_i$ in $G_i$ whenever $b_i = 1$.

## Special cases for $n = 2$

- Cartesian product $G_1 \square G_2$ when $\mathcal{B} = \{(0,1),(1,0)\}$;
- Direct product $G_1 \times G_2$ when $\mathcal{B} = \{(1,1)\}$;
- Strong product $G_1 \boxtimes G_2$ when $\mathcal{B} = \{(0,1),(1,0),(1,1)\}$.

- $|V_{G_i}| = n_i$;
- $A_{G_i}$ = adjacency matrix of $G_i$, with eigenvalues $\lambda_{i1}, \ldots, \lambda_{in_i}$

**Adjacency matrix of the NEPS with basis $\mathcal{B}$:**

$$\sum_{b \in \mathcal{B}} A_{G_1}^{b_1} \otimes \cdots \otimes A_{G_n}^{b_n},$$

with $A_{G_i}^0 = I_{G_i}$ and $A_{G_i}^1 = A_{G_i}$.

**Spectrum of the NEPS with basis $\mathcal{B}$:**

$$\Lambda_{i_1,\ldots,i_n} = \sum_{b \in \mathcal{B}} \lambda_{1i_1}^{b_1} \cdots \lambda_{ni_n}^{b_n}$$

for $i_k = 1, \ldots, n_k$; $k = 1, \ldots, n$.

[D. Cvetković, M. Doob, H. Sachs, Spectra of graphs. Theory and applications. *Johann Ambrosius Barth, Heidelberg*, 1995]

However, there are many graph operations which cannot be interpreted as NEPS.

$\Rightarrow$ Adjacency matrices and spectra can be harder to be computed!

This is the case of the *wreath product of graphs* that we are going to investigate.

Motivations

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## Wreath product of graphs

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two finite graphs.
The *wreath product* $G \wr H$ is the graph with vertex set

$$V_H^{V_G} \times V_G = \{(f, v) \mid f : V_G \to V_H,\ v \in V_G\},$$

where $(f, v) \sim (f', v')$ if:

1. either $v = v' =: \overline{v}$ and $f(w) = f'(w)$, $\forall\ w \neq \overline{v}$, and $f(\overline{v}) \sim f'(\overline{v})$ in $H$; (*edges of type I*)
2. or $f(w) = f'(w)$, $\forall\ w \in V_G$, and $v \sim v'$ in $G$. (*edges of type II*)

**Remark:**

- $G$ is $d_G$-regular on $n$ vertices and $H$ is $d_H$-regular on $m$ vertices $\Rightarrow$ $G \wr H$ is $(d_G + d_H)$-regular on $nm^n$ vertices
- $G \wr H$ is connected $\Leftrightarrow$ $G$ and $H$ are both connected

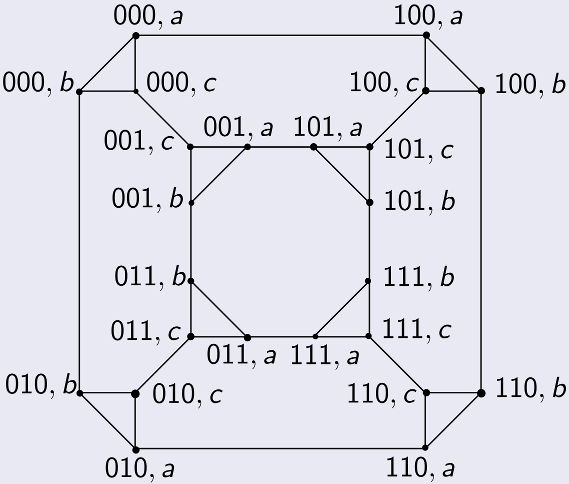
## The Lamplighter random walk (*Walk or switch* model)

The simple random walk on $G \wr H$ is called *Lamplighter random walk*:

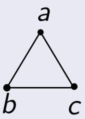
at each vertex of $G$ there is a lamp, whose possible states (or colors) are represented by the vertices of $H$ (the *color graph*), so that the vertex $(f, v)$ of $G \wr H$ represents the configuration of the $|V_G|$ lamps at each vertex of $G$ (for each vertex $u \in V_G$, the lamp at $u$ is in the state $f(u) \in V_H$), together with the position $v$ of a lamplighter walking on $G$. At each step, the lamplighter may:

- either stay at the vertex $v \in G$, but he changes the state of the lamp which is in $v$ to a neighbor state in $H$ (**edges of type I** in $G \wr H$)
- or go to a neighbor of the current vertex $v$ and leave all lamps unchanged (**edges of type II** in $G \wr H$)

# Example: $K_3 \wr K_2$



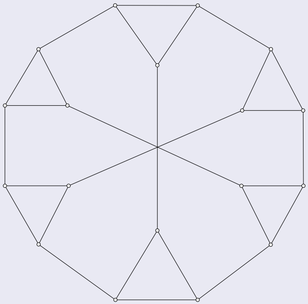

$K_3 \wr K_2$




$K_3$


$K_2$

## The graph $K_2 \wr K_3$



**Remark:** The wreath product is not commutative!

## Transitivity properties

Let $Aut(G)$ denote the automorphism group of $G = (V_G, E_G)$.

