A NOTE ON POLYCYLIC RESIDUALLY FINITE-p GROUPS

GIOVANNI CUTOLO AND HOWARD SMITH

ABSTRACT. A subgroup H of a residually finite-p group G is almost p-closed in G if H has finite p'-index in $\bar{H}$, its closure with respect to the pro-p topology on G. We characterise polycyclic residually finite-p groups in which all subgroups are almost p-closed and discuss a few conditions that are sufficient for particular subgroups H to be almost p-closed. We also present, for each prime p, an example of a polycyclic residually-p group G for which $|\bar{H} : H|$ takes on all possible values, including infinity, as H varies.

1. Introduction

If p is a prime, G is residually a finite p-group and H is a subgroup of G, we shall say that H is *almost p-closed* in G if the index of H in its closure with respect to the pro-p topology of G is a finite p'-number. Recall that the pro-p topology on G is that topology having the set of all normal subgroups of finite p-power index as a fundamental system of neighbourhoods of the identity. If G is soluble-by-finite and finitely generated then $\{G^{p^n} \mid n \in \mathbb{N}\}$ is another, often more convenient, equivalent fundamental system. Throughout the paper, we will use bars to denote closures with respect to this topology.

It is a well-known result due to Mal'cev that in a polycyclic-by-finite group all subgroups are closed with respect to the profinite topology (see for instance [2], 5.4.16). It is easy to see that every infinite polycyclic group G in the class $\mathbf{R}\mathfrak{F}_p$ of groups that are residually (finite-p) has some subgroup H which is not closed with respect to the pro-p topology. However, it is equally easy to show that if G is also abelian then every subgroup of G is almost p-closed in G. The second author was asked some time ago whether this is the general rule, that is: if G is a polycyclic group in $\mathbf{R}\mathfrak{F}_p$, is every subgroup H of G almost p-closed? The general answer is in the negative; we will characterise the groups for which the answer is in the positive as those in which all finite quotients are p-nilpotent (Theorem 2.2). This includes the case of nilpotent-by-finite groups (see Corollary 2.3), hence, for instance, that of supersoluble groups. By contrast, in the case of arbitrary polycyclic groups in $\mathbf{R}\mathfrak{F}_p$ the index $|\bar{H} : H|$ need not even be finite and, if it is finite, it can be any positive integer (see Remark 3.2). Nevertheless, it turns out that every polycyclic group G has a residually-p subgroup of finite index in which all subgroups are almost p-closed (Corollary 2.4). Finally, in Theorem 3.7 we will find a condition that is sufficient for all subgroups contained in the hypercentre of a polycyclic group in $\mathbf{R}\mathfrak{F}_p$ to be almost p-closed.

2. Groups in which all subgroups are almost p-closed

We begin by recalling some elementary, well-known facts about the (Hausdorff) topology that we are discussing. If G is a group in $\mathbf{R}\mathfrak{F}_p$ and $H \leq G$ then the closure of H in the pro-p topology of G is $\bar{H} = \bigcap\{HN \mid N \lhd G \text{ and } N \in \mathcal{N}\}$, where $\mathcal{N}$ is any fundamental system of neighbourhoods of the identity. In particular, $\bar{H} = \bigcap\{HN \mid N \lhd G \text{ and } G/N \text{ is a finite } p\text{-group}\}$; and $\bar{H} = \bigcap\{HG^{p^n} \mid n \in \mathbb{N}\}$ if G is polycyclic-by-finite. It follows that if $\mathfrak{V}$ is a variety and H is maximal among the subgroups (resp. normal subgroups) of G in $\mathfrak{V}$ then H is closed. When $H \lhd G$ this latter condition means exactly that $G/H \in \mathbf{R}\mathfrak{F}_p$. Thus, a soluble-by-finite group G in $\mathbf{R}\mathfrak{F}_p$ for some prime p is necessarily soluble. For, the soluble radical S of G is closed in the pro-p

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topology, being a maximal normal subgroup of G in the variety generated by itself. Hence G/S is a finite group in $\mathbf{R}\mathfrak{F}_p$, that is, a finite p-group. By a similar argument, the Fitting subgroup $\operatorname{Fit} G$ of any polycyclic group in $\mathbf{R}\mathfrak{F}_p$ is closed. Also, all finite subgroups and all centralizers are closed, hence such is $Z(G)$ and, in our case, $Z(\operatorname{Fit} G)$. All these properties will be used throughout the paper without further comments.