1. $G$ is vertex-transitive if, given any two vertices $u, v \in V_G$, there exists $\phi \in Aut(G)$ s.t. $\phi(u) = v$;
2. $G$ is edge-transitive if, given any two edges $e = \{u, v\}$, $e' = \{u', v'\} \in E_G$, there exists $\phi \in Aut(G)$ s.t. $\{\phi(u), \phi(v)\} = \{u', v'\}$;
3. $G$ is arc-transitive if, given any two pairs of adjacent vertices $u \sim v$ and $u' \sim v'$, there exists $\phi \in Aut(G)$ s.t. $\phi(u) = u'$ and $\phi(v) = v'$.

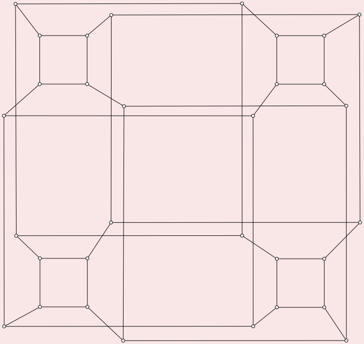
Arc-transitive $\Rightarrow$ edge-transitive $+$ vertex-transitive
Edge-transitive $\not\Rightarrow$ vertex-transitive (semisymmetric graphs)

## Fact

$G, H$ vertex-transitive $\Rightarrow$ $G \wr H$ vertex-transitive graph

However, edge-transitivity and arc-transitivity are not inherited by the wreath product!

## Example: $K_2 \wr C_4$ is not edge-transitive



The wreath product of graphs represents a graph-analogue of the classical wreath product of groups!

## Wreath product of finite groups

Let $H$ and $K$ be two finite groups. The set $K^H = \{f : H \to K\}$ has a group structure w.r.t. the pointwise multiplication

$$(f_1 f_2)(h) = f_1(h) f_2(h).$$

The *wreath product* $H \wr K$ is the semidirect product $K^H \rtimes H$, where $H$ acts on $K^H$ by shifts, i.e., if $f \in K^H$, one has

$$f^h(h') = f(h^{-1}h'), \quad \text{for all } h, h' \in H.$$

## Cayley graphs

Let $G$ be a group generated by a symmetric finite set $S$ (i.e. $s \in S \Rightarrow s^{-1} \in S$).
The *Cayley graph* $Cay(G, S)$ of $G$ w.r.t. $S$ is the graph with vertex set $G$, where $g \sim g'$ if $\exists$ a generator $s \in S$ s.t. $gs = g'$.
The graph $Cay(G, S)$ is a connected regular graph of degree $|S|$.

## Theorem

Let $G_1$ and $G_2$ be two finite groups and let $S_1$ and $S_2$ be symmetric generating sets for $G_1$ and $G_2$, respectively. Then

$$Cay(G_1, S_1) \wr Cay(G_2, S_2) = Cay(G_1 \wr G_2, S),$$

where $S$ is the generating set of $G_1 \wr G_2$ given by

$$S = \{((s_2, 1_{G_2}, \ldots, 1_{G_2}), 1_{G_1}), ((1_{G_2}, \ldots, 1_{G_2}), s_1) \mid s_1 \in S_1, s_2 \in S_2\}.$$

## Wreath product of matrices

Let $A$ and $B$ be two square matrices of order $n$ and $m$, respectively. For each $i = 1, \ldots, n$, let $C_i = (c_{h,k})_{h,k=1,\ldots,n}$ be the matrix defined by

$$c_{h,k} = \begin{cases} 1 & \text{if } h = k = i \\ 0 & \text{otherwise.} \end{cases}$$

The **wreath product** of $A$ and $B$ is the square matrix of order $nm^n$ defined by

$$A \wr B = I_m^{\otimes^n} \otimes A + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes B \otimes I_m^{\otimes^{n-i}} \otimes C_i.$$

[D. D'Angeli, A. Donno, Wreath product of matrices, *Linear Algebra Appl.* **513** (2017), 276–303]

## Theorem [D'Angeli, Donno]

Let $A_G$ (resp. $A_H$) be the adjacency matrix of $G$ (resp. $H$), with $|V_G| = n$, $|V_H| = m$.
Then the wreath product

$$A_G \wr A_H = I_m^{\otimes^n} \otimes A_G + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes A_H \otimes I_m^{\otimes^{n-i}} \otimes C_i$$

is the adjacency matrix of the graph $G \wr H$.

## Sketch of the proof

The first summand corresponds to edges of type II in $G \wr H$:
$(f, v_h) \sim (f, v_k)$, for some $f : V_G \to V_H$ and with $v_h \sim v_k$ in $G$.

The second summand corresponds to edges of type I in $G \wr H$:
$(f, v_i) \sim (g, v_i)$, with $f(v_j) = g(v_j)\ \forall\ v_j \neq v_i$ and $f(v_i) \sim g(v_i)$ in $H$.

The matrix $C_i$ takes into account the fact that the vertex $v_i$ does not change.

## Corollary

Let $G = (V_G, E_G)$ be a $d_G$-regular graph, and let $H = (V_H, E_H)$ be a $d_H$-regular graph, with adjacency matrix $A_G$ and $A_H$, respectively.
Then

$$\frac{1}{d_G + d_H} A_G \wr A_H$$

is the **transition matrix** of the "Walk or switch" Lamplighter random walk on the base graph $G$, with color graph $H$.

## Some references

**Infinite setting:**

[1] L. Bartholdi, W. Woess, Spectral computations on lamplighter groups and Diestel-Leader graphs, *J. Fourier Anal. Appl.* **11** (2005), no. 2, 175–202;

[2] W. Woess, A note on the norms of transition operators on lamplighter graphs and groups, *Internat. J. Algebra Comput.* **15** (2005), no. 5–6, 1261–1272;

[3] F. Lehner, On the Eigenspaces of Lamplighter Random Walks and Percolation Clusters on Graphs, *Proc. AMS* **137** (2009), no. 8, 2631-2637.

**Finite setting:**

[4] F. Scarabotti, F. Tolli, Harmonic Analysis of finite lamplighter random walks, *J. Dyn. Control Syst.* **14** (2008), no. 2, 251–282;

[5] F. Scarabotti, F. Tolli, Harmonic analysis on a finite homogeneous space, *Proc. Lond. Math. Soc. (3)* **100** (2010), no. 2, 348–376.

Motivations

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Results

The adjacency spectrum of $G \wr H$ is hard to be computed, in general!
By specializing the structure of the composite graphs, the spectrum can be elegantly computed for some infinite classes of graphs.

## References

[1] F. Belardo, M. Cavaleri, A. Donno, Spectral analysis of the wreath product of a complete graph with a Cocktail Party graph, to appear in *Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis. Mat. Natur.*
[2] F. Belardo, M. Cavaleri, A. Donno, Wreath product of a complete graph with a cyclic graph: topological indices and spectrum, *Appl. Math. Comput.* **336**, 288–300
[3] A. Donno, Spectrum, distance spectrum, and Wiener index of wreath products of complete graphs, *Ars Math. Contemp.* **13** (2017), no. 1, 207–225.

## Spectral analysis of $A \wr B$, with $B$ circulant

Let $B = \begin{pmatrix} b_0 & b_1 & & & b_{m-1} \\ b_{m-1} & b_0 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ b_1 & & & b_{m-1} & b_0 \end{pmatrix}$.

## Theorem (D'Angeli, Donno)

The spectrum $\Sigma$ of $A \wr B$, with $B$ circulant, is obtained by taking the union of the spectra $\Sigma_{i_1,\ldots,i_n}$ of the $m^n$ matrices of order $n$ given by

$$\widetilde{M}^{i_1,i_2,\ldots,i_n} = A + \sum_{t=1}^{n} \sum_{h=0}^{m-1} b_h \rho^{hi_t} C_t,$$

where $i_t \in \{0, 1, \ldots, m-1\}$, $\forall\ t = 1, \ldots, n$, and $\rho = \exp\left(\frac{2\pi i}{m}\right)$.

## Spectrum of the graph $K_n \wr K_m$

**Theorem (Donno)**

The adjacency spectrum of the graph $K_n \wr K_m$ is the union of the following partial spectra $\Sigma_k$, each with multiplicity $\binom{n}{k} \cdot (m-1)^{n-k}$:

$$\Sigma_0 = \left\{(-2)^{n-1}; n-2\right\}$$

$$\Sigma_k = \left\{(m-2)^{k-1}; (-2)^{n-k-1}; \frac{m+n-4 \pm \sqrt{(m-n)^2+4km}}{2}\right\},$$

for $k = 1, \ldots, n-1$, and

$$\Sigma_n = \left\{(m-2)^{n-1}; m+n-2\right\}.$$

## Spectrum of the graph $K_n \wr CP_{2m}$

**Theorem (Belardo, Cavaleri, Donno):**
The adjacency spectrum of the graph $K_n \wr CP_{2m}$ is the union of the following $\frac{(n+1)(n+2)}{2}$ partial spectra $\Sigma_{k,h,q}$, where $k, h, q$ are nonnegative integers satisfying the condition $k+h+q=n$, each having multiplicity $\binom{n}{k,h,q} m^h (m-1)^q$:

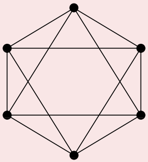
$$\Sigma_{k,h,q} = \{(2m-3)^{k-1}, (-1)^{h-1}, (-3)^{q-1}, \alpha, \beta, \gamma\},$$

where $\alpha, \beta, \gamma$ are the zeros of the polynomial of degree 3

$$\begin{aligned} P(\lambda) &= \lambda^3 + (-h-k-2m-q+7)\lambda^2 \\ &+ (2hm+2mq-6h-4k-8m-4q+15)\lambda \\ &+ 6hm+2mq-9h-3k-6m-3q+9. \end{aligned}$$