We also make use of the following very elementary remark.

Lemma 2.1. *Let G be a polycyclic group and let $H \leq G$. If $|G : H|$ is infinite then for every prime p there exists a subgroup K of finite index in G such that $H < K$ and p divides $|G : K|$.*

Proof. We may assume that H is maximal among the subgroups of G for which the statement fails. Then every subgroup of G properly containing H has finite index. If $|G : G^{(n)}H|$ is infinite, for some positive integer n, then $G^{(n)} \leq H$. Thus there exists a term A of the derived series of G such that $|G : AH|$ is finite and $A' \leq H$. Let $B = A \cap H$; then A/B is (finitely generated) infinite abelian and, for every prime p, we have that A/A^pB is a nontrivial finite p-group. Now, $|AH : A^pH| = |A : A^pB|$, so A^pH has the the property required for K. □

Theorem 2.2. *Let p be a prime. For a polycyclic group $G \in \mathbf{R}\mathfrak{F}_p$ the following are equivalent conditions:*

(i) *every subgroup of G is almost p-closed;*
(ii) *every normal subgroup of finite index in G is almost p-closed;*
(iii) *every finite quotient of G is p-nilpotent.*

Proof. Obviously (i) implies (ii). Suppose that (ii) holds and let G/N be a finite quotient of G. Then the closure $\bar{N}$ of N is normal in G and $\bar{N}/N$ is a p'-group. On the other hand, $G/\bar{N}$ is a p-group, as a finite group in $\mathbf{R}\mathfrak{F}_p$. Thus G/N is p-nilpotent and (iii) holds. Finally, assume (iii) and let $H \leq G$. By Lemma 2.1, if $|\bar{H} : H|$ is infinite there exists K such that $H < K < \bar{H}$ (so that $\bar{K} = \bar{H}$) and $|\bar{H} : K|$ is finite and divisible by p. So, aiming at proving that $|\bar{H} : H|$ is a finite p'-number, we may assume that $|\bar{H} : H|$ is finite. Since H is closed in the profinite topology of G there exists a normal subgroup N of finite index in G such that $NH \cap \bar{H} = H$. Let Q/N be the Hall p'-subgroup of G/N. Then G/Q is a finite p-group, hence $\bar{H} \leq HQ$. Therefore $|\bar{H} : H| = |N\bar{H} : NH|$ divides $|QH : NH|$, which is a p'-number. This completes the proof. □

For the sake of brevity, for every prime p we will call $\mathfrak{N}(p)$ the class of those groups whose finite quotients are all p-nilpotent. It is easy to check that all finite sections of a polycyclic $\mathfrak{N}(p)$-group must be p-nilpotent. Also, every nilpotent-by-finite group G in $\mathbf{R}\mathfrak{F}_p$ is in $\mathfrak{N}(p)$, for $F = \operatorname{Fit}(G)$ is closed and so G/F is a p-group. Thus we have:

Corollary 2.3. *Let p be a prime and G a finitely generated nilpotent-by-finite group in $\mathbf{R}\mathfrak{F}_p$. Then every subgroup of G is almost p-closed.*

On the other hand there exist polycyclic $\mathfrak{N}(p)$-groups that are not nilpotent-by-finite, as the following remarks make clear.

It is well known that, for every prime p, every polycyclic group has a residually-p subgroup of finite index (this is a result by Šmelkin [5], also see [4], p. 19, Theorem 4(ii)). On the other hand, as remarked by Roseblade ([3], also see [1], Lemma 11.2.16), for each prime p, every polycyclic group has a characteristic $\mathfrak{N}(p)$-subgroup of finite index, namely the intersection of the centralizers of all chief factors of p-power order. Thus we obtain:

Corollary 2.4. *Let G be polycylic group and let p be a prime. Then G has a characteristic subgroup G_1 of finite index with the property that all subgroups of G_1 are almost p-closed in G_1.*

We can also look again at Theorem 2.2 and note that, as a consequence of Šmelkin's result (rather than of Corollary 2.4), polycyclic groups in $\mathfrak{N}(p)$ are themselves close to being residually-p, in yet another sense.

Corollary 2.5. *Let p be a prime and G a polycyclic-by-finite $\mathfrak{N}(p)$-group. Then G has a finite normal p'-subgroup F such that G/F is residually-p and has all subgroups almost p-closed.*

Proof. There exists $H \lhd G$ such that G/H is finite and $H \in \mathbf{R}\mathfrak{F}_p$. For every normal subgroup N of finite index in G let N^*/N be the Hall p'-subgroup of G/N. Let $q = |H^*/H|$. For all $n \in \mathbb{N}$, let $H_n = H^{p^n}$; then $H^* = HH_n^*$ and $(H_n^*)^q \leq H_n$. Let F be the (finite-p) residual of G; then

$$F = \bigcap_{\substack{N \lhd G \\ G/N \text{ is finite}}} N^* \qquad \text{and so} \qquad F^q \leq \bigcap_{\substack{N \lhd G \\ G/N \text{ is finite}}} (N^*)^q \leq \bigcap_{n \in \mathbb{N}} (H_n^*)^q \leq \bigcap_{n \in \mathbb{N}} H_n = 1.$$

Thus F is finite, of exponent at most q. The final observation follows now from Theorem 2.2. □

3. Examples and further results

The following construction contrasts with the results in the previous section and with Corollary 2.4 in particular, by showing how far away from being almost p-closed subgroups of polycyclic residually-p groups can be.

Proposition 3.1. *Let p be a prime and let $G = A \rtimes \langle x \rangle$, where A is free abelian of finite rank but not cyclic and x has infinite order, such that the following properties hold:*

- (P_1) $\big|A/A^p[A,x]\big| = p$;
- (P_2) *x acts rationally irreducibly on A;*
- (P_3) *p does not divide the order of the automorphism induced by x on A/A^p.*

Then $G \in \mathbf{R}\mathfrak{F}_p$ and $A = \overline{\langle g \rangle}$ for every $g \in A \setminus A^p[A,x]$, and if n is either a positive integer or $\aleph_0$ then there exists $H \leq A$ such that $|\bar{H} : H| = n$.

Moreover, for every $n \in \mathbb{N}$ there exists $H \lhd G$ such that $H \leq A$ and $|\bar{H} : H| = p^{n(r-1)}$, where $r = \mathrm{rk}(A)$.

Proof. For every $t \in \mathbb{N}$ let $B_t = A^{p^t}[A, x^{p^t}]$ and $G_t = \langle x^{p^t} \rangle B_t$. Then it is clear that B_t and G_t are normal in G and $B_t = A \cap G_t$; moreover $G^{p^{2t}} \leq G_t \leq G^{p^t}$.

Since $\bigcap_{t \in \mathbb{N}} G_t \leq A$, in order to prove that $G \in \mathbf{R}\mathfrak{F}_p$ it will suffice to show that $B := \bigcap_{t \in \mathbb{N}} B_t = 1$. As $B \lhd G$, by (P_2) this amounts to proving that A/B is infinite or, equivalently, that the (descending) sequence of subgroups $(B_t)_{t \in \mathbb{N}}$ does not stop after finitely many steps. Fix $t \in \mathbb{N}$, let asterisks denote images modulo B_t^p and let $y = (x^*)^{p^t}$. Then $G_t^* = B_t^* \rtimes \langle y \rangle$. If $(G_t^*)' = B_t^*$ then the endomorphism θ of A^* defined by $\bar{u} \mapsto [\bar{u}, y]$ maps the socle B_t^* of A^* onto itself. Since A^* is finite this yields that θ is an automorphism, hence that $[A^*, y] = A^*$. However, this is false, since $A^p[A,x] < A$, by (P_1). Therefore $(G_t^*)' < B_t^*$. This shows that $G_t^*/(G_t^*)'$ has a quotient isomorphic to a rank-2 elementary abelian p-group, hence the same is true of G_t. Therefore $B_t \not\leq G_t^p$ (recall that G_t/B_t is cyclic). As $G_{2(t+1)} \leq G^{p^{2(t+1)}} \leq (G^{p^{2t}})^p \leq G_t^p$ we have that $B_t \not\leq G_{2(t+1)}$, hence $B_{2(t+1)} = A \cap G_{2(t+1)} < B_t$. This shows that A/B is infinite, so $B = 1$, as we claimed. Therefore $G \in \mathbf{R}\mathfrak{F}_p$.