# Example: Spectrum of $K_2 \wr CP_6$

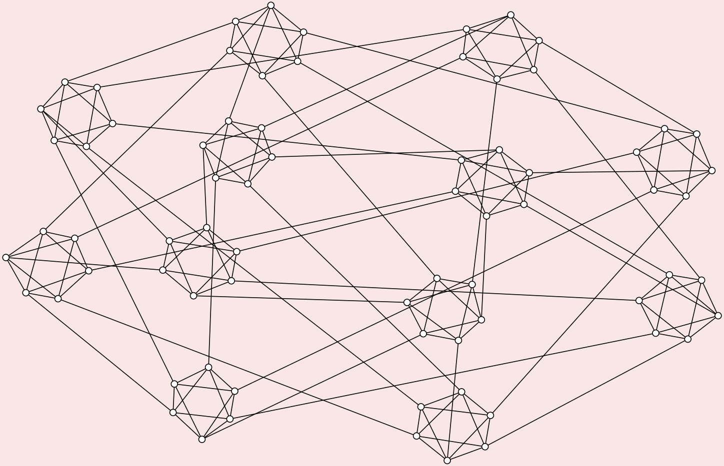
$K_2$



$CP_6$

| $k$ | $h$ | $q$ | Multiplicity | $\Sigma_{k,h,q}$ |
|---|---|---|---|---|
| 2 | 0 | 0 | 1 | $3, 5$ |
| 1 | 1 | 0 | 6 | $2 \pm \sqrt{5}$ |
| 1 | 0 | 1 | 4 | $1 \pm \sqrt{10}$ |
| 0 | 2 | 0 | 9 | $\pm 1$ |
| 0 | 1 | 1 | 12 | $-1 \pm \sqrt{2}$ |
| 0 | 0 | 2 | 4 | $-3, -1$ |

## The graph $K_2 \wr CP_6$



## Spectrum of the graph $K_n \wr C_m$

**Theorem (Belardo, Cavaleri, Donno):**
The adjacency spectrum of the graph $K_n \wr C_m$ is the union of $m^n$ partial spectra, consisting of the zeros of the following $m^n$ polynomials of degree $n$ in the variable $\lambda$:

$$\lambda^n + \sum_{i=1}^{n} (e_i(x_1,\ldots,x_n) - (n-i+1)e_{i-1}(x_1,\ldots,x_n))\lambda^{n-i},$$

with $x_t = 1 - 2\cos\frac{2\pi i_t}{m}$, and $i_t \in \{0,1,\ldots,m-1\}$ $\forall$ $t = 1,\ldots,n$.

Here, $e_j(x_1,\ldots,x_n)$ is the $j$-th elementary symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$ defined as:

$$e_0(x_1,\ldots,x_n) = 1, \qquad e_1(x_1,\ldots,x_n) = x_1 + x_2 + \cdots + x_n,$$
$$e_j(x_1,\ldots,x_n) = \sum_{1\le i_1<\ldots<i_j\le n} x_{i_1}\cdots x_{i_j}, \quad \ldots, e_n(x_1,\ldots,x_n) = \prod_{1\le i\le n} x_i.$$

## Zagreb indices

**Definition:**
Let $G = (V_G, E_G)$ be a finite simple connected graph.
The *first Zagreb index* of $G$ is defined as

$$M_1(G) = \sum_{v \in V_G} (\deg v)^2.$$

The *second Zagreb index* of $G$ is defined as

$$M_2(G) = \sum_{u \sim v} \deg u \deg v.$$

[I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total $\pi$-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972), 535–538]

**Theorem (Cavaleri, Donno)**

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ with $|V_G| = n$ and $|V_H| = m$. Then

$$M_1(G \wr H) = m^{n-1}(mM_1(G) + nM_1(H) + 8|E_G||E_H|)$$

and

$$\begin{aligned} M_2(G \wr H) &= 3m^{n-1}|E_H|M_1(G) + 2|E_G|m^{n-1}M_1(H) \\ &+ m^nM_2(G) + nm^{n-1}M_2(H) + 4m^{n-2}|E_G||E_H|^2 \end{aligned}$$

## The special cases $G = K_n$ and $G = P_n$

$K_n$ **complete graph:** $M_1(K_n) = n(n-1)^2$; $M_2(K_n) = \frac{n(n-1)^3}{2} \Longrightarrow$

$$M_1(K_n \wr H) = nm^n(n-1)^2 + nm^{n-1}M_1(H) + 4n(n-1)m^{n-1}|E_H|$$

$$\begin{aligned} M_2(K_n \wr H) &= 3nm^{n-1}(n-1)^2|E_H| + n(n-1)m^{n-1}M_1(H) \\ &+ \frac{nm^n(n-1)^3}{2} + nm^{n-1}M_2(H) + 2nm^{n-2}(n-1)|E_H|^2 \end{aligned}$$

$P_n$ **path graph:** $M_1(P_n) = 4n-6$; $M_2(P_n) = 4n-8 \Longrightarrow$

$$M_1(P_n \wr H) = m^n(4n-6) + nm^{n-1}M_1(H) + 8(n-1)m^{n-1}|E_H|$$

$$\begin{aligned} M_2(P_n \wr H) &= 3m^{n-1}(4n-6)|E_H| + 2(n-1)m^{n-1}M_1(H) \\ &+ m^n(4n-8) + nm^{n-1}M_2(H) + 4m^{n-2}(n-1)|E_H|^2 \end{aligned}$$

## Distances and Wiener index in $G \wr H$

Put $V_G = \{x_1, x_2, \ldots, x_n\} \Rightarrow$ a vertex of $G \wr H$ can be written as

$$u = (y_1, \ldots, y_n)x_i, \quad \text{with } y_j \in V_H, \text{ and } \ x_i \in V_G.$$

**Lamplighter interpretation:**
the lamp placed at the $j$-th vertex $x_j$ of $G$ has color $y_j \in V_H$, and the lamplighter is in position $x_i$ in $G$.

We are interested in computing the distance between two vertices $u = (y_1, \ldots, y_n)x_i$ and $v = (y'_1, \ldots, y'_n)x_k$.

**Remark:**
In a **shortest path from $u$ to $v$**, the lamplighter has to take a path of minimal length in $G$ from $x_i$ to $x_k$, and visiting all vertices $x_j$ where the lamp configurations do not coincide. Moreover, for each of such vertices, he has to take a shortest path from $y_j$ to $y'_j$ in $H$.

## Distances in $G \wr H$

For any $A \subseteq V_G$, for any $u, v \in V_G$, we define $\rho_A(u, v)$ as the length of a shortest path from $u$ to $v$ visiting each vertex of $A$ (not necessarily once).

In the case $A = V_G$, we write $d_{Ha} := \rho_{V_G}$ (Hamiltonian distance).

**Property:**
Let $\emptyset \neq A \subseteq V_G$, with $A = \{a_1, \ldots, a_k\}$. Then

$$\rho_A(u, v) = \min_{\sigma \in Sym(k)} \left\{ d_G(u, a_{\sigma(1)}) + \sum_{i=1}^{k-1} d_G(a_{\sigma(i)}, a_{\sigma(i+1)}) + d_G(a_{\sigma(k)}, v) \right\}$$

**Remark:** If $|V_G| = n$:

$$\exists u \in V_G \ (\forall u \in V_G): \ d_{Ha}(u,u) = n \iff G \text{ is Hamiltonian}$$

**Definition:**
For any $u \in V_G$, the *Hamiltonian eccentricity* of $u$ is

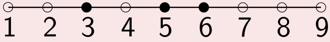
$$e_{G,Ha}(u) := \max_{v \in V_G}\{d_{Ha}(u,v)\}.$$

The *Hamiltonian diameter* of $G$ is

$$diam_{Ha}(G) := \max_{u \in V_G}\{e_{G,Ha}(u)\}.$$

In particular, if $G$ is Hamiltonian, then $diam_{Ha}(G) = n$ and all the shortest paths starting and ending at the same vertex, visiting any other vertex, realize the Hamiltonian diameter.

## The example of the path graph $P_n$




Observe that $\rho_A = \rho_{\{\min A, \max A\}}$. In this case $A = \{3, 5, 6\}$:

$$\rho_A = \left(\begin{array}{cc|cccc|ccc} 10 & 9 & {\color{blue}8} & 7 & 6 & {\color{red}5} & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 & 5 & 4 & 5 & 6 & 7 \\ \hline 8 & 7 & 6 & 5 & 4 & 3 & 4 & 5 & 6 \\ 7 & 6 & 5 & 6 & 5 & 4 & 5 & 6 & 7 \\ 6 & 5 & 4 & 5 & 6 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 6 & 5 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 6 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 7 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}\right)$$

## Theorem (Cavaleri, Donno)

Suppose $|V_G| = n$ and $|V_H| = m$.
Let $u = (y_1, \ldots, y_n)x$, $v = (y'_1, \ldots, y'_n)x' \in G \wr H$. Then:

$$d_{G \wr H}(u, v) = \sum_{i=1}^{n} d_H(y_i, y'_i) + \rho_{\delta(y,y')}(x, x'),$$

with $y = (y_1, \ldots, y_n)$, $y' = (y'_1, \ldots, y'_n)$,
and $\delta(y, y') := \{x_i \in V_G : y_i \neq y'_i\}$

## Corollary

$$e_{G \wr H}(u) = \sum_{i=1}^{n} e_H(y_i) + e_{G,Ha}(x)$$

$$diam(G \wr H) = n\, diam(H) + diam_{Ha}(G)$$

## Definition

Let $G = (V_G, E_G)$ be a connected graph. The *Wiener index* $W(G)$ of $G$ is the sum of the distances between all the unordered pairs of vertices, i.e.,

$$W(G) = \frac{1}{2} \sum_{u,v \in V_G} d_G(u, v),$$

where $d_G(u, v)$ denotes the geodesic distance between $u$ and $v$, that is, the length of a shortest path from $u$ to $v$ in $G$.

**Reference:**

[H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* **69**, 17–20 (1947)]

## Wiener index of $G \wr H$

**Theorem (Cavaleri, Donno):**
Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two connected graphs with $|V_G| = n$, $|V_H| = m$. Then:

$$W(G \wr H) = n^3 m^{2(n-1)} W(H) + m^n \sum_{A \subseteq V_G} (m-1)^{|A|} W_{\rho_A}(G),$$

where

$$W_{\rho_A}(G) := \frac{1}{2} \sum_{u,v \in V_G} \rho_A(u,v).$$

The coefficient of $W_{\rho_A}(G)$ in $W(G \wr H)$ only depends on the cardinality of $A$!