We now compute closures (with respect to the topology on G) of subgroups of A. Note that $\{G_t \mid t \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 1. If $H \leq A$ then $\bar{H} = \bigcap_{t \in \mathbb{N}} HG_t \leq \bar{A} = A$ and so $\bar{H} = \bigcap_{t \in \mathbb{N}} HB_t$. Let $t \in \mathbb{N}$. By (P_3), x acts with p'-order on A/A^p, hence $A^pB_t = A^p[A, x^{p^t}] = A^p[A,x]$ is maximal in A, by (P_1). Therefore A/B_t is cyclic with maximal subgroup $A^p[A,x]/B_t$. It follows that $\langle g \rangle B_t = A$ for all $g \in A \setminus A^p[A,x]$. This proves that $\overline{\langle g \rangle} = A$ for all such g, in agreement with our statement. Moreover, for every $n \in \mathbb{N}$ and for every such g there exists $H \leq A$ such that $g \in H$ and $|A/H| = n$. As $\bar{H} = A$ this proves the first part of the proposition. To complete the proof, fix $n \in \mathbb{N}$ and compute the closure of A^{p^n}. Let m be the least positive integer such that $|A/B_m| \geq p^n$ (such m does exist, by the first part of the proof). For all integers $t \geq m$ we have that $A^{p^n}B_t/B_t$ is the subgroup of index p^n in A/B_t, and hence $A^{p^n}B_t = A^{p^n}B_m$. Therefore

$$\overline{A^{p^n}} = \bigcap_{m \leq t \in \mathbb{N}} A^{p^n}B_t = A^{p^n}B_m$$

and

$$\big|\overline{A^{p^n}}/A^{p^n}\big| = \frac{|A/A^{p^n}|}{|A/\overline{A^{p^n}}|} = \frac{p^{nr}}{p^n} = p^{n(r-1)}.$$

Since $A^{p^n} \lhd G$ the proof is complete. □

The proposition just proved prompts a number of remarks. Firstly, we claim that in the case when the rank r of the subgroup A is 2 or 3, verification of condition (P_2) reduces to checking that no nontrivial cyclic subgroup of A is normal in G, that is to say, that no nontrivial element of A is centralised or inverted by the action of x. In fact, if A is not rationally irreducible and $r \leq 3$ then A has a G-invariant subgroup N such that either N or A/N is infinite cyclic. In the latter case the image of one of the endomorphisms of A defined by $u \mapsto [u, x]$ or $u \mapsto uu^x$ is contained in N, hence this endomorphism's kernel is nontrivial and contains an infinite cyclic G-invariant subgroup. So, in either case, A has a nontrivial cyclic G-invariant subgroup. This justifies our claim.

Remark 3.2. Examples of groups satisfying the hypotheses of the previous proposition are now easily obtained for every prime p. If p is odd let $G = A \rtimes \langle x \rangle$, where $A = \langle a \rangle \times \langle b \rangle$ is free abelian of rank 2 and the action of x on A is defined by $a^x = ab^p$ and $b^x = ab^{p-1}$. Then x acts on the factor A/A^p as an automorphism of order 2 and $A^p[A, x] = A^p\langle ab^{p-2} \rangle$ is maximal in A. Moreover the matrix expressing the action of x with respect to the basis (a, b) has neither 1 nor -1 as an eigenvalue, so that conditions (P_1)–(P_3) in Proposition 3.1 are satisfied. This gives, for every odd prime p, an example of a polycyclic residually-p group with subgroups which are not almost p-closed (also in some rather strong sense), thus answering the original question.

For $p = 2$ the construction must be slightly different: in this case conditions (P_1) and (P_3) are incompatible with A having rank 2. However, an example is obtained as $G = A \rtimes \langle x \rangle$ where A is free abelian of rank 3 and x acts, with respect to a fixed basis of A, according to the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

of order 3 modulo 2.

Remark 3.3. It is also worth observing that the proof of Proposition 3.1 is somewhat simpler and more informative when $p > 2$. In fact, in the notation used in the proof the factors G/G_t are metacyclic; hence, if $p > 2$, they are regular p-groups. It follows that they have exponent p^t, so $G_t = G^{p^t}$ for all $t \in \mathbb{N}$. A consequence that can be read off from the proof is that $B_{t+1} < B_t$ for all t, hence $|A/B_t| = p^t$, and so $\overline{A^{p^n}} = B_n$ for all $n \in \mathbb{N}$.