$\Rightarrow$ for any $k$, set $W_{\rho_k}(G) := \sum_{A \subseteq V_G, |A|=k} W_{\rho_A}(G)$

$$\Longrightarrow \quad W(G \wr H) = n^3 m^{2(n-1)} W(H) + m^n \sum_{k=0}^{n} (m-1)^k W_{\rho_k}(G)$$

**Remark:**

$$W_{\rho_0}(G) = W_{\rho_\emptyset}(G) = W(G);$$
$$W_{\rho_1}(G) = 2nW(G);$$
$$W_{\rho_n}(G) = W_{\rho_{V_G}}(G)$$

**Definition:**
The *Wiener vector* of $G$ is the $(n+1)$-component vector:

$$\mathbf{W}_{\rho}(G) := (W_{\rho_0}(G), W_{\rho_1}(G), \ldots, W_{\rho_n}(G)).$$

Therefore:

$$\mathbf{W}_{\rho}(G_1) = \mathbf{W}_{\rho}(G_2) \Rightarrow W(G_1 \wr H) = W(G_2 \wr H) \quad \text{for every } H$$

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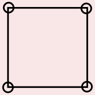
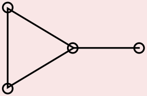
Actually, also the converse is true!

**Proposition:**
For any pair of connected graphs $G_1$ and $G_2$:

$$\mathbf{W}_\rho(G_1) = \mathbf{W}_\rho(G_2) \iff W(G_1 \wr H) = W(G_2 \wr H) \quad \text{for every } H.$$

## The cycle $C_4$ and the paw graph $P$





$\mathbf{W}_\rho(C_4) = (8, 64, 132, 104, 28)$ $\mathbf{W}_\rho(P) = (8, 64, 134, 110, 32)$

**Question:**

Are there pairs of non-isomorphic graphs $G_1$, $G_2$ s.t. $\mathbf{W}_\rho(G_1) = \mathbf{W}_\rho(G_2)$?

Equivalently, are there pairs of non-isomorphic graphs $G_1$, $G_2$ s.t. $W(G_1 \wr H) = W(G_2 \wr H)$ for every graph $H$?

## The Wiener vectors of $K_n$ and $P_n$

$$
\begin{aligned}
W_{\rho_0}(K_n) =& \frac{n(n-1)}{2} \\
W_{\rho_1}(K_n) =& n^2(n-1) \\
W_{\rho_k}(K_n) =& \frac{1}{2}\binom{n}{k}(kn^2 - 2kn + k + n^2), \qquad \text{with } 2 \le k \le n
\end{aligned}
$$

$$
\begin{aligned}
W_{\rho_k}(P_n) \quad = & \quad \binom{n+1}{k+1}\frac{1}{6(k+2)(k+3)}(5k^3n^2 + k^3n + 18k^2n^2 - 18k^2n \\
- & \quad 12k^2 + 19kn^2 - 25kn + 12k + 6n^2 - 6n)
\end{aligned}
$$

## The case of the wreath product $K_n \wr K_m$

1. The diameter of the graph $K_n \wr K_m$ is $2n$.
   Such a maximal distance is obtained when the vertices $u, v$ have the form
   $$u = (y_1, \dots, y_n)x_k \qquad v = (y'_1, \dots, y'_n)x_k,$$
   with $y_j \neq y'_j$, for each $j = 1, \dots, n$.
2. The Wiener index of the graph $K_n \wr K_m$ is
   $$\frac{nm^n}{2}(2m^n n^2 - nm^n - 2n^2 m^{n-1} + m^n + 2nm^{n-1} - m^{n-1} - m).$$

## The case of the wreath product $K_n \wr C_m$

1. The diameter of the graph $K_n \wr C_m$ is $n\left(1 + \lfloor \frac{m}{2} \rfloor\right)$. Such a maximal distance is obtained when the vertices $u, v$ have the form
$$u = (y_1, \ldots, y_n)x_k \qquad v = (y_1', \ldots, y_n')x_k,$$
where the distance $d_{C_m}(y_j, y_j')$ is maximal for each $j = 1, \ldots, n$.
2. The Wiener index of the graph $K_n \wr C_m$ is
$$\frac{nm^n}{2}\left(n^2 m^{n-1}\left\lfloor \frac{m^2}{4} \right\rfloor + n^2 m^n - n^2 m^{n-1} - nm^n + 2nm^{n-1} - m + m^n - m^{n-1}\right)$$

## The Szeged index of $G \wr H$

**Definition:** Let $G = (V_G, E_G)$. Given $e = \{u, v\} \in E_G$, put:

$$B_u(e) = \{w \in V_G : d_G(w, u) < d_G(w, v)\}$$

$$B_v(e) = \{w \in V_G : d_G(w, v) < d_G(w, u)\}.$$

If $d_G(w, u) = d_G(w, v)$, then $w$ is neither in $B_u(e)$ nor in $B_v(e)$.

Then:

$$Sz(G) = \sum_{e=\{u,v\}\in E_G} |B_u(e)||B_v(e)|.$$

[A. Dobrynin, I. Gutman, On a graph invariant related to the sum of all distances in a graph, *Publ. Inst. Math. (Beograd) (N.S.)* **56** (70) (1994), 18–22]

In order to compute $Sz(G \wr H)$ we decompose $E_{G \wr H}$ into the subset $E_I$ of edges of type I, and the subset $E_{II}$ of edges of type II:

$$Sz_I(G \wr H) := \sum_{e=\{u,v\} \in E_I} |B_u(e)||B_v(e)|$$

$$Sz_{II}(G \wr H) := \sum_{e=\{u,v\} \in E_{II}} |B_u(e)||B_v(e)|$$

$$\Longrightarrow \qquad Sz(G \wr H) = Sz_I(G \wr H) + Sz_{II}(G \wr H).$$

## Edges of type I

$$Sz_I(G \wr H) = n^3 m^{3n-3} Sz(H)$$

## Edges of type II

If $E = \{u, v\} \in E_{II}$, with $u = (y_1, \ldots, y_n)x_i$ and $v = (y_1, \ldots, y_n)x_j$ and $e = \{x_i, x_j\} \in E_G$, then:

$$|B_u(E)| = \sum_{A \subseteq V_G} (m-1)^{|A|} |\{x_k \in V_G : \rho_A(x_k, x_i) < \rho_A(x_k, x_j)\}|$$

and it does not depend of the particular lamp configuration.
If $G$ is edge-transitive:

$$Sz_{II}(G \wr H) = m^n |E_G||B_u(E)||B_v(E)|, \quad \text{for any } E = \{u, v\} \in E_{II}$$

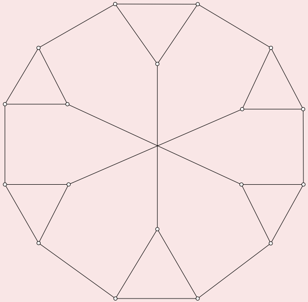
If $G$ is also arc-transitive: $Sz_{II}(G \wr H) = m^n |E_G||B_u(E)|^2$.

## The example of $K_n \wr H$

Let $K_n$ be the complete graph on $n$ vertices and let $H$ be a connected graph with $m$ vertices. Then:

$$Sz(K_n \wr H) = n^3 m^{3n-3} Sz(H) + \frac{1}{2} m^n n(n-1) \left(m + m^{n-2}(m^2 + mn - 3m - n + 2)\right)^2$$

**The graph** $K_2 \wr C_3$



$K_2 \wr C_3$ is 3-regular on 18 vertices.

- 18 edges of type I (in each triangle, representing a change of configuration, that is, a step in $C_3$)
- 9 edges of type II (connecting distinct triangles, representing a change of position, that is, a step in $K_2$).

For each $e = \{u, v\} \in E_I$, we have $|B_u(e)| = |B_v(e)| = 6$ and 6 vertices are equidistant from $u$ and $v$.
For each $e = \{u, v\} \in E_{II}$, we have $|B_u(e)| = |B_v(e)| = 9$.

Therefore:

$$Sz(K_2 \wr C_3) = \sum_{e=\{u,v\} \in E_{K_2 \wr C_3}} |B_u(e)||B_v(e)| = 18 \cdot 6^2 + 9 \cdot 9^2 = 1377.$$

**Reference**

[M. Cavaleri, A. Donno, Some degree and distance-based invariants of wreath products of graphs, preprint, arXiv:1805.08989]

## Total distance and distance-balanced property

Let $G = (V_G, E_G)$ be a graph, and let $e = \{u, v\} \in E_G$:

$$B_u(e) = \{w \in V_G : d_G(w, u) < d_G(w, v)\}$$

$$B_v(e) = \{w \in V_G : d_G(w, v) < d_G(w, u)\}.$$

**Definition:**
$G$ is *distance-balanced* if $|B_u(e)| = |B_v(e)|$, for every pair of adjacent vertices $u, v \in V_G$.

**Example**:
a graph $G$ of diameter 2 is distance-balanced iff it is regular;
a vertex-transitive graph is distance-balanced.

[J. Jerebic, S. Klavžar, D.F. Rall, Distance-balanced graphs, *Ann. Combin.* **12** (2008), 71–79]:
the distance-balanced property is investigated for the Cartesian and the lexicographic product.

**Definition:**
Let $v \in V_G$. The *total distance* of $v$ is

$$W(v,G) = \sum_{u \in V_G} d_G(v,u),$$

The *median* $M(G)$ of $G$ is the set of vertices of $G$ for which the value $W(v,G)$ is minimal among all vertices of $G$.

**Remark:** $W(G) = \frac{1}{2}\sum_{v \in V_G} W(v,G)$.

**Proposition:**
$G = (V_G, E_G)$ is distance-balanced $\iff$ $M(G) = V_G$.

[K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl, A.R. Subhamathi, Strongly distance-balanced graphs and graph products, *European J. Combin.* **30** (2009), 1048–1053]

## Total distances in $G \wr H$

**Theorem:**
Let $|V_G| = n$, $|V_H| = m$, and $u = (y_1, \dots y_n)x \in V_{G \wr H}$:

$$W(u, G \wr H) = nm^{n-1} \sum_{i=1}^{n} W(y_i, H) + \sum_{A \subseteq V_G} (m-1)^{|A|} W_{\rho_A}(x, G),$$

where $W_{\rho_A}(x, G) = \sum_{x' \in V_G} \rho_A(x, x')$.