Corollary 3.4. *Let p be an odd prime and n a positive integer. Then there exist an abelian-by-cyclic, polycyclic group G in $\mathbf{R}\mathfrak{F}_p$ and $H \lhd G$ such that $|\bar{H}/H| = n$.*

Proof. Let $n = qp^\lambda$, where p does not divide q and λ is a non-negative integer. Let G_0 be the residually-p group constructed in Remark 3.2. Then, by the last part of Proposition 3.1, G_0 has a normal subgroup H_0 of index p^λ in its closure K. Let $G = G_0 \times C$ where C is infinite cyclic, and let $H = H_0C^q$. Then K is also the closure of H_0 in G, and it is easy to check that the closure of H is $\bar{H} = KC$, so that $|\bar{H}/H| = n$. □

It remains open whether, in our usual setting, normal subgroups can have infinite index in their closures. Note that the subgroups considered in Proposition 3.1 are subnormal of defect 2 at most.

Next we will look at sufficient conditions for a subgroup of a polycyclic residually-p group to be almost p-closed. First we state and prove a simple lemma.

Lemma 3.5. *Let X be a polycyclic group having a normal p-subgroup P contained in the hypercentre of X and such that $X/P \in \mathbf{R}\mathfrak{F}_p \cap \mathfrak{N}(p)$. Then $X \in \mathbf{R}\mathfrak{F}_p \cap \mathfrak{N}(p)$.*

Proof. Note first that $X \in \mathfrak{N}(p)$: every finite quotient of X is p-nilpotent modulo its hypercentre and is therefore p-nilpotent. As X is residually finite, there is a normal subgroup N of finite index in X such that $N \cap P = 1$. If Q/N is the Hall p'-subgroup of X/N then X/Q is a finite p-group and $Q \cap P = 1$. Thus Q embeds in X/P; therefore $Q \in \mathbf{R}\mathfrak{F}_p$ and so $X \in \mathbf{R}\mathfrak{F}_p$. □

In [3], Corollary A3, Roseblade proves that the weak Artin-Rees property holds for all finitely generated modules over a group ring JG, where J has prime characteristic p and G is polycyclic, if and only if $G \in \mathfrak{N}(p)$. We will use only the 'if' part of this statement, in the following slightly

simplified form that can also be proved by elementary, standard arguments. We use multiplicative notation for the module operation.

Lemma 3.6. *Let p be a prime, G an $\mathfrak{N}(p)$-group and M a finite G-module of p-power order. Then for every submodule U of M there exists $n \in \mathbb{N}$ such that $[M, {}_nG] \cap U \leq [U, G]$.*

Theorem 3.7. *Suppose that p is a prime and G is a polycyclic, abelian-by-$\mathfrak{N}(p)$ group in $\mathbf{R}\mathfrak{F}_p$ and that H is a subgroup of the hypercentre of G. Then H is almost p-closed in G.*

Proof. Suppose that H is not almost p-closed in G. First we settle the case when $H \leq Z(G)$. We may assume that H is maximal among the subgroups of $Z(G)$ which are not almost p-closed in G; then $|\bar{H} : H| = p$. Let A be a maximal normal abelian subgroup of G such that $G/A \in \mathfrak{N}(p)$. Then $H \leq A = \bar{A}$. It is clear that $A/H \in \mathbf{R}\mathfrak{F}_p$, so there exists $B \leq A$ such that $B \lhd G$, A/B is a p-group and $H = B \cap \bar{H}$. By applying Lemma 3.6 to the (G/A)-module A/B and its submodule $\bar{H}B/B$ we obtain a G-invariant subgroup L of A containing B such that $L \cap \bar{H}B = B$ and A/L lies in the hypercentre of G/L. Now $G/L \in \mathbf{R}\mathfrak{F}_p$ by Lemma 3.5, and $L \cap \bar{H} = B \cap \bar{H} = H$, so that H is closed in G. This contradiction establishes the result for $H \leq Z(G)$.

Consider now the general case, where $H \leq Z_n(G)$ for some positive integer n. We may again assume that H is a maximal counterexample and so $|\bar{H} : H| = p$ (note that $\bar{H} \leq Z_n(G)$ since the latter is closed); we may further assume that the statement holds for all proper quotients of G which are residually-p. Suppose that $D := H \cap Z(G) \neq 1$. By the previous paragraph $\bar{D}/D$ is a finite p'-group and so $\bar{D} \leq H$. By assumption $H/\bar{D}$ is almost p-closed in $G/\bar{D}$, hence $|\bar{H}/H|$ is a finite p'-number. It follows that also $|\bar{H} : H|$ is a finite p'-number. By this contradiction, $H \cap Z(G) = 1$. Suppose that $Z(G)$ is infinite and let S be an infinite cyclic subgroup of $Z(G)$. If $\bar{H} \cap \bar{S} = 1$ then $|\bar{H}\bar{S} : H\bar{S}| = p$, contradicting the assumption that the result holds for $G/\bar{S}$. Thus $\bar{H} \cap \bar{S} \neq 1$. As $H \cap \bar{S} = 1$ it follows that $|\bar{H} \cap \bar{S}| = p$. Since $\bar{S}/S$ is a p'-group, as $S \leq Z(G)$, we have $\bar{H} \cap \bar{S} \leq S$. This is a contradiction, and we conclude that $Z(G)$ is finite. But then $Z_n(G)$ is finite, hence H is finite, which is impossible because H is not closed. This final contradiction completes the proof. □

Remark 3.8. Theorem 3.7 holds, as a special case, for abelian-by-(nilpotent-by-finite) polycyclic groups. It is a natural question to ask whether it actually holds for all polycylic groups, that is to say, whether the hypothesis that G is abelian-by-$\mathfrak{N}(p)$ can be dismissed. Note that our proof uses this hypothesis only in the case when H is central, so, only this case would need care. Also, one can reduce the problem to the case of torsion-free nilpotent-by-abelian groups. In fact, it can be proved that, *for every prime p, every polycyclic-by-finite group G in $\mathbf{R}\mathfrak{F}_p$ has a torsion-free normal subgroup J such that G/J is a finite p-group and $J/\operatorname{Fit} J$ is torsion-free abelian*; in this situation, a subgroup H is almost p-closed in G if and only if $H \cap J$ is almost p-closed in J.

One difficulty in tackling the problem of extending Theorem 3.7 to arbitrary polycyclic groups is that the corresponding extension of Lemma 3.5 does not hold: the following example seems to suggest that some kind of nilpotency requirement is necessary there. For a positive integer λ consider the class-2 nilpotent group $F = \langle a \rangle \times \langle b, c \rangle$, where $d := [b, c]$ has order $q = 2^\lambda$ while a, b and c have infinite order. Let $G = F \rtimes \langle x \rangle$, where x also has infinite order, $a^x = a^{q-1}b^q$, $b^x = ac$ and $c^x = bc^{-1}$. Then $G/\langle d \rangle \in \mathbf{R}\mathfrak{F}_2$, this follows from Proposition 3.1; as a matter of fact, if $\lambda = 1$ then $G/\langle d \rangle$ is isomorphic to the residually-2 group constructed in Remark 3.2. However, $G \notin \mathbf{R}\mathfrak{F}_2$. Indeed, x acts on F/F^2 by means of an automorphism of order 3, hence, for all $n \in \mathbb{N}$, either x^{2^n} or x^{-2^n} acts on F/F^2 in the same way as x. Now, $[F^2, F] \leq \langle d^2 \rangle$ and so $d \in [F, x]'\langle d^2 \rangle = [F, x^{2^n}]'\langle d^2 \rangle$. It follows that d lies in every normal subgroup of finite 2-power index in G, so that $G \notin \mathbf{R}\mathfrak{F}_2$. Therefore $X := F\langle x^2 \rangle \notin \mathbf{R}\mathfrak{F}_2$, while $X/\langle d \rangle \in \mathbf{R}\mathfrak{F}_2$ and $\langle d \rangle \leq Z(X)$; this is the counterexample sought.

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Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Via Cintia — Monte S. Angelo, I-80126 Napoli, Italy
E-mail address: cutolo@unina.it
URL: http://www.dma.unina.it/~cutolo/

Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837, USA.
E-mail address: howsmith@bucknell.edu