For any $k$ put $W_{\rho_k}(x, G) := \sum_{A \subseteq V_G, |A|=k} W_{\rho_A}(x, G)$

$$\Rightarrow W(u, G \wr H) = nm^{n-1} \sum_{i=1}^{n} W(y_i, H) + \sum_{k=0}^{n} (m-1)^k W_{\rho_k}(x, G)$$

**Remark:** Computing $W_{\rho_k}(x, G)$ for $G$ allows to immediately deduce the total distances in $G \wr H$, when the total distances in $H$ are known.

## Total distances and Wiener index in $S_n \wr K_m$

[M. Cavaleri, A. Donno, A. Scozzari, Total distance, Wiener index, and opportunity index in wreath products of star graphs, preprint]

**Theorem:** Let $u = (y_1, \ldots, y_n)x \in V_{S_n \wr K_m}$. Then

$$W(u, S_n \wr K_m)$$
$$= \begin{cases} 3m^n n^2 - 4m^n n + 6m^{n-1} n - 3m^{n-1} n^2 - 4m^{n-1} + 3m^n - 2m & \text{if } x \neq c \\ 3m^n n^2 - 3m^n n + 4m^{n-1} n - 3m^{n-1} n^2 - 2m^{n-1} + m^n & \text{if } x = c \end{cases}$$

where $c$ is the central vertex of the star $S_n$.

**Corollary:**
Let $u = (y_1, \ldots, y_n)x \in V_{S_n \wr K_m}$.
If $m = 2$, or if $m = n = 3$, the vertex $u$ is median if and only if $x = c$.
In all the other cases, the vertex $u$ is median if and only if $x \neq c$.

**Corollary:**
The Wiener index of the graph $S_n \wr K_m$ is

$$m^{2n-1}(m-1)\left(\frac{3}{2}n^3 - 1\right) + m^{n+1}(n-1)(3m^{n-2}n - 2m^{n-1}n - 1)$$

## Total distances and Wiener index in $S_n \wr S_m$

**Theorem:**
Put:

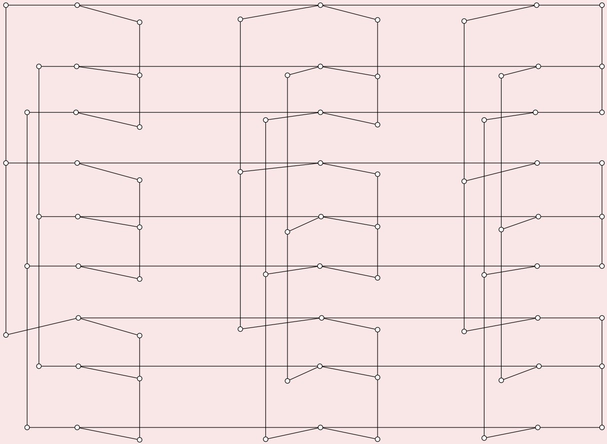
$$W_{min} = 3m^n - 3m^{n-1}n^2 - 4m^n n + 3m^n n^2 + 6m^{n-1}n - 4m^{n-1} - 2m,$$
$$\Delta = m^{n-1}n(m-2),$$
$$\Delta_c = -2m^n + m^n n - 2m^{n-1}n + 2m^{n-1} + 2m.$$

Then, for each $u = (y_1, \ldots, y_n)x \in V_{S_n \wr S_m}$, we have:

$$W(u, S_n \wr S_m) = \begin{cases} W_{min} + \ell(u)\Delta & \text{if } x \neq c \\ W_{min} + \ell(u)\Delta + \Delta_c & \text{if } x = c, \end{cases}$$

where $\ell(u) = |\{i \in \{1, \ldots, n\} : y_i \neq c\}|$.

## The graph $S_3 \wr S_3$



**Corollary:**

If $\max\{n,m\} > 3$, a vertex $u = (y_1, \ldots, y_n)x \in S_n \wr S_m$ is a median vertex $\iff x \neq c$ and $\ell(u) = 0$.

That is:

$$M(S_n \wr S_m) = \{(c, \ldots, c)x : x \in V_{S_n}, x \neq c\}$$

**Corollary:**

The Wiener index of $S_n \wr S_m$ is

$$W(S_n \wr S_m) = \frac{nm^n}{2}\left(W_{min} + \frac{1}{n}\Delta_c + \Delta\frac{(m-1)n}{m}\right)$$

## Further developments

1. To compute the adjacency spectrum for more general classes of graphs.
2. To determine the Wiener vector of graphs with high symmetries.
3. To extend the computation of distances and total distances to *weighted graphs*.
4. The analysis of total distances is the first step to understand the *distance-balanced property of a wreath product*. Having conditions on the factors for the distance-balance of $G \wr H$ would produce new examples and counterexamples for many *centrality problems*.
5. To investigate the automorphism group of $G \wr H$